Composite asymptotic expansions and scaling wall turbulence

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In this article, the assumptions and reasoning that yield composite asymptotic expansions for wall turbulence are discussed. Particular attention is paid to the scaling quantities that are used to render the variables non-dimensional and of order one. An asymptotic expansion is proposed for the streamwise Reynolds stress that accounts for the active and inactive turbulence by using different scalings. The idea is tested with the data from the channel flows and appears to have merit.

Keywords: composite expansions; Reynolds stresses; streamwise Reynolds stress; inactive turbulent motion

1. Introduction

Mathematically, the theory of turbulence is the leading term in asymptotic expansions for high Reynolds number. Wall turbulence has two regions and thus requires two matched asymptotic expansions. A composite of these two expansions is needed to approximate the data for any position and to show the effect of varying Reynolds number. In this paper, standard methods of asymptotic expansion mathematics are applied to the turbulence data. First, the Reynolds shear stress is discussed, followed by the mean velocity. Commentary on the essential features of Poincaré expansions, the matching process and composite expansions is also included. Finally, the streamwise Reynolds stress is considered, and a two-term expansion is proposed. The motivation for this expansion is to treat the active and inactive motions with different scalings. Tennekes & Lumley (1972) discuss scaling, while Panton (2005) deals with composite expansions.

2. The flow conditions

Consider the flow in a slot formed by plane smooth walls. Assume that a fluid of density $\rho$ and viscosity $\nu$ is in a flow driven by a pressure gradient $dP_0/dx$ along the wall. Additionally, assume that the flow is fully developed, so that the mean velocity $U(y)$ and the Reynolds shear stress $-\langle uv \rangle$ (time averaging is indicated

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by ( ) are independent of the flow direction coordinate \( x \). The transverse coordinate \( y \) is measured from the lower wall and the half-height, \( y = h \), is the centreline where the velocity is \( U_0 \). The mean velocity and Reynolds stress profiles are

\[
U = U\left(y, \nu, \frac{1}{\rho} \frac{dP_0}{dx}, h\right),
\]

and

\[
-\langle uw \rangle = -\langle uv \rangle\left(y, \nu, \frac{1}{\rho} \frac{dP_0}{dx}, h\right),
\]

respectively. Several non-dimensional forms are possible. The proper form is of order one in the Reynolds number limit \( Re_\ast \to \infty \).

As boundary conditions, one assumes that

\[
\frac{dU}{dy}\bigg|_{y = h} = 0; \quad U(y = h) = U_0, \quad (2.3)
\]

\[
-\langle uv \rangle(y = h) = 0, \quad (2.4)
\]

\[
U(y = 0) = U(y = 2h) = 0, \quad (2.5)
\]

and

\[
-\langle uv \rangle(y = 0) = -\langle uv \rangle(y = 2h) = 0. \quad (2.6)
\]

Both the mean velocity and the Reynolds stress vanish at the walls.

### 3. Governing equations and the scaling parameters

The centreline velocity \( U_0 \) is related to the other parameters, in particular the pressure gradient, by evaluating equation (2.1) at \( y = h \).

\[
U_0 = U\left(h, \nu, \frac{1}{\rho} \frac{dP_0}{dx}, h\right). \quad (3.1)
\]

The Reynolds-averaged \( y \)-momentum equation yields \( P_0(x) = p(x, y) + \rho \langle uv \rangle(y) \), and hence we find that \( \partial p/\partial x = dP_0/dx \). The \( x \)-direction momentum equation integrated from the wall, where \( \tau_0 = \mu \frac{dU}{dy}\big|_0 \), to a position \( y \) yields

\[
0 = -\frac{y}{\rho} \frac{dP_0}{dx} - \langle uv \rangle(y) + \nu \frac{dU(y)}{dy} - \frac{\tau_0}{\rho}. \quad (3.2)
\]

Equation (3.2) can be evaluated at the opposite wall \( y = 2h \), where \( \tau_0 = -\mu \frac{dU}{dy}\big|_{2h} \), yielding

\[
-\frac{h}{\rho} \frac{dP_0}{dx} = \frac{\tau_0}{\rho}. \quad (3.3)
\]

The friction velocity is defined by

\[
u_s^2 = \frac{\tau_0}{\rho} = \nu \frac{dU}{dy}\bigg|_0 \equiv -\frac{h}{\rho} \frac{dP_0}{dx}. \quad (3.4)
\]

The \( x \)-momentum equation (3.2) now reads

\[
0 = u_s^2\left(\frac{y}{h} - 1\right) - \langle uv \rangle(y) + \nu \frac{dU(y)}{dy}. \quad (3.5)
\]
Thus, with equation (3.1) or (3.4), the pressure gradient can be replaced by either $U_0$ or $u_*$. The following two Reynolds numbers are possible:

$$Re = \frac{U_0 h}{\nu}; \quad Re_* = \frac{u_* h}{\nu}.$$  \hspace{1cm} (3.6)

Experimentally, it is known that $U_0 / u_* \rightarrow \infty$ as $Re_* \rightarrow \infty$. Theoretically, it is found that

$$\frac{U_0}{u_*} = \frac{1}{\kappa} \ln Re_* + C.$$  \hspace{1cm} (3.7)

The two velocities $U_0$ and $u_*$ are not equivalent parameters to use as scales because their ratio is not constant.

4. Outer region variables

For the velocity profile, equation (2.1), let us replace the pressure gradient by the centreline velocity, thus $U = U(y, \nu, U_0, h)$. Non-dimensional variables of order one are

$$F = \frac{U}{U_0}; \quad Y = \frac{y}{h}.$$  \hspace{1cm} (4.1)

However, since $Re$ and $Re_*$ are equivalent, we may consider

$$\frac{U}{U_0} = F = F(Y, Re_*).$$  \hspace{1cm} (4.2)

Next, consider the Reynolds shear stress. The intensities of turbulent fluctuations are a smaller fraction of $U_0$ as the Reynolds number increases. For the Reynolds stress function (2.2), replace the pressure gradient by $u_*$,

$$-\langle uv \rangle = -\langle uv \rangle(y, \nu, u_*, h).$$  \hspace{1cm} (4.3)

Non-dimensional variables of order one are

$$G = -\frac{\langle uv \rangle}{u_*^2}; \quad Y = \frac{y}{h},$$  \hspace{1cm} (4.4)

$$G = G(Y, Re_*).$$  \hspace{1cm} (4.5)

The $x$-momentum equation confirms (motivates) these scalings. The non-dimensional form of equation (3.5) is

$$0 = (Y - 1) + G + \frac{\ln Re_* + C}{Re_*} \frac{dF}{dY}.$$  \hspace{1cm} (4.6)

Since the indeterminate form $\ln Re_* / Re_* \rightarrow 0$ as $Re_* \rightarrow \infty$, equation (4.6) shows that $G$ will be of order one as $Re_* \rightarrow \infty$.

5. Outer region asymptotic expansions

A formal way to proceed is to propose Poincaré asymptotic expansions for the limit $Re_* \rightarrow \infty$,

$$\frac{U(y)}{U_0} = F(Y, Re_*) \sim F_0(Y) + \Delta_1(Re_*)F_1(Y) + \cdots,$$  \hspace{1cm} (5.1)

$$\frac{-\langle uv \rangle(y)}{u_*^2} = G(Y, Re_*) \sim G_0(Y) + \delta_1(Re_*)G_1(Y) + \cdots.$$  \hspace{1cm} (5.2)
If the proper gauge functions are chosen for $\Delta_1$ and $\delta_1$, then $F_1$ and $G_1$ are of order one.

When equations (5.1) and (5.2) are substituted into the momentum equation (4.6) and the terms of the same order are grouped together, each group must be zero. The group at zero order is

$$G_0 = 1 - Y.$$  \hspace{1cm} (5.3)

Thus, asymptotic scaling arguments produce the Reynolds stress profile.

The Reynolds stress profile satisfies the centreline condition, but at the wall, $G_0=1$ and the no-slip condition is not met. This is the first indication that the problem is a singular perturbation and that a layer with different physics exists near the wall.

In order to find the velocity profile, consider the turbulent kinetic energy equation in outer variables,

$$0 = -G \frac{dF}{dY} - \frac{u_*}{U_0} \frac{d}{dY} \left[ \frac{1}{2} \langle v^2 \rangle + \frac{\rho w^3}{\rho u_*^3} \right] + \frac{1}{Re_*} \frac{u_*}{U_0} \frac{d^2}{dY^2} \left( \frac{k}{u_*^2} \right) - \frac{u_*}{U_0} \varepsilon^*, \hspace{1cm} (5.4)$$

where $k = (1/2) \langle uu + vv + ww \rangle = (1/2) q^2$; $\varepsilon^*$ is the non-dimensional dissipation; and turbulent fluctuations are assumed to scale with $u_*$. For high Reynolds numbers, $u_*/U_0 \to 0$ as $Re \to \infty$, and we find that the zero-order group is

$$0 = G_0 \frac{dF_0}{dY}. \hspace{1cm} (5.5)$$

Since $G_0$ is not zero, $F_0$ is constant, and from $U(Y=1)=U_0$, $F_0=1$. The first finding is that the velocity is a uniform flow. This is another failure to meet the no-slip condition, and again indicates a singular region near the wall.

We may improve on the answer by adding the second term, $F_1$. Reconsider the kinetic energy equation. If we chose $\Delta_1 = (u_*/U_0)$, then it means that production, diffusion and dissipation are all possible effects on the profile

$$0 = -G_0 \frac{dF_1}{dY} - \frac{d}{dY} \left[ \frac{1}{2} \langle v(uu + vv + ww) \rangle + \frac{\rho w^3}{\rho u_*^3} \right] - \varepsilon^*. \hspace{1cm} (5.6)$$

The outer-layer velocity profile is

$$\frac{U}{U_0} (Y, Re_*) \sim 1 + \frac{u_*}{U_0} (Re_*) F_1(Y). \hspace{1cm} (5.7)$$

Rearranging equation (5.7) yields the defect law that was originally proposed as an empirical correlation,

$$F_1(Y) = \frac{U(y) - U_0}{u_*}. \hspace{1cm} (5.8)$$

6. Inner region variables and expansions

The first step is to determine the scales for $\langle uv \rangle$, $U$, and $y$. The inner region Reynolds stress will vary from 0 to 1 if $u_*$ is the scale,

$$g \equiv -\frac{\langle uv \rangle}{u_*^2} = g(y^+, Re_*) \sim g_0(y^+) + \cdots \text{ as } Re_* \to \infty. \hspace{1cm} (6.1)$$
Let $d$ be the inner-layer length-scale and $u_s$ the velocity scale, then

$$\frac{y^+}{d},$$

$$(6.2)$$

$$\frac{U(y)}{u_s} = f(y^+, Re_*) f_0(y^+) + \cdots.$$  

$$(6.3)$$

The function $f_0(y^+)$ is the law of the wall.

The definitions of $u_s$ and $d$ are chosen so that the governing equations contain significant and different physical effects than those in the outer layer. If we set $u_s = u_*$ and $d/h = 1/Re_*$, the inner variables are

$$y^+ = \frac{u_* y}{v}; \quad f_0 = \frac{U}{u_*}; \quad \text{and} \quad g_0 = -\frac{\langle uv \rangle}{u_*^2}.$$  

$$(6.4)$$

Then, the zero-order inner region equations will be

$$g_0 + \frac{df_0}{dy^+} = 1,$$  

$$(6.5)$$

$$0 = -g_0 \frac{df_0}{dy^+} - \frac{d}{dy^+} \left[ \frac{1}{2} \left\langle v(uu + vv + wv) \right\rangle \frac{u_*^3}{\rho u_*^3} \right]_0 + \frac{d^2}{dy^+} \left( \frac{k_0}{u_*^2} \right) - \epsilon_0^+.$$  

$$(6.6)$$

If all the wall layers have the same inner velocity profiles, then equation (6.5) shows that they also have the same Reynolds stress profiles.

Note that

$$y^+ = Y Re_*.$$  

$$(6.7)$$

Because $Re_*$ is the ratio of the outer scale $h$ to the inner scale $v/u_*$, it is the natural normalized perturbation parameter.

### 7. Matching and composite expansion for the Reynolds stress

There is a region where the Reynolds stress functions $G_0(Y)$ and $g_0(y^+)$ match to a constant common part,

$$G_0(Y \Rightarrow 0) \equiv G_{cp} = 1,$$  

$$(7.1)$$

$$G_{cp} = g_{cp},$$  

$$(7.2)$$

$$g_0(y^+ \Rightarrow \infty) \equiv g_{cp} = 1.$$  

$$(7.3)$$

The matching requirement (7.2) shows that $g_{cp} = 1$.

Neither $G_0(Y)$ nor $g_0(y^+)$ is valid for all $y$, nor do they show the influence of $Re_*$. A complete profile is given by a composite expansion, a concept introduced by Latta (1951), who proved that such a profile is uniformly valid for all $y$. Coles noted (Coles & Hirst 1968) that his wake law had a formal connection to a composite expansion of asymptotic theory. Consider the additive composite

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expansion, the sum of the inner and outer expansions minus the common part,
\[ -\frac{\langle u'v' \rangle}{u^*_{comp}} \sim g_0(y^+) + G_0(Y \Rightarrow y^+/Re^*) - G_{cp}. \tag{7.4} \]

For \( y \) positions near the wall, \( G_0 \) is equal to \( G_{cp} \) and the Reynolds stress is correctly given by \( g(y^+) \). For \( y \) positions far from the wall, \( g_0 \) is equal to \( G_{cp} \) from the matching condition (7.2), and the Reynolds stress is correctly given by \( G_0(Y) \). At the intermediate \( y \) positions, the Reynolds stress makes a smooth transition between the inner and outer functions. Fortunately, for pipe and channel flows, we know the exact result for \( G_0 \), equation (5.3). Hence, in these flows, equation (7.4) becomes
\[ -\frac{\langle u'v' \rangle}{u^*_{comp}} \sim g_0(y^+) + (1 - Y) - 1 = g_0(y^+) - y^+/Re^*. \tag{7.5} \]

The \( g_0(y^+) \) function can be determined by solving equation (7.5) and substituting the data for \( \langle u'v' \rangle/u^*_c \equiv \langle u'v' \rangle^+ \). Typical channel flow experiments (Zanoun et al. 2002; Zanoun 2003) and DNS (Del Álamo & Jimenez 2003; Hoyas & Jimenez 2005) are shown in figure 1. An empirical formula (Panton 1997) is also shown in figure 1,
\[ g_0(y^+) = \frac{2}{\pi} \arctan \left( \frac{2\kappa y^+}{\pi} \right) \cdot \left[ 1 - \exp \left( \frac{-y^+}{C^+} \right) \right]^2. \tag{7.6} \]

Equation (7.6) has two constants: \( \kappa \) the von Kármán constant and \( C^+ \), which can be related to the additive constant in the log law by integrating equation (6.5). For the above data, \( C^+ = 6.78 \) when \( \kappa = 0.37 \).

The Reynolds stress profiles computed using equations (7.5) and (7.6) are shown in figure 2, along with the DNS data. Asymptotically, the behaviour of the maximum is
\[ y_{max}^+ \sim \sqrt{\frac{Re^*_s}{\kappa}}; \quad -\langle u'v' \rangle_{max} \sim 1 - \frac{2}{\sqrt{\kappa Re^*_s}} \quad Re^*_s \to \infty. \tag{7.7} \]

Sreenivasan (1987) found the \( y_{max}^+ \) relation empirically with a coefficient of 1.8–2, while a simplified momentum analysis by Antonia et al. (1992) produced the correct coefficient, \( \kappa^{-1/2} \sim 1.58 \).

8. The matching problem

In the simplest matching, the common part is a constant. In more complicated situations, the common part is a function of \( y \). This happens if the dependent variable changes scaling between the inner and outer regions. The Kolmogorov spectrum law is of this type. Another situation is when logarithms occur as gauge functions.

Rigorous matching theory is done in terms of an intermediate independent variable. Consider
\[ \gamma \equiv YRe^*_s^\alpha = y^+ Re^*_s^{\alpha-1} \quad \text{with } 0 < \alpha < 1. \tag{8.1} \]
Figure 1. Inner Reynolds stress data of Zanoun et al. (2002) and equation (7.6).

Figure 2. Composite expansion for the Reynolds stress. Data from DNS (Hoyas & Jimenez 2005) and equations (7.5) and (7.6).
If \( \alpha \) were exactly zero, then \( \mathcal{Y} \) would be identical with \( Y \). If \( \alpha \) were exactly 1, then \( \mathcal{Y} = YRe_0^1 = y^+ \) would be identical with \( y^+ \). In the limit \( Re_0 \to \infty \), \( \mathcal{Y} \) fixed, the inner and outer asymptotic expansions match up to some specified gauge function,

\[
\lim_{Re_0 \to \infty} F_{\text{EXPANSION}}(Y \Rightarrow \mathcal{Y}Re_0^{-\alpha}) = \lim_{Re_0 \to \infty} f_{\text{EXPANSION}}(y^+ \Rightarrow \mathcal{Y}Re_0^{-\alpha+1}).
\] (8.2)

The functions that are equal are called common parts. This procedure is simplified by using \textit{Van Dyke’s (1975) rule}. He proposed truncating the inner and outer expansions at orders \( m \) and \( n \), respectively, re-expressing them in the other variable, re-truncating and then equating. However, there are problems with Van Dyke’s rule when the gauge functions contain logarithms. The source of the difficulty is that logarithms are very slow functions. As \( Re_0^1 \to 1 \), much faster than \( 1/\ln(Re_0) \). Van Dyke revised his rule so that if logarithms occur, they are counted with the nearest gauge function. For example, \( 1;1/\ln(Re_0);1/(\ln(Re_0))^2 \), etc. are counted together. \textit{Hinch (1991)} gives several model examples that also illustrate how \( m \) and \( n \) must be chosen judiciously in order to apply Van Dyke’s rule.

### 9. Matching the mean velocity

In the case of the velocity profile, the law of the wall \( f_0(y^+) \) cannot match \( F_0(Y) = 1 \) and the next term has a logarithmic gauge function \( (u_*/U_0 \sim 1/\ln(Re_0)) \). Consider matching two terms of the outer expansion with one term of the inner expansion for a range of the intermediate variable \( \mathcal{Y} \). This is essentially an argument given first by \textit{Isakson (1937)} and \textit{Millikan (1938)},

\[
1 + \frac{u_*}{U_0}(Re_0)F_{1,cp}(Y \Rightarrow \mathcal{Y}Re_0^{-\alpha}) = \frac{u_*}{U_0}(Re_0)f_{0,cp}(y^+ \Rightarrow \mathcal{Y}Re_0^{-\alpha+1}).
\] (9.1)

Take the derivative of equation (9.1) with respect to \( \mathcal{Y} \). Because \( Y \) and \( y^+ (= YRe_0) \) can vary independently, each side must be a constant \( 1/\kappa \),

\[
\frac{Y}{dY} \frac{dF_{1,cp}}{dY} = \frac{y^+}{dy^+} \frac{df_{0,cp}}{dy^+} = \frac{1}{\kappa}.
\] (9.2)

The appropriate solutions are the common parts of \( F_1 \) and \( f_0 \),

\[
f_{0,cp}(y^+) = \frac{U(y)}{u_*} = \frac{1}{\kappa} \ln y^+ + C_1, \tag{9.3}
\]

\[
F_{1,cp}(Y) = \frac{U(y) - U_0}{u_*} = \frac{1}{\kappa} \ln Y + C_2. \tag{9.4}
\]

The final result of matching is determined by subtracting equation (9.4) from (9.3),

\[
\frac{U_0}{u_*} = \frac{1}{\kappa} \ln Re_0 + C_1 - C_2. \tag{9.5}
\]

It is interesting to note that \textit{Lundgren (submitted)} has derived the log laws from the instantaneous Navier–Stokes equations. The fact that \( F_0 = 1 \) and the next term has a logarithmic gauge function requires a match of two terms of the outer expansion with one term of the inner expansion. This results in a logarithmic overlap law. If \( F_0 \) were a non-trivial function of \( Y \), then a power-law matching could result.
Prior to Millikan and Isakson, the log law was regarded, and often still is, as an approximating equation for pipe flow velocity profiles. The Millikan–Isakson derivation is a change in viewpoint. The log law is the outer asymptote of the inner function and the inner asymptote of the outer function. The approximating equation for profiles is a composite expansion. The log law is only the common part. It has the same theoretical role that $G_{cp} = 1 = g_{cp}$ does for the Reynolds stress.

10. Composite expansion for the velocity

For the mean velocity, we have the additive composite expansion

$$\frac{U(y)}{u_*} = f_0(y^+) + F_1(Y) - [F_1(Y)]_{cp}. \quad (10.1)$$

Define the law of the wake as $W(Y) = F_1(Y) - [F_1(Y)]_{cp}$. Thus, an equivalent notation is

$$\frac{U(y)}{u_*} = f_0(y^+) + W(Y) \quad \text{where} \quad Y = y^+ / Re^*. \quad (10.2)$$

Coles (1956) introduced the parameter $\Pi$ to account for different pressure gradients, and proposed that all equilibrium boundary layers could be represented by $W(Y; \Pi) = (\Pi / \kappa) w(Y)$. Coles’ law has a ‘corner defect’, in that it does not have a slope of $-1/\kappa$ at $Y=1$. Lewkowicz (1982) corrected the corner defect with an equation that would not contribute to the displacement integral,

$$W(Y; \Pi) = \frac{\Pi}{\kappa} 2 Y^2 (3 - 2 Y) - \frac{1}{\kappa} Y^2 (1 - 3 Y + 2 Y^2). \quad (10.3)$$

Although not originally intended for use in pipes and channels, Coles’ wake law has been used to approximate the wake profile. Figure 3 shows the pipe data of Morrison et al. (2004), with $\Pi = 0.25$. Figure 4 shows the channel flow DNS data, with $\Pi = 0.1$. The wake data are sensitive to the log-law coefficients that are used to compute the values (as well as initial conditions, tripping, etc.).

11. Accuracy, gauge functions, alternate scales and higher-order terms

The numerical accuracy of a Poincaré expansion as an approximation to the true answer is of interest. The inner-expansion curve $g_0$ is valid for low $y^+$. The exact value depends on the Reynolds number. Consider the $-\langle uw \rangle$ DNS data for $Re_* = 550$ shown in figure 5. The inner-expansion curve $g_0$ is a good approximation for $-\langle uw \rangle$ to perhaps $y^+ \leq 20$ ($Y \leq 0.036$). The outer curve $G_0$ is a good approximation past perhaps $Y \geq 0.2$ ($y^+ \geq 110$). There is a region, $20 \leq y^+ \leq 110$, where neither curve is valid, and the maximum is about 0.85. This is very far from the common function value of $-\langle uw \rangle = 1$. However, the entire curve is approximated well by a composite expansion, even though there is actually no overlap. The major point is that neither the inner nor the outer expansions by themselves approximate the data. The composite expansion carries that task. On the other hand, the velocity profile has a different functional character and lies closer to the common function for a larger region.

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\[ \kappa = 0.42; \quad B = 5.6 \]

Figure 3. Velocity wake law for pipe flow. Data are from Morrison et al. (2004).

Figure 4. Velocity wake law for channel flow, with \(\kappa = 0.40\) and \(B = 4.9\). DNS data are from Moser et al. (1999), Del Álamo & Jimenez (2003), Del Álamo et al. (2004) and Hoyas & Jimenez (2005).

The behaviour of the maximum of the Reynolds stress led to the idea of a mesolayer with an intermediate independent variable \((y/h)\sqrt{Re_e}\). Long & Chen (1981) were the first to publish the idea. Since that time, many others have
revived the idea in several forms. If one counts regions by the scalings required for the independent variable, then there are only two regions. The \( \langle uv \rangle^+ \) maximum value is at the interaction or the transition between the regions. The entire behaviour is explained by two expansions that are combined into one composite expansion. A mesolayer is not required.

After we find a scale that makes a dependent variable of order one, a Poincaré expansion with a leading non-zero term is a viable representation (the velocity profile of Barenblatt et al. (1997) is not a Poincaré expansion). Given a set of gauge functions, the expansion is unique. The physics of the situation determines the general nature of the gauge functions that are needed. However, even then, the gauge functions are not unique, as many functions are asymptotically equal. For example, in the limit \( \text{Re}_* \rightarrow \infty \), the gauge functions \( 1/\text{Re}_* \rightarrow 0 \) and \( 1/(\text{Re}_* + C) \rightarrow 0 \) are asymptotically equal and either would make a proper gauge function. Consider that as \( \text{Re}_* \rightarrow \infty \),

\[
\frac{1}{\text{Re}_* + C} = \frac{1}{\text{Re}_*} \left( 1 - \frac{C}{\text{Re}_*} + \cdots \right) = \frac{1}{\text{Re}_*} - \frac{C}{\text{Re}_*^2} + \cdots \tag{11.1}
\]

Using \( 1/(\text{Re}_* + C) \) instead of \( 1/\text{Re}_* \) includes a contribution from higher-order terms. Van Dyke called this procedure ‘telescoping’. Sometimes, it has practical advantage, as the resulting series can be numerically better to lower Reynolds numbers.

In turbulent wall layers, there are two distinct velocity scales: \( U_0 \) and \( u_* \). It is essential to choose the correct scale between \( U_0 \) and \( u_* \). There are asymptotically equal choices. Consider, for example, the average velocity. Assuming a composite expansion for the velocity, where the wake is represented by Coles’ wake law, one

\[ \text{Figure 5. Comparison of the Reynolds stress DNS at } \text{Re}_* = 550, \text{ inner (7.6), outer (5.3) and composite (7.5) expansions.} \]
can find that
\[ U_{\text{ave}} = U_0 - u_* \left[ -C_2 + \frac{3}{2\kappa} \frac{3\Pi}{5\kappa} + \frac{1}{Re_*} \left( 540 - \frac{145}{\kappa} - 50C_1 \right) \right]. \] (11.2)

Hence, the use of $U_0$ and $U_{\text{ave}}$ is asymptotically equivalent, but numerically distinct. As a scale, one or the other may give more accurate results to lower Reynolds numbers. Another example is using $U_0 - U_{\text{ave}}$ in place of $u/C_3$. From equation (11.2), we see that
\[ U_0 - U_{\text{ave}} = u_* \left[ C_3 + \frac{C_4}{Re_*} \right]. \] (11.3)

Zagarola & Smits (1998) empirically discovered that using $U_0 - U_{\text{ave}}$ gave better results for scaling the velocity in pipes. McKeon et al. (2004) showed that the improvement occurred for $Re_D < 2 \times 10^5$. Morrison et al. (2004) found that the scaling was also useful for $\langle uu \rangle$.

Several people have proposed adding higher-order terms to the expansions (5.1) and (5.2). The first question is what are the appropriate gauge functions? Some chose $(1/Re_*)^n$, while others $(u_*/U_0)^n \sim (1/\ln Re_*)^n$. There is a big difference in ordering here, as all powers of $(1/\ln Re_*)^n$ come before $(1/Re_*)^n$. Theoretical questions aside, a higher-order theory will be successful, because it will have more constants to fit the data.

Additive composite expansions have been used above to produce approximations to the data. There are several types of composite expansions. A multiplicative composite for the Reynolds stress is
\[ -\frac{\langle uu \rangle}{u_*^2} \sim g_0(y^+) G_0(Y = y^+/Re^*) \frac{G_{cf}}{G_0} = g_0(y^+)(1 - y^+/Re^*). \] (11.4)

All composite expansions are asymptotically equal; however, for lower finite Reynolds number, one may be numerically more accurate than the other.

12. Streamwise Reynolds stress

The streamwise Reynolds stress $\langle uu \rangle$ has some unusual trends. The effect of Reynolds number is shown in figure 6 that displays the DNS data. Here, it is seen that the simple $u_*^2$ scaling is incorrect in both the inner and outer regions. Previous researchers reached this same conclusion. For pipes, Morrison et al. (2004) found no consistent scaling, while for boundary layers, Degraaff & Eaton (2000) found a mixed scale $\langle uu \rangle/(u_* U_0)$. The fact that the peak is always about $y^+ = 15–20$ implies that the correct inner region distance scale is $y^+$.

Consider figure 7 that shows the peak value of $\langle uu \rangle/u_*^2$ as a function of $U_0/u_*$ ($\sim \ln Re^*$). Many early workers thought that the peak value was independent of Reynolds number. When data over a larger range of Reynolds number became available, the issue returned. For boundary layers, Metzger & Klewicki (2001) and Metzger et al. (2001) provided key measurements at very high Reynolds numbers from the Utah salt flats. They showed that $\langle uu \rangle^+$ increases linearly with $\ln Re$. This is in agreement with the previous proposal of DeGraaff & Eaton (2000). In figure 7, a line with a linear dependence is shown for reference. Within the data scatter, one cannot distinguish a different trend for channels, pipes and boundary layers.
Data that are in the outer region at $Y=0.4$ are shown in figure 8. One might reasonably expect differences here between channels, pipes and boundary layers; however, within the scatter, there are no apparent trends. A line with a linear increase is also shown. A possible exception is the pipe data of Morrison et al. (2004), which tend to be flat, especially at the last three points. Also, the points of Zanoun (2003) tend to form two groups at different levels. In both cases, the hot-wire length is of concern (see Fernholtz & Findley 1996).

13. The active and inactive motions

Townsend (1976) attributed the large values of $\langle uu \rangle$ to ‘inactive’ motions. He envisioned swirling $u-w$ (streamwise–spanwise) motions of large scale. With self-similar models of attached wall eddies, Townsend’s idea was extended by Perry & Abell (1977), Perry et al. (1986) and Marusic et al. (1997). Marusic & Kunkel (2003) produced equations representing inactive effects in boundary layers for all distances.

In order to motivate a scaling law, consider Townsend’s idea in more detail. Let the streamwise velocity be an active motion $u_A$ plus an inactive motion $u_I$, 

$$u = u_I + u_A. \quad (13.1)$$

Active motions make an essential contribution to the Reynolds shear stress $-\langle uv \rangle$, while inactive motions do not produce Reynolds shear stress. Another viewpoint would be that $u_I$ and $v$ are statistically independent, so

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that the cross-correlation is equal to the product of the individual means, \( \langle u_1 v \rangle = \langle u_1 \rangle \langle v \rangle = 0 \) and thus \( \langle uv \rangle = \langle u_A v \rangle \). Note that the mean of \( u \) is zero, but the separate motions can have non-zero means,

\[
\langle u \rangle = -\langle u_A \rangle.
\] (13.2)

The streamwise Reynolds stress correlation is

\[
\langle uu \rangle = \langle u_1 u_1 \rangle + 2 \langle u_1 u_A \rangle + \langle u_A u_A \rangle.
\] (13.3)

If \( u_1 \) and \( u_A \) are statistically independent, \( \langle u_1 u_A \rangle = \langle u_1 \rangle \langle u_A \rangle = -\langle u_A \rangle^2 \). Thus,

\[
\langle uu \rangle = \langle u_1 u_1 \rangle - 2 \langle u_A \rangle^2 + \langle u_A u_A \rangle.
\] (13.4)

The terms in equation (13.4) need different scalings in order to be of order one for the limit \( Re_a \to \infty \).

To proceed in the most general manner, assume that the power \( \alpha \) gives an order-one streamwise Reynolds stress, denoted with a superscript symbol ‘\#’. In the limit \( Re_a \to \infty \), the inactive motion dominates, and therefore it has the same scaling,

\[
\langle uu \rangle^\# = \left( \frac{u*}{U_0} \right)^{\alpha} = \langle uu \rangle^+ \left( \frac{u*}{U_0} \right)^{\alpha},
\] (13.5)

\[
\langle u_1 u_1 \rangle^\# = \left( \frac{u_1 u_1}{u_*^2} \right)^{\alpha} = \langle u_1 u_1 \rangle^+ \left( \frac{u_*}{U_0} \right)^{\alpha}.
\] (13.6)
The cross-correlation term has a different scaling, 
\[ \langle u_1 u_A \rangle^\# = -\frac{\langle u_A \rangle^2}{u_s^2} \left( \frac{u_s}{U_0} \right)^\beta = -\langle u_A \rangle^2 \left( \frac{u_s}{U_0} \right)^\beta. \]  
(13.7)

For \( \beta = 0 \), \( \langle u_1 u_A \rangle^\# = -\langle u_A \rangle^2 \). Because \( u_A \) determines the Reynolds stress, assume that \( \langle u_A u_A \rangle \) scales with \( u_s \),
\[ \langle u_A u_A \rangle^+ = \frac{\langle u_A u_A \rangle}{u_s^2}. \]  
(13.8)

With the definitions (13.5)–(13.8), the non-dimensional form of equation (13.4) is
\[ \langle uu \rangle^\# = \langle u_1 u_1 \rangle^\# - 2\langle u_A \rangle^+ \left( \frac{u_s}{U_0} \right)^{\alpha-\beta} + \langle u_A u_A \rangle^+ \left( \frac{u_s}{U_0} \right)^\alpha. \]  
(13.9)

The work of DeGraaff & Eaton (2000) and Metzger & Klewicki (2001), and figures 7 and 8, suggest that \( \alpha \) is nearly equal to 1. Moreover, assuming all active motion scales on \( u_s \) would mean \( \beta = 0 \).

### 14. Asymptotic expansion for the streamwise Reynolds stress

With \( \alpha = 1 \) and \( \beta = 0 \), the streamwise stress has an asymptotic expansion
\[ \langle uu \rangle^\# \sim f_0(y) + f_1(y) \frac{u_s}{U_0} (Re_s). \]  
(14.1)
We have already seen that the outer velocity profile has two terms with gauge functions 1 and \( \frac{u_c}{C_3} U_0 \). The form (14.1) will be used for both an inner expansion \( (f_{0_{\text{in}}}(y^+), f_{1_{\text{in}}}(y^+)) \) and an outer expansion \( (f_{0_{\text{out}}}(Y), f_{1_{\text{out}}}(Y)) \).

Regardless of the correctness of the physical interpretation of the terms in equation (14.1), in theory, one can produce the coefficient functions by limit processes

\[
\begin{align*}
\lim_{u_c/U_0 \to 0} \langle uu \rangle^# &= \frac{f_0}{(u_c/U_0)}, \\
\lim_{u_c/U_0 \to 0} \left( \langle uu \rangle^# - f_0 \right) &= \frac{f_1}{(u_c/U_0)}.
\end{align*}
\]

Outer functions are produced when the limit process is done with \( Y \) fixed, and inner functions are produced with \( y^+ \) fixed. Unfortunately, the data for \( \langle uu \rangle^#(y, u_c/U_0) \) are not extensive or precise enough to apply this method.

### 15. Some approximations

In order to test the proposed formalism, we adopt the physical interpretation of equations (13.5)–(13.7). The strategy will be to relate \( \langle u_A \rangle^+ \) and \( \langle u_A u_A \rangle^+ \) to the Reynolds shear stress and then, using data for \( \langle uu \rangle^# \), solve equation (14.1) for \( f_0(y) = \langle u_1 u_1 \rangle^# \). This needs to be done for both the inner and outer regions. The final uniformly valid composite expansion will be

\[
\langle u_1 u_1 \rangle^\text{Comp} = \langle u_1 u_1 \rangle^\text{#}(y^+) + \langle u_1 u_1 \rangle^\text{#}(Y \Rightarrow y^+ / Re_s) - \langle u_1 u_1 \rangle^\text{#} \text{cp}.
\]

Equation (15.1), together with the Reynolds stress equations (7.5) and (7.6), allows one to predict the streamwise Reynolds stress.

The active motions, by definition, determine the Reynolds shear stress. Assume that the active motion autocorrelation is some number \( M \) times the Reynolds shear stress,

\[
\langle u_A u_A \rangle^+ = M(\langle u v \rangle^+).
\]

Also, assume that its mean value \( \langle u_A \rangle^+ \) is some fraction of the Reynolds shear stress itself,

\[
-2\langle u_A \rangle^{+2} = 2N(\langle u v \rangle^+).
\]

Our experience is that \( \langle u v \rangle \) is negative, so \( M \) is positive and \( N \) is negative.

Substituting equations (15.2) and (15.3) directly into equation (13.9) and solving for the inactive auto correlation yield

\[
\langle u_1 u_1 \rangle^# = \langle uu \rangle^# - (2N + M)(\langle u v \rangle^+)(u_c/U_0).
\]

To continue, let us concentrate on channel flows, where in addition to experiments, DNS offers an internally consistent dataset.

### 16. Correlation and composite expansions for the channel flows

The inactive–inactive correlation \( \langle u_1 u_1 \rangle^# \) was computed from equation (15.4) using a value of \( 2N + M = -1 \). No attempt was made to optimize these coefficients. In order to treat all the data consistently, \( -\langle u v \rangle \) was computed using the theory of §15.
First, the outer region will be correlated, ignoring the data near the wall and at very low Reynolds number. Figure 9 shows both $h_{uu}$ and $h_{ui}$. If $2N+M$ were positive, $h_{ui}$ curves would be below $h_{uu}$ and spread apart. We do not want the $h_{ui}$ curves to be too much higher than the $h_{uu}$ curves, because in the limit $Re/C^3$, $h_{uu}$ approaches $h_{ui}$. In general, the correlation is good. Lundgren (submitted) noted in his analysis that the common functions of all Reynolds stresses are constants. This appears to be true for $h_{ui}$ in figure 9.

In order to form a composite expansion, an equation was chosen to represent the data. The equation rises from the centreline value, $C_{cl}$, to the common part $h_{ui}$ at $Y=0$,

$$\langle u_1 u_1 \rangle_{cp} = C_{cp}$$

It is questionable whether the same equation would apply to pipe or boundary-layer flows, because those flows have different velocity wake components and, in the case of a boundary layer, a different $-\langle uw \rangle$ correlation (figure 10).

Next, the inner region will be correlated. By solving equation (5.1) for the inner function, we find that

$$\langle u_1 u_1 \rangle_{in} = \langle u_1 u_1 \rangle - \left( \langle u_1 u_1 \rangle_{out} - \langle u_1 u_1 \rangle_{cp} \right).$$

The terms in parentheses are given by equation (16.1) and the DNS data are used for $\langle u_1 u_1 \rangle$. Figure 11 displays $\langle u_1 u_1 \rangle_{in}$. There is a slight shift in the peak with $Re$, but overall the correlation is good. The DNS data at $Re=2000$ were

Figure 9. The inactive streamwise Reynolds stress $\langle u_1 u_1 \rangle$ DNS and curve equation (16.1). DNS data sources are the same as given in figure 4 legend. Streamwise Reynolds stress $\langle uu \rangle$ is also shown.

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Figure 10. The inactive streamwise Reynolds stress $\langle u_1 u_1 \rangle^*$. LDV data are from Wei & Willmarth (1989), Harder & Tiederman (1991) and Christensen & Adrian (2001). Equation (16.1) is shown for reference.

Figure 11. Curve fit of equation (16.3) for the inner expansion of the inactive streamwise Reynolds stress $\langle u_1 u_1 \rangle_{\text{in}}^*$. DNS data are from the same sources as given in figure 4 legend.
approximated by

$$\langle u_1 u_1 \rangle_{1n}^\# = C_0 \left[ 1 - \exp\left( -\frac{y^+}{C_1} \right) \right]^2 - (C_0 - C_{cp}) \left[ 1 - \exp\left( -\frac{y^+}{C_2} \right) \right]$$

$$C_0 = 0.724; \quad C_1 = 5.41; \quad C_2 = 18.3.$$  \hspace{1cm} (16.3)
Equation (16.3) and the experimental data measured with LDV are compared in figure 12. As a matter of interest, the prediction of the traditionally scaled streamwise stress $\frac{u^2}{C^3}$ is provided in figure 13 in the inner variable $y^+$ and in figure 14 in the outer variable $Y$. This chart was produced using the composite expansion

$$\langle uu \rangle_{\text{Comp}}^+ = \left[ \frac{u_1 u_1}{C^2} + (2N + M)(-\langle uv \rangle_{\text{Comp}}^+) \left( \frac{u_+}{U_0} \right) \right] \div \left( \frac{u_+}{U_0} \right). \quad (16.4)$$

The Reynolds stress is given by equations (11.1) and (11.2), and the inactive stress is given by equations (16.1)–(16.3).

17. Summary

Asymptotic expansions require variables scaled to order one. The velocity parameters $U_0$ and $u_+$ are essentially different for scaling purposes. Other choices are possible and can be useful at low Reynolds numbers, but are asymptotically equivalent. Comparison with data requires a composite expansion, because the inner or outer expansions by themselves have limited validity. A composite expansion is not only uniformly valid, but also displays the effect of varying Reynolds number. Data for the mean velocity and the Reynolds shear stress were examined in terms of composite expansions.

Data for the streamwise Reynolds stress $\langle uu \rangle$ were also reviewed. Townsend conjectured that the streamwise fluctuations consist of the active and inactive motions. Here, it is proposed that the additive active and inactive motions scale differently. This implies that the inner and outer asymptotic expansions of $\langle uu \rangle$
each consist of two terms. In order to test the idea, some crude assumptions were made concerning the active–active and active–inactive correlations. Using data from channel flows to compute the inactive–inactive correlation yields a reasonably good collapse with Reynolds number. This occurs in two regions corresponding to the traditional $y^+$ and $Y$ regions. A composite expansion shows that the theory is useful to interpolate and predict the streamwise Reynolds stress. The active–inactive viewpoint also has consequences for other turbulent intensities, the turbulent kinetic energy and the fluctuating pressure and fluctuating wall shear stress.


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