Edge states intermediate between laminar and turbulent dynamics in pipe flow

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We studied the dynamics near the boundary between laminar and turbulent dynamics in pipe flow. This boundary contains invariant dynamical states that are attracting when the dynamics is confined to the boundary. These states can be found by controlling a single quantity, in our case the energy content. The edge state is dominated by two downstream vortices and shows intrinsic chaotic dynamics. With increasing Reynolds number the separation between the edge state and turbulence increases. We can track it down to $Re=1900$, where the turbulent lifetimes are short enough that spontaneous decay can also be seen in experiments.

Keywords: turbulence transition; pipe flow; coherent states

1. Introduction

Turbulence in pipe flow has to coexist with the laminar profile since the latter is linearly stable for all Reynolds numbers. Triggering turbulence hence requires not only a sufficiently high Reynolds number but also a perturbation of sufficient amplitude. The determination of this ‘double threshold’ in Reynolds number and perturbation amplitude has been the focus of many experimental, numerical and theoretical studies (Reynolds 1883; Boberg & Brosa 1988; Darbyshire & Mullin 1995; Grossmann 2000; Hof et al. 2003). The double threshold suggests the existence of a surface in the state space of the system such that trajectories on one side return to the laminar profile and those on the other side become turbulent. Ideally, this surface should be of co-dimension one, so that it fully divides state space. Experiments in which controlled perturbations are used to trigger turbulence can be used to probe this surface locally.

For many perturbations one can separate the spatial dependence of the perturbation, as defined, for example, by the arrangement of jets, from its amplitude, as determined by the rate or amount of fluid injected (for a discussion of the subtleties involved in determining amplitudes, see Trefethen et al. 2000). Keeping the spatial dependence fixed one may determine a critical amplitude above which the flow becomes turbulent, and then by varying the spatial structure of the perturbation determine the entire surface. An indicator that can be used to decide whether one is on the laminar or turbulent side is the lifetime of...
a perturbation, determined as the time it takes to return to a value close to that of the laminar profile (Faisst & Eckhardt 2004). Lifetimes will be short when the perturbation amplitude is small, will increase as one moves away from the laminar state and will diverge when the boundary is crossed. Beyond this point, the lifetimes will usually vary in a chaotic manner. This led us to call the point of divergence a point on the edge of chaos (Skufca et al. 2006). This concept generalizes the usual basin boundaries between attractors to situations in which transient chaos is also present, as will be shown below (Vollmer et al. 2008).

It is our aim here to study the dynamics of velocity fields that are intermediate between laminar and turbulent flows and evolve inside the edge. They are unstable, since a slightly stronger perturbation will trigger the transition to turbulence, and a slightly weaker one will cause decay towards the laminar profile. Using the algorithms described in Skufca et al. (2006) and Schneider et al. (2007a), it is, however, possible to confine the velocity fields to this intermediate situation. Interestingly, these studies suggest that all edge points at a given Reynolds number are dynamically connected. Thus, at a fixed Reynolds number the edge of chaos is a subset of the state space that carries its own dynamics. Moreover, the work by Skufca et al. (2006) suggests that there is an invariant dynamic object embedded in the edge that resides in state space between the laminar flow and turbulent dynamics. The edge of chaos then coincides with the stable set of this invariant dynamical object, which we call the edge state.

In §2 we describe how the dynamics inside the edge of chaos can be tracked and how the invariant edge state emerges. In §3 we discuss the unstable directions of the edge state. The variation of the edge state with Reynolds number is the topic of §4. We close with a few concluding remarks in §5.

## 2. Tracking the dynamics inside the edge

The procedure that allows one to track the dynamics on the dividing surface is as follows. Consider a perturbation on top of the laminar profile and imagine that the form and spatial structure of the velocity field are fixed, but that the amplitude of the perturbation can be varied. Then, for sufficiently small amplitude, this perturbation will decay as a consequence of the linear stability of the parabolic profile. The possible transient growth due to non-normal amplification does not invalidate this conclusion, but suggests that the amplitude of a perturbation that can trigger turbulence may be very small, and may become even smaller when the Reynolds number is higher. A perturbation with strong enough amplitude will increase further in amplitude and will become turbulent. Intermediate between these two situations will be an amplitude for which the velocity field neither relaxes to the laminar one nor becomes turbulent. A pair of trajectories on either side of the edge, i.e. where one trajectory decays and the other becomes turbulent, can then be used to approximate an edge trajectory on the edge.

We used two kinds of perturbations to study and to follow the dynamics inside the edge. For the first kind, we took a velocity field that came out of a turbulent flow at a Reynolds number of $Re=3000$. For the second kind, we took a pair of vortices, as in the optimally growing modes identified by Zikanov (1996),
modulated in streamwise direction by applying a $z$-dependent twist in order to break translational symmetry

$$u_0(r, \varphi, z) = u_{Zik}(r, \varphi + \varphi_0 \sin \left(\frac{2\pi L}{z} \right), z),$$

where $u_{Zik}$ is Zikanov’s mode and $L=5D$ is the length of the computational domain used in our direct numerical simulation. The initial condition is described by two parameters: the amplitude $A_0 = \sqrt{E_0}$, where $E_0$ is the kinetic energy of the initial perturbation, and the angle $\varphi_0$, which controls the breaking of translational invariance. In the studies below, we fix $\varphi_0 = 0.05$ and vary the amplitude $A_0$ only.

The pair of initial conditions and trajectories that approximates the one within the edge is found along a ray connecting the laminar profile and an initial condition that becomes turbulent. We determine a pair of close-by initial conditions, one of which becomes turbulent and the other relaminarizes. Since the separation between them grows exponentially in time, new pairs have to be found after some time. Technically, we follow the trajectory starting from $u(t=0) = u_{HP} + A_0 v_0$ for 200 time units to the point $u(t) = u_{HP} + v_1$ and determine a new pair of amplitudes $(A'_l, A'_h)$ such that the trajectory starting from $u_{HP} + A'_l v_1$ decays and the one starting from $u_{HP} + A'_h v_1$ becomes

Figure 1. The sequence of instantaneous snapshots shows the evolution towards the invariant edge state along an edge trajectory at $Re=2875$. The downstream velocity relative to the parabolic laminar profile is shown in color ranging from −0.25 (blue) to 0.25 (red). In-plane velocity components are indicated by arrows. The evolution starts from a perturbed Zikanov mode (a) at $t=0$. The other snapshots from (a)–(f) show the state at times $t=10$, 280, 355, 880 and 1680 later. The counter-rotating vortices of the initial flow field first generate streaks via the known lift-up process. The high-speed streak then breaks apart and rearranges. The last snapshot is already close to the invariant object embedded inside the edge.
turbulent. Given a typical energy $E \approx 2.5 \times 10^{-2}$ of the edge trajectory and a relative difference of $10^{-5}$ of the amplitude, the initial separation in energy is $\Delta E \approx 2.5 \times 10^{-12}$. The relative separation grows in time as $\Delta E = \exp(0.05t)$ and reaches $\Delta E \approx 5 \times 10^{-8}$ after 200 time units. The relative accuracy of the approximation is thus $\Delta E/E \approx 2 \times 10^{-6}$ in energy or $\sqrt{\Delta E/E} \leq 2 \times 10^{-3}$ in amplitude (Schneider et al. 2007a).

The method described differs somewhat from the one used in Skufca et al. (2006). They interpolate between the two approximating solutions and use bisection to find new initial conditions along the connection of the two interpolating states. Instead, here the new pair of initial conditions is found by controlling in the direction defined by scaling the energy of one state only. If the edge trajectory has only one unstable direction, both methods are equivalent. In pipe flow, however, in addition to the unstable direction perpendicular to the edge there is evidence for additional unstable directions within the edge which can cause jumps in the edge trajectory when an interpolation between the approximating trajectories is used. The large separation in energy between the edge of chaos and the turbulent state shows that deviations from the edge are linked to changes in energy and suggest a considerable overlap of the direction defined by scaling the energy of a velocity field and the ones pointing away from the edge. Consequently, our method is tailored towards preventing the trajectory from becoming turbulent or from decaying while at the same time minimizing control inside the edge.

As long as the direction in state space defined by the amplitude scaling is not tangent to the edge of chaos, this operational technique can be used to obtain arbitrarily long traces of a state that neither decays nor becomes turbulent: this trajectory lives inside the edge of chaos and should approach the invariant ‘state’ embedded within the edge of chaos, the edge state.

Various stages during the evolution of the dynamics inside the edge are shown in figure 1 for a perturbation of the parabolic profile with a modified Zikanov mode. The different stages that one can identify are (i) a modulation of the downstream velocity field due to the vortices, (ii) a noticeable breaking of reflection symmetry, (iii) the formation of two high-speed streaks close to the walls and a low-speed streak in the central region, and (iv) a persistent dynamics that is predominantly confined to the central region.

A similar sequence of events is observed when the perturbing velocity field is more complex, as is the case for a turbulent velocity field (see Schneider & Eckhardt 2006). The final structure that is obtained is similar to the one described here, except for the obvious continuous symmetries of azimuthal and axial translations. All initial conditions were observed to converge to the same edge state, up to these two symmetries.

We conclude that the edge of chaos separating laminar and turbulent dynamics at fixed Reynolds number carries an internal dynamics, in that all points are dynamically connected. Embedded within the edge is a locally attracting, invariant object, the edge state. Quite generally, this invariant object can be a simple fixed point, as in the familiar situation of a saddle-node bifurcation, or it can be a periodic orbit or more complicated (see Skufca et al. (2006), Pringle & Kerswell (2007), Mellibovsky & Meseguer (2009) and Vollmer et al. (2008) for examples). In the case of pipe flow the edge state turns out to be chaotic. Consequently, the edge of chaos is formed by the stable set of a chaotic saddle,
which is the edge state of pipe flow. While the edge of chaos is extended throughout the complete state space the statistical weight on the chaotic attractor appears to be centred in a small region of state space which leads to the simple and universal global structure of the observed edge state.

3. Unstable directions of the edge state

In the following, we analyse how a trajectory evolves from the edge state to turbulence and how it decays from the edge. A typical example of the time evolution for two trajectories that approximate the edge trajectory is shown in figure 2. One can clearly identify a time interval up to about $t \approx 300$ during which the energies of both trajectories are very close together. Thereafter, both trajectories start to separate noticeably in energy: while one (shown in black) decays, the other (shown in grey) shows a sharp increase in energy.

The difference between the two trajectories increases exponentially (Schneider et al. 2007b). The associated unstable direction may be determined from normalized differences between the trajectory that becomes turbulent $u_t(t)$ and the decaying one $u_d(t)$,

$$
\delta u(t) \equiv \frac{u_t(t) - u_d(t)}{\|u_t(t) - u_d(t)\|},
$$

where the norm is the square root of the energy in units of the kinetic energy of the laminar profile, $\|u\| = (\int_{\text{vol}} u \cdot u \, dV^{1/2})/(\int_{\text{vol}} (1 - y^2)^2 \, dV^{1/2})$.

In figure 3 instantaneous snapshots of the normalized flow field $\delta u$ are presented. Since the initial conditions of both approximating trajectories are chosen by scaling the energy of the same velocity field, the difference vector by construction initially corresponds to the normalized edge state. However, it then turns into the direction of the largest Lyapunov exponent. In the centre, vortical structures appear, get stronger and modulate the streaks, which are finally drawn apart. Note

Figure 2. Energy $E$ of the deviation from the laminar profile plotted as a function of time $t$ since the start of the approximation step. Shown is a pair of trajectories starting from neighbouring initial conditions. One swings up to turbulence (grey line), the other one decays (black line). Both trajectories start to deviate considerably in energy at $t \approx 300$. Snapshots of the flow field that becomes turbulent are presented in figure 4.
Figure 3. Unstable direction of the edge state at $Re=2875$. Shown is the temporal evolution of the normalized difference between the two approximating trajectories $\delta u$. (a(i)–(iv)) shows isosurfaces of the downstream velocity at levels $\pm 0.5$ as indicators for the position of the streaks (low speed in blue, high speed in red). (b(i)–(iv)) highlights vortical structures by showing isosurfaces of the downstream vorticity at the levels $\pm 2.5$. The four snapshots are taken (from top to bottom) at times $t=0.1, 5, 10$ and $50$ after the new pair of initial conditions was chosen. Since the two chosen initial conditions differ only in amplitude, by construction $\delta u$ in the beginning corresponds to the normalized edge state itself. However, the difference vector then turns in the direction of the largest Lyapunov exponent. Vortical structures in the centre appear, start to modulate the streaks and finally tear them apart. Note that the difference field becomes complex and shows ruptured streak structures on a time scale of less than 50 time units, where the approximation of the edge trajectory is still very accurate (cf. figure 2).

Figure 4. Instantaneous snapshots of the velocity field along the trajectory that becomes turbulent. The snapshots are taken at times (from (a) to (h)) $t=300, 435, 440, 445, 450, 455, 460$ and $500$. Shown are cross sections perpendicular to the pipe axis. The downstream velocity relative to the laminar profile is shown in colour ranging from $0.25$ (blue) to $-0.25$ (red). In-plane velocity components are indicated by vectors. The sharp increase of energy by almost one order of magnitude is caused by vortical structures that are tilted inside the volume and tear the streaks apart.
that the separation vector turns into a direction that corresponds to a velocity field with ‘ruptured’ streaks on a time scale $t \leq 50$, where the confinement to the edge is still very good (figure 2). The growth rate associated with this instability as estimated from four separations is approximately $5 \times 10^{-2} U_{cl}/R$ (see Schneider et al. 2007b). The separation field does not show a simple large-scale structure but complex dynamics, which can be attributed to the chaotic dynamics of the edge state and to additional unstable directions within the edge.

Figure 5. Variation of the edge state with Reynolds number. From top to bottom the Reynolds number is $Re = 2160, 2320, 2500$ and 4000. (a(i)–(iv)) Shown are energy traces of trajectories approximating the edge dynamics. One can estimate the typical energy both of the edge state and the turbulent ‘state’. Energetically the separation of the edge trajectory and turbulence grows with $Re$ (cf. figure 7). (b(i)–(iv)) Time-averaged cross sections of the edge state flow field are presented. The colour runs linearly from $-0.2$ (blue) to $+0.2$ (red). One observes that the topology of the flow field does not change much. This is further supported by the instantaneous snapshots presented in (c(i)–(viii)). (a–c) Isosurfaces of the downstream velocity indicate the arrangement of the streaks (levels are $0.1$ for $Re = 2160$ and 2300 and $0.08$ for $Re = 2500$ and 4000). (a,b(iv),c(viii)) Isosurfaces of the downstream vorticity (levels are $0.25$ for $Re = 2160$ and 2300 and $0.22$ for $Re = 2500$ and 4000) reveal vortical structures in the centre region.
For times longer than approximately 300, the difference vector leaves the region where it can be described by the linearized vectors, and the further time evolution of the trajectories that decay and become turbulent differs noticeably.

For the decaying trajectory, one observes that first the intensity of vortical structures in the centre is reduced. Transverse modulations drop below a threshold where the energy feedback to the streaks can no longer take place and the streak arrangement decays as its kinetic energy is dissipated. The monotonic decay suggests that the edge state is located at the boundary of the laminar state’s basin of attraction. There is no evidence for additional state-space structures that deflect the decaying trajectory from its direct path towards the laminar state.

The further evolution of the trajectory which becomes turbulent is governed by the interaction of downstream oriented vortices and the streak structure. Since this interaction is best visualized in instantaneous cross sections of the field, we present in figure 4 eight snapshots of the trajectory swinging up to turbulence (cf. figure 2).

In the first step (up to $t \approx 400$) the vortical structures in the centre grow in intensity but the overall topology of the state is not affected. The stronger vortices then start to modulate the downstream streaks. In the final stage of this process, the sharp increase in energy at $t = 430 \cdots 470$ is caused by strong vortices, tilted inside the volume with respect to the downstream direction. They effectively draw energy from the streak gradients and tear the two characteristic high-speed streaks of the edge state apart. Thus, as the decay of the edge state is caused by reduced intensity of the vortices in the centre, the evolution towards turbulence is initiated by growing intensity of the vortices. The final ‘blow-up’ is then caused by vortices tilted against the downstream direction similar to hairpin vortices that are known to be linked to the most energetic events in wall-bounded turbulence.

4. Edge states for different $Re$

The results in the previous sections describe the edge state at $Re = 2875$. Upon variation of the Reynolds number the overall appearance of the edge state does not change much (see figure 5, and the results by Viswanath (2009), which...
discusses even higher Reynolds numbers). We did not find any bifurcations or transitions between flow topologies, as opposed to the low-dimensional model studied in Skufca et al. (2006).

The energy of the edge state and turbulent trajectories are shown in figure 5(a)–(iv). They clearly show that the two are well separated for larger Reynolds numbers, but that the difference becomes smaller for lower Reynolds numbers. Thus, it becomes increasingly difficult for our algorithm to detect when a trajectory has left the edge and has entered the turbulent region. However, when not the full energy but only the energy content of the modes that are not translationally invariant is used, the separation becomes much clearer (figure 6). Nevertheless, following the edge state to lower Reynolds number requires manual interference and decisions on many levels and cannot be automated as easily as at higher Reynolds numbers.

In figure 7, we show the variation of the energies of the edge state and the turbulent dynamics with Reynolds number. As $Re$ decreases, the energy of the turbulent state decreases and the one from the edge state increases and they seem to merge at some Reynolds number near 1800. For Reynolds numbers near 1900, the edge state can be identified (figure 8). Note that numerical simulations for this domain show that the turbulence at this Reynolds number is transient (Hof et al. 2006), so that the coalescence between the turbulence and the edge state cannot be correlated with a possible crisis that would turn the transient turbulent saddle into a permanent attractor. Moreover, the ability to determine an edge state when the turbulence is transient and not permanent shows that the edge state tracking algorithm as presented here generalizes the concept of basin boundaries to situations of transient chaos (for a discussion within a model system, see Vollmer et al. 2008).

As a consequence of the observed energy variation the saddle and the edge approach each other as one reduces the Reynolds number. Thus, fluctuations of the turbulent dynamics will more likely reach the edge, thereby rendering the stabilization of the edge state more difficult at lower Reynolds numbers.

It is interesting to see that while the energy separation between laminar and turbulent flow decreases with decreasing Reynolds number, the unstable eigenvalues of the edge state increase. The Lyapunov exponent for the transverse
dynamics increases by approximately 15 per cent when the Reynolds number is reduced from 4000 to 2160. This is to be contrasted with the variation of the Lyapunov exponent of the turbulent state, which decreases with decreasing \( Re \) (Faisst 2003; Faisst & Eckhardt 2004).

5. Conclusions

The form and topology of the edge state suggests that it could be induced experimentally, for example by removing fluid at one point and injecting fluid at two points to the left and right of the injection point. Perturbations of this type will induce a pair of counter-rotating vortices as in the optimally amplifying mode found by Zikanov (1996). The vortices then draw energy from the laminar base profile and induce high- and low-speed streaks in the downstream velocity. As in the case of the self-sustained cycle for near wall turbulence (Hamilton et al. 1995; Waleffe 1995), one might anticipate a shear flow instability of the streak arrangement if the edge state breaks down and the flow turns turbulent. However, the unstable direction presented in figure 3 shows that the leading instability favours the formation of small-scale structures in the centre region.

For large Reynolds numbers, the edge state is well separated from the turbulent dynamics. When the Reynolds number is reduced, the separation becomes less well defined, until approximately near 1800 the edge and turbulence can no longer be distinguished with the algorithms used here. Since this value is lower than those at which turbulence has been shown to be transient, this coalescence does not point to a crisis bifurcation by which a repeller could become an attractor or vice versa. However, it does suggest that the phase-space structure of the flow is reorganized, though the details of that change remain to be explored.

We thus conclude that the transition to turbulence in pipe flow is mediated by a flow structure that is neither located in the neighbourhood of the stable laminar profile nor is part of the statistically relevant ‘core region’ of the chaotic saddle. Explaining the transition to turbulence in pipe flow and other related linearly stable flow situations thus requires the role and the interaction of three dominant objects in the system’s state space to be understood: the laminar state; the strange chaotic saddle; and the edge state presented here.
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