Symmetry of steady periodic water waves with vorticity

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The symmetry and monotonicity properties of steady periodic gravity water waves are established for arbitrary vorticities if the wave profile is monotone near the trough and every streamline attains a minimum below the trough. The proof uses the method of moving planes.

Keywords: water waves; vorticity; symmetry; nonlinear elliptic; maximum principles

1. Introduction

We discuss certain a priori geometric properties of two-dimensional steady gravity water waves with vorticity. The main result states that for an arbitrary distribution of vorticity, any periodic wave of finite depth with a single trough (a minimum over one period) is symmetric about a single crest (a maximum over one period) and the wave profile decreases (strictly) monotonically from crest to trough if every streamline attains its minimum below the trough and the wave profile is monotone near the trough. The proof involves the method of moving planes as adapted to nonlinear elliptic boundary value problems.

The mathematical existence theory for periodic waves with vorticity dates back to the construction by Dubreil-Jacotin (1934) of small amplitude waves of infinite depth, and it includes the works by Constantin & Strauss (2002) and Constantin & Strauss (2004) in the finite depth case and Hur (2006) in the infinite depth case on the global bifurcation of large amplitude waves. The construction by Constantin & Strauss (2002) and Constantin & Strauss (2004) assumes that the wave profiles are symmetric; our result establishes a priori their symmetry and monotonicity properties.

In the irrotational setting, Garabedian (1965) considered the symmetry property of periodic waves of finite depth with a variational approach provided that each streamline has a single crest and a single trough per wavelength except for the flat bottom; a direct proof is due to Toland (2000), which combines with the divergence theorem and Dirichlet’s principle for harmonic functions. Further demonstration of symmetry appeared with the advent of the so-called method of moving planes. In particular, with its extension by Berestycki & Nirenberg (1988) to nonlinear elliptic problems, Craig & Sternberg (1988) proved the

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symmetry and monotonicity properties of irrotational solitary waves. In the present work, the method of moving planes as in Craig & Sternberg (1988) is applied to the periodic water-wave problem with vorticity. The proof is given in §3. Recorded in §2 are a precise account of the steady gravity free-surface water-wave problem and its reformulation as a nonlinear elliptic boundary value problem in a fixed domain.

The symmetry result presented here holds for a general class of nonlinear elliptic boundary value problems. In particular, the same argument, as adapted in the infinite strip setting, leads to the symmetry and monotonicity properties of supercritical solitary waves for arbitrary vorticities (Hur 2006). An a priori exponential decay estimate provides a fairly precise knowledge on the behaviour of solitary-wave solutions at infinity, while in our theorem on periodic waves the monotonicity of their wave profiles near the trough is assumed.

For a certain class of vorticities, Constantin & Escher (2004a, b) showed that periodic waves are symmetric if their profile is monotone between crest and trough. This result extends that for irrotational flows by Okamoto & Shoji (2001) to rotational flows. An important difference of the present work from Constantin & Escher (2004a, b) is that our proof uses a quasi-linear elliptic boundary value problem in which the vorticity function appears only as part of coefficient functions, and thus it is free from restrictions on the vorticity. Instead, we require that the wave profile is monotone near the trough and every streamline has a single minimum per wavelength located below the trough.

2. Formulation and the main result

(a) The steady water-wave problem in physical variables

The classical (gravity) water-wave problem concerns the motion of an incompressible inviscid fluid with a free surface acted on only by gravity. We consider a two-dimensional flow which at time \( t \) is contained in a channel in the \((x, y)\)-plane bounded from above by a free surface \( y = \eta(t; x) \) and from below by a rigid horizontal bottom \( y = -d \), where \( 0 < d < \infty \). In the fluid region \( \{(x, y); -d < y < \eta(t; x)\} \), the velocity field \((u(t; x, y), v(t; x, y))\) and the pressure \(P(t; x, y)\) satisfy the following Euler equations:

\[
\frac{\partial u}{\partial t} + uu_x + vu_y = -P_x, \quad \frac{\partial v}{\partial t} + uv_x + vv_y = -P_y - g, \quad u_x + v_y = 0. \quad (2.1)
\]

Here, \( g > 0 \) denotes the gravitational constant of acceleration. The flow is supposed to be rotational and characterized by the vorticity \( \omega = v_x - u_y \).

The dynamic and kinematic boundary conditions hold on the free surface \( y = \eta(t; x) \)

\[
P = P_0 \quad \text{and} \quad v = \eta_t + u\eta_x, \quad (2.2)
\]

where \( P_0 \) is the constant atmospheric pressure. The impermeability condition at the bottom is

\[
v = 0 \quad \text{on } y = -d. \quad (2.3)
\]

The steady periodic wave problem is then to find solutions to (2.1)–(2.3) for which the wave profile, the velocity field and the pressure have space-time dependence \((x-ct, y)\) and period \( L \) in the \( x \)-variable; \( c > 0 \) is the speed of wave
propagation and $L > 0$ is the wavelength. In the frame of reference moving with speed $c$, the steady flow occupies the domain

$$D_\eta = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\},$$

which lies between the wave profile $y = \eta(x)$ and the flat bottom $y = -d$.

Throughout this work, solutions are assumed to satisfy the regularity requirement $\eta \in C^2(\mathbb{R})$ and $(P, u, v) \in C^1(\bar{D}_\eta) \times C^1(\bar{D}_\eta) \times C^1(\bar{D}_\eta)$. Further assumed is that $u - c < 0$ throughout the fluid region, namely, no stagnation. Indeed, experimental evidence (Lighthill 1978) indicates that for wave patterns which are not near the spilling or breaking state, the speed of wave propagation is in general considerably larger than the horizontal velocity of any water particle.

It is traditional in the theory of steady water waves to define the (relative) stream function $\psi \in C^2(\bar{D}_\eta)$ by

$$\psi_x = -v, \quad \psi_y = u - c,$$

from which $-\Delta \psi = v_x - u_y = \omega$ follows. As will be shown in §2b, $\psi_y = u - c < 0$ guarantees that $\omega = \gamma(\psi)$ for some function $\gamma \in C([0, |p_0|])$ throughout the fluid. The vorticity function $\gamma$ measures the strength of the vorticity. Under this physically motivated assumption of no stagnation, $\psi_y = u - c < 0$, the Euler equations (2.1) reduces to the Poisson equation

$$-\Delta \psi = \gamma(\psi).$$

The kinematic boundary condition allows us to normalize $\psi$, so that $\psi = 0$ on the free surface and $\psi = -p_0 > 0$ on the flat bottom; the relative mass flux $p_0$ is defined by

$$p_0 = \int_{-d}^{\eta(x)} \psi_y(x, y) dy,$$

which is independent of $x$.

From the equations of motion follows Bernoulli’s law, which states that the quantity

$$E = \frac{1}{2}\lVert \nabla \psi \rVert^2 + gy + P - \Gamma(-\psi)$$

is constant throughout the fluid; here,

$$\Gamma(p) = \int_0^p \gamma(-s) ds.$$

In view of Bernoulli’s law, the dynamic boundary condition (2.2) takes the form that

$$\lVert \nabla \psi \rVert^2 + 2gy = Q$$

is constant on $y = \eta(x)$.

Finally, $\psi$ is $L$-periodic in the $x$-variable. Indeed, $\int_0^L \psi(s, y) ds$ is independent of $y$ and is zero by (2.3).

In summary, there results in the following formulation for the steady periodic water-wave problem, equivalent to the original one: for $p_0 < 0$ and $\gamma \in C([0, |p_0|])$
given, there exist a parameter value \( Q \in \mathbb{R} \) and functions \( \eta \in C^2(\mathbb{R}) \) and \( \psi \in C^2(D_\eta), \psi \in C^2(D_\eta) \), such that \( \eta(x) \) and \( \psi(x, y) \) are \( L \)-periodic in the \( x \)-variable, and

\[
-\Delta \psi = \gamma(\psi) \quad \text{in} \quad -d < y < \eta(x),
\]

\[
\psi = -p_0 \quad \text{on} \quad y = -d,
\]

\[
\psi = 0 \quad \text{on} \quad y = \eta(x),
\]

\[
|\nabla \psi|^2 + 2gy = Q \quad \text{on} \quad y = \eta(x).
\]

(2.5a) (2.5b) (2.5c) (2.5d)

Our main result is the following theorem on the symmetry and the monotonicity properties of periodic waves for arbitrary vorticities.

**Theorem 2.1. (Main theorem).** Given a relative mass flux \( p_0 < 0 \) and a vorticity function \( \gamma \in C([0, |p_0|]) \), let \( Q \in \mathbb{R} \) and \((\eta, \psi) \in C^2(\mathbb{R}) \times C^2(D_\eta)\) be a solution pair of the steady periodic water-wave problem (2.5a)–(2.5d) with \( \psi_y < 0 \) throughout \( D_\eta \).

If \( \eta \) has a single minimum (trough) per wavelength near which \( \eta \) is monotone and for every \( p \in (p_0, 0) \) a lowest position \((x, y)\) where the \( y \)-value attains a minimum among those satisfying \( \psi(x, y) = p \) is located below the trough, then \( \eta \) also has a single maximum (crest) per wavelength. Moreover, \( \eta \) and \( \psi \) are symmetric about the crest and \( \eta \) decreases monotonically from crest to trough.

(b) Reduction to the elliptic boundary value problem

It is convenient for our analysis to reformulate (2.5a–d) as a nonlinear elliptic boundary value problem in a (fixed) domain. Its origin is the ‘partial-hodograph’ transform by Dubreil-Jacotin (1934).

Observe that \( \psi \) is constant on the free surface and on the flat bottom and that \( \psi \) decreases with \( y \), i.e. the \( y \)-coordinate is a single-valued function of \( \psi \) for each fixed \( x \). This suggests the introduction of the new independent variables

\[ q = x \quad \text{and} \quad p = -\psi(x, y), \]

which map the fluid region \( D_\eta \) to an infinite strip \((-\infty, \infty) \times (p_0, 0)\) in the \((q, p)\)-plane and the free surface \( \{(x, \eta(x)) : x \in \mathbb{R}\} \) to the horizontal line \( p = 0 \). Let

\[ R = \left\{(q, p) : -\frac{L}{2} < q < \frac{L}{2}, \quad p_0 < p < 0\right\} \]

be the domain of one period.

The vertical elevation measured from the flat bottom \( h(q, p) = y + d \) replaces the dependent variables. An explicit calculation yields

\[
h_q = -\frac{\psi_x}{\psi_y}, \quad h_p = -\frac{1}{\psi_y}. \tag{2.6}
\]

The assumption \( \psi_y < 0 \) throughout the fluid region guarantees that \( \omega \) is a single-valued function of \( p \). Indeed,

\[
\partial_q \omega = \left(\partial_x + \frac{h_q}{h_p} \partial_p\right) \omega = \left(\partial_x + \frac{\psi_x}{\psi_y} \partial_y\right) \omega = 0.
\]

We say \( \omega = y(-p) \).

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The vorticity-stream formulation (2.5a–d) is then reformulated as an elliptic boundary value problem in a fixed domain

\[ h_p^2 h_{qq} - 2 h_q h_p h_{pq} + (1 + h_q^2) h_{pp} = -\gamma(-p) h_p^3 \quad \text{in } R, \]  
\[ 1 + h_q^2 + (2gh - Q) h_p^2 = 0 \quad \text{for } p = 0, \]  
\[ h = 0 \quad \text{for } p = p_0, \]

where \( h \) is required to be \( L \)-periodic in the \( q \)-variable.

The above formulation is equivalent to (2.5a–d). The proof is nearly identical to that of lemma 2.1 in Constantin & Strauss (2004).

In the proceeding discussion, let us define the nonlinear differential operators

\[ F_1(\nabla h, \nabla^2 h) = h_p^2 h_{qq} - 2 h_q h_p h_{pq} + (1 + h_q^2) h_{pp} + \gamma(-p) h_p^3, \]  
\[ F_2(h, \nabla h) = 1 + h_q^2 + (2gh - Q) h_p^2 |_{p=0}. \]

The principal part of \( F_1 \) is denoted by

\[ A_0(\nabla h) = h_p^2 \partial_q^2 - 2 h_q \partial_q h_p \partial_p + (1 + h_q^2) \partial_p^2. \]

Clearly, \( F_1 \) and \( F_2 \) are smooth. Note that the differential operator \( F_1 \) is uniformly elliptic; the coefficients of the principal part satisfy

\[ h_p^2 \xi_1^2 - 2 h_q h_p \xi_1 \xi_2 + (1 + h_q^2) \xi_2^2 \geq 4\delta^2 (\xi_1^2 + \xi_2^2) \]

for all \( (\xi_1, \xi_2) \in \mathbb{R}^2 \), where \( h_p > \delta > 0 \) throughout \( \bar{R} \). Note that the boundary operator \( F_2 \) is uniformly oblique in the sense that it is bounded away from being tangential:

\[ \left| \frac{\partial F_2}{\partial h_p} \right| = |2(2gh - Q) h_p| > 2 ||h_p||_{C(R)}^{-1} > 0. \]

## 3. Symmetry of periodic waves

Our goal in this section is to prove the symmetry and monotonicity properties of periodic waves for arbitrary vorticities. The proof uses the method of moving planes, which was initially developed by Alexandrov and then used extensively by Serrin (1971), Gidas et al. (1979), Berestycki & Nirenberg (1988) and others.

Throughout this section, the wavelength \( L > 0 \) is held fixed. We choose the wave trough at \( q = \pm L/2 \).

As is shown in §2, under the condition \( h_p > \delta > 0 \), the periodic wave problem is formulated as

\[
\begin{aligned}
F_1(\nabla h, \nabla^2 h) &= 0 \quad \text{in } R, \\
F_2(h, \nabla h) &= 0 \quad \text{for } p = 0, \\
h &= 0 \quad \text{for } p = p_0,
\end{aligned}
\]

where \( h \) is \( L \)-periodic in the \( q \)-variable.
The boundary conditions on the sides \(q = \pm L/2\) of \(R\) are

\[
\begin{align*}
(\pm L/2, 0) < h(q, 0) & \quad \text{for} \quad -L/2 < q < L/2, \\
h(\pm L/2, p) & \leq h(q, p) \quad \text{for} \quad -L/2 < q < L/2, p_0 \leq p \leq 0,
\end{align*}
\]

which express, respectively, that \(q = \pm L/2\) is the single wave trough and that every streamline attains a minimum below the trough. In addition, the wave profile is required to be monotone near the trough, i.e.

\[
\begin{align*}
h(q_1, 0) & \leq h(q_2, 0) \quad \text{for} \quad -L/2 < q_1 < -L/2 + \epsilon, \\
h(q_1, 0) & \geq h(q_2, 0) \quad \text{for} \quad L/2 - \epsilon < q_1 < q_2 < L/2,
\end{align*}
\]

where \(\epsilon > 0\) is small.

Theorem 3.1 is the precise statement of the sense in which periodic water waves are symmetric and monotone.

**Theorem 3.1.** For \(p_0 < 0\) and \(y \in C([0, |p_0|])\) given, let \(Q \in \mathbb{R}\) and \(h \in C^2(\bar{R})\) be a solution pair to (3.1) and (3.2) and let \(h\) satisfy (3.3). Then, \(h\) has a single maximum (crest) at \(q = 0\) and is symmetric about the crest in the \(q\)-variable, i.e.

\[
h(q, p) = h(-q, p) \quad \text{for} \quad (q, p) \in \bar{R}.
\]

Moreover, \(h\) is monotonically increasing for \(-L/2 < q < 0\) and monotonically decreasing for \(0 < q < L/2\), for every \(p_0 < p \leq 0\), i.e.

\[
h_q(q, p) > 0 \quad \text{for} \quad -L/2 < q < 0, \quad p_0 < p \leq 0.
\]

The basic analytic tools of the proof are the maximum principles and their sharp form at corner points for linear elliptic partial differential operators, which are presented below in a form suitable for our purposes.

For \(a^{ij}, b^i \in C(\bar{R}) \quad (i, j = 1, 2)\), we define the elliptic partial differential operator

\[
\mathcal{A} = a^{11}(q, p)\partial_q^2 + 2a^{12}(q, p)\partial_p \partial_q + a^{22}(q, p)\partial_p^2 + b^1(q, p)\partial_q + b^2(q, p)\partial_p,
\]

that is

\[
\sum_{i,j} a^{ij} \xi_i \xi_j \geq \delta^2 (\xi_1 + \xi_2)^2,
\]

for some constant \(\delta > 0\).

**Lemma 3.2.**

(i) (The maximum principle). If \(h \in C^2(R) \cap C(\bar{R})\) is such that \(\mathcal{A}h \leq 0\) in \(R\) and \(h \geq 0\) on \(\partial R\), then \(h > 0\) throughout \(R\) unless \(h \equiv 0\).

(ii) (The Hopf boundary lemma). Let \(h \in C^2(R) \cap C(\bar{R})\) satisfy \(\mathcal{A}h \leq 0\) and \(h \geq 0\) in \(\bar{R}\). Suppose \(h = 0\) at some point \((q^*, p^*) \in \partial R\). If \((q^*, p^*)\) satisfies an interior sphere condition, i.e. there exists an open ball contained in \(R\) with \((q^*, p^*)\) on its boundary, then the outer normal derivative \(\partial h/\partial n\) of \(h\) at \((q^*, p^*)\) (if it exists) satisfies the inequality \(\partial h/\partial n(q^*, p^*) < 0\) unless \(h \equiv 0\).
(iii) (The edge-point lemma). Let $R(\mu) = \{(q, p) \in R : -L/2 < q < \mu, \ p_0 < p < 0\}$ and let $h \in C^2(R(\mu))$ satisfy $Ah \leq 0$ in $R(\mu)$ and $h \geq 0$ in $R(\mu)$. Suppose $h=0$ at some corner point $(q^*, p^*)$ of $R(\mu)$. If $A(q^*, p^*)$ satisfies a bluntness condition, i.e. $a_p^2(q^*, p^*) = 0$ and there is a constant $K>0$ such that

$$|a_{12}^{12}(q, p^*)| \leq K(\mu - q) \quad \text{for} \quad -L/2 < q < \mu,$$

then

$$\frac{\partial h}{\partial s}(q^*, p^*) > 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial s^2}(q^*, p^*) > 0,$$

unless $h \equiv 0$. Here, $s$ is any direction vector at $(q^*, p^*)$ which enters $R(\mu)$ non-tangentially.

Assertions (i) and (ii) are the weak maximum principle and the Hopf boundary lemma, respectively. The proofs are found, for instance, in Gilbarg & Trudinger (2001). Assertion (iii) is the edge point lemma due to Serrin (1971), which extends the Hopf boundary lemma at a corner point. For the proof, consult Serrin (1971) and Gidas et al. (1979).

To proceed with the detailed symmetrization analysis, we need to describe several notations. For $-L/2 < \mu < 0$ to be a parameter, let us define

$$R(\mu) = \{(q, p) \in R : q < \mu\}.$$

The idea is to compare, at $(q, p) \in R(\mu)$, the values $h(q, p)$ and

$$h^\mu(q, p) = h(q, p, \mu) := h(2\mu - q, p).$$

We consider the functions

$$v^\mu(q, p) = v(q, p; \mu) := h^\mu(q, p) - h(q, p),$$
$$f^\mu(q, p) = u(q, p; \mu) := h^\mu(q, p) + h(q, p),$$

defined in $R(\mu)$. A straightforward calculation yields that $v^\mu$ is a solution of the linear elliptic boundary value problem

$$A[v^\mu] = A_0(\nabla h)[v^\mu] + (h^\mu_{pq} - 2h^\mu_{pp}h^\mu_{pq})v^\mu_q + (2h^\mu_{pq} + f^\mu_{p}h^\mu_{qp} + f^\mu_{p}h^\mu_{pq})v^\mu_p$$

$$\equiv A_0(\nabla h)[v^\mu] + b^1(h, h^\mu)v^\mu_q + b^2(h, h^\mu)v^\mu_p = 0 \quad \text{in} \quad R(\mu),$$

$$f^\mu_qv^\mu_q + (2gh^\mu - Q)f^\mu_pv^\mu_p + 2gh^\mu_pv^\mu = 0 \quad \text{for} \quad p = 0,$$

where $A_0(\nabla h)$ is defined in (2.10). (Therefore, $v^\mu$ obeys the maximum principle and its corollaries.) Other boundary conditions for $v^\mu$ are

$$\begin{align*}
v^\mu(q, p_0) &= 0 \quad \text{for} \quad -L/2 \leq q \leq \mu, \\
v^\mu(\mu, p) &= 0 \quad \text{for} \quad p_0 \leq p \leq 0, \\
v^\mu(-L/2, p) &\geq 0 \quad \text{for} \quad p_0 \leq p \leq 0.
\end{align*}$$

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The boundary conditions on the bottom $p = p_0$ and on the right side $q = \mu$ are of Dirichlet type. Note that

$$v^\mu(q, 0) \geq 0 \quad \text{for} \quad -L/2 < q < -L/2 + \epsilon,$$

(3.11) where $\epsilon > 0$ is as in (3.3). Hence, for $0 < \mu + L/2 < \epsilon$ with $\epsilon > 0$ small $v^\mu \geq 0$ on $\partial R(\mu)$.

**Proof of theorem 3.1.** Let

$$\mu^* = \sup \{ \mu : v^\mu(q, p) \geq 0 \quad \text{in} \quad R(\mu) \quad \text{whenever} \quad -L/2 < \mu < \mu < 0 \}.$$  

We claim that $\mu^*$ exists. Indeed, since $v^\mu \geq 0$ on $\partial R(\mu)$ for $0 < \mu + L/2 < \epsilon$, where $\epsilon > 0$ is as in (3.3), the maximum principle asserts that $v^\mu > 0$ in $R(\mu)$.

Our goal is to show that $\mu^* = 0$ and that

$$v^*(q, p) := v(q, p; \mu^*) \equiv 0 \quad \text{in} \quad R(\mu^*).$$

Suppose the contrary that $\mu^* < 0$. Since $v^*(\cdot - L/2, 0) > 0$ by assumption, continuity and the maximum principle assert that $v^*(q, p) > 0$ in $R(\mu^*)$. It is then easy to see that $v^*(\mu^*, 0) = 0$ is a minimum of the function $v^*(q, p)$ in $R(\mu^*)$. It follows from the boundary condition on $q = \mu^*$ in (3.10) that

$$v^*_p(\mu^*, 0) = v^*_p(\mu^*, 0) = 0.$$  

(3.12)

To obtain a contradiction, we will show that $v^*_q(\mu^*, 0) = 0$ and $v^*_p(\mu^*, 0) = 0$ (thereby it violates the edge-point lemma).

The first step is to show by the extremal property of $\mu^*$ that $\mu^*(q^*, p^*) = 0$ and $v^*_q(q^*, p^*) = 0$ for some $(q^*, p^*) \in \partial R(\mu^*)$.

The maximaity of $\mu^*$ allows us to choose sequences

$$\mu^k \downarrow \mu^* \quad \text{and} \quad (q^k, p^k) \in R(\mu^k) \text{ such that } v^k(q^k, p^k) < 0.$$  

Let us denote $v^k(q, p) = v(q, p; \mu^k)$. We may choose $v^k(q^k, p^k) = \inf_{R(\mu^k)} v^k(q, p)$.

Note that $v^k(\mu^k, p) = 0$ and $v^k(-L/2, p) \geq 0$ for $p_0 < p \leq 0$. Since $v^k(q, p_0) = 0$, a negative minimum of $v^k$ must be attained either in $R(\mu^k)$ or on the smooth part of the top boundary. If the minimum point $(q^k, p^k)$ is in the interior of $R(\mu^k)$, it follows as a simple consequence of an interior minimum that

$$\nabla v^k(q^k, p^k) = 0, \quad \det \nabla^2 v^k(q^k, p^k) \geq 0.$$  

(3.13)

Here, $\nabla^2 v^k(q^k, p^k) = [\partial_{ij} v^k]_{1 \leq i, j \leq 2}$ denotes the Hessian matrix of $v^k$. If $(q^k, p^k)$ is on the smooth part of the top boundary, it then follows that

$$v^k_q(q^k, p^k) = 0, \quad v^k_p(q^k, p^k) \leq 0.$$  

(3.14)

From (3.13) and (3.14), we infer that $v^k(q^k, p^k) < 0$ and $v^k(q^k, p^k) = 0$ for each $k$.

We now take a convergent subsequence $k_n \rightarrow \infty$ such that $(q^k_n, p^k_n) \rightarrow (q^*, p^*) \in R(\mu^*)$. By continuity, it follows that $v^*(q^*, p^*) = 0$ and

$$v^*_q(q^*, p^*) = 0.$$  

(3.15)

Since $v^*(q, p) > 0$ in $R(\mu^*)$, the limit point $(q^*, p^*)$ must be on the boundary $\partial R(\mu^*)$ and $v^*(q^*, p^*) = 0$.
The next step is to prove that \((q^*, p^*)\) is the corner point \((\mu^*, 0)\). (Then, it follows from (3.15) that \(v_q^*(\mu^*, 0) = 0\).)

By (3.15), the Hopf boundary lemma rules out the possibility that \((q^*, p^*)\) is on the smooth part of the side boundaries \(q^* = -L/2\) or \(q^* = \mu^*\). Indeed, \(v_q^*(q^*, p^*) = 0\). Similarly, \((q^*, p^*)\) cannot be achieved on the smooth part of the bottom boundary \(p^* = p_0\); otherwise, we would find a sequence \((q^k, p^k) \in R(\mu^k)\), such that \((q^k, p^k) \to (q^*, p^*)\) with \(p^* = p_0\) and (3.13) would hold for each \(k\). In particular, \(v_p^*(q^k, p^k) = 0\) for each \(k\). By continuity, then \(v_p^*(q^*, p^*) = 0\), which again contradicts the Hopf boundary lemma.

If \((q^*, p^*)\) were on the smooth part of the top boundary \(p^* = 0\), then the boundary condition (3.9) at such a point would reduce to

\[
(2gh^* - Q)f_p^* v_p^*(q^*, p^*) = 0;
\]

here, \(h^*(q, p) = h(q, p; \mu^*)\) and \(f^*(q, p) = f(q, p; \mu^*)\). This uses that \(v^*(q^*, p^*) = 0\) and \(v_q^*(q^*, p^*) = 0\). Since \[2(2gh^* - Q)h_p^* > 2\|h\|_{C^1(\overline{R})} > 0\] and \(f_p^*(q^*, p^*) > 2\delta > 0\), this in turn would imply that \(v_p^*(q^*, p^*) = 0\), which contradicts the Hopf boundary lemma. Therefore, \((q^*, p^*)\) cannot be on the smooth part of the boundaries of \(R(\mu^*)\).

We now consider the corner points of \(R(\mu^*)\). It is clear that \((q^*, p^*)\) is not the wave trough \(-L/2, 0\) since \(v^*(-L/2, 0) > 0\) by assumption. Let us assume that \((q^*, p^*)\) is a lower corner point, i.e. \(p^* = p_0\) and either \(q^* = -L/2\) or \(q^* = \mu^*\). Then, there exists a sequence \((q^k, p^k) \in R\mu^k\), such that \((q^k, p^k) \to (q^*, p^*)\), and thus (3.13) holds for each \(k\). By continuity, we obtain

\[
\nabla v^*(q^*, p^*) = 0, \quad \det(\nabla^2 v^*)(q^*, p^*) \geq 0.
\]

On the other hand, the boundary condition (3.10) of \(v^*\) for \(p = p_0\) implies that \(v_q^*(q^*, p^*) = v_q(q^*, p^*) = 0\). Since \(\det(\nabla^2 v^*)(q^*, p^*) = v_{qq}^*(q^*, p^*) v_{pp}^*(q^*, p^*) - (v_{qp}^*)^2 (q^*, p^*) \geq 0\), it follows that \(v_{qp}^*(q^*, p^*) = 0\). We use (3.8) to express \(v_{pp}^*\) in terms of other derivatives of order 1 or 2. Evaluated at \((q^*, p^*)\), it asserts that \(v_{pp}^*(q^*, p^*) = 0\). Indeed, \(\nabla v^*(q^*, p^*) = 0\) and \(v_{qp}^*(q^*, p^*) = v_{qp}(q^*, p^*) = 0\). Since \(h(q^*, p^*) = 0\), the edge-point lemma leads to a contradiction, and therefore, \((q^*, p^*)\) must be \((\mu^*, 0)\).

It follows that \(v^*(\mu^*, 0) = 0\) and \(v_q^*(\mu^*, 0) = 0\). Recall that (3.12) holds at \((\mu^*, 0)\), i.e.

\[
v_p^*(\mu^*, 0) = v_{pp}(\mu^*, 0) = 0. \tag{3.16}
\]

The third step is to demonstrate \(v_{qp}(\mu^*, 0) = 0\) and to draw a contradiction. Note that the bluntness condition (3.7) is fulfilled since \(h_q(\mu^*, 0) = 0\). We differentiate the top boundary condition (2.9) to obtain

\[
2(2gh - Q)h_p h_{qp} + 2h_q h_{qq} + 2gh_p^2 h_q = 0.
\]

Since \(h_q(\mu^*, 0) = -(1/2)v_q^*(\mu^*, 0) = 0\), the above equation at \((\mu^*, 0)\) reduces to

\[
2(2gh - Q)h_p h_{qp}(\mu^*, 0) = 0,
\]

whence \(h_{qp}(\mu^*, 0) = 0\). This in turn implies \(v_{qp}^*(\mu^*, 0) = -2h_{qp}(\mu^*, 0) = 0\).
We use (3.8) to express \( v_{qq} \) in terms of the other derivatives of order 1 or 2. Recall that (3.16) holds. Since \( v_q^*(\mu^*, 0) = v_{qp}^*(\mu^*, 0) = 0 \), it follows that \( v_{qq}^*(\mu^*, 0) = 0 \). This violates the edge point lemma, and therefore asserts \( \mu^* = 0 \).

By continuity, \( v^*(q, p) \geq 0 \) in \( \bar{R}(\mu^*) \); or equivalently,

\[
h(-q, p) - h(q, p) \geq 0 \quad \text{for} \quad (q, p) \in \bar{R} \quad \text{with} \quad -L/2 \leq q \leq 0.
\]

Since \( F_1 \) and \( F_2 \) are symmetric in \( q \), the function \( h^-(q, p) = h(-q, p) \) is also a solution to (3.1) and (3.2). We repeat the same argument for \( h^- \) to conclude

\[
h(-q, p) - h(q, p) \geq 0 \quad \text{for} \quad (q, p) \in \bar{R} \quad \text{with} \quad 0 \leq q \leq L/2.
\]

This proves the symmetry assertion (3.4).

The last step is to prove the monotonicity assertion (3.5). For \( \mu < 0 \), since \( v^*(-L/2, 0) > 0 \), the maximum principle shows that \( v^* > 0 \) in \( R(\mu) \). The Hopf boundary lemma then ensures that

\[
h_q(\mu, p) = -\frac{1}{2} v^*_q(\mu, p) > 0 \quad \text{for} \quad \mu < 0 \quad \text{and} \quad p_0 < p < 0.
\]

By continuity follows \( h_q(\mu, 0) \geq 0 \). To state (3.5), it remains to show that \( h_q(\mu, 0) > 0 \).

Suppose that \( h_q(\mu, 0) = 0 \). The boundary condition \( F_{2h}(h, \nabla h)[h_q] = 0 \) at \( (\mu, 0) \) would reduce to

\[
2(2gh - Q)h_p h_{qp} = 0.
\]

Note that the elliptic differential equation \( A[v^*] = 0 \) holds and \( v^*(\mu, 0) = 0 \) is a minimum of \( v^* \) in \( \bar{R}(\mu) \). On the other hand,

\[
\begin{align*}
v_q^*(\mu, 0) &= -2h_q(\mu, 0) = 0, \\
v_{qp}^*(\mu, 0) &= -2h_{qp}(\mu, 0) = 0,
\end{align*}
\]

hold at \( (\mu, 0) \). The same argument as in the third step contradicts the edge point lemma. This completes the proof.

Remark. In order to initiate the method of moving planes, it is important to have control over the behaviour of solutions at the wave trough. In the solitary-wave problem (Hur 2006), a priori asymptotic description of solutions at infinity provides the positivity property of \( v^* \) at infinity. Our proof for the periodic waves, in contrast, requires the monotonicity of \( v^* \) on the (top) boundary near the trough. An interesting open question is a priori monotonicity estimate of periodic-wave solutions near the trough.

Remark. The boundary conditions (3.2) control the sign of \( v^* \) at the left side \( q = -L/2 \), so that the maximum principles be applicable. The same boundary conditions appear in Berestycki & Nirenberg (1988). A similar constraint is imposed in the irrotational case (Garabedian 1965; Toland 2000). Another interesting open question is to prove that if the free surface profile has a single maximum and a single minimum, then every streamline exhibits a similar geometry.
4. Discussion

Our main result is that for arbitrary vorticities, periodic water waves of finite depth are symmetric and monotone if their profile has a single minimum (trough) per wavelength near which it is monotone and every streamline attains a minimum below the trough. Garabedian (1965) and Toland (2000) proved the symmetry of periodic waves in case of zero vorticity under the assumption that every streamline has a maximum and a minimum per wavelength except for the flat bottom. Our result extends that in Garabedian (1965) and Toland (2000) to an arbitrary distribution of vorticity.

Another symmetry result for irrotational periodic waves, Okamoto & Shoji (2001), states that periodic waves are symmetric if their profile is a single-valued function and monotone crest to trough. Constantin & Escher (2004a,b) extended this result to a certain class of vorticites. With the same argument adapted in a semi-infinite strip, they also proved that periodic deep-water waves (infinite depth) are symmetric if the vorticity is positive and decreases with depth (Constantin & Escher 2004a,b). While the approach taken here is similar to that of Okamoto & Shoji (2001) and Constantin & Escher (2004a,b) in which it is based on the maximum principles, our proof uses a nonlinear elliptic boundary value problem (3.1) in a fixed domain in which the vorticity function appears only in places of coefficient functions, and thus it indicates that the vorticity has no contribution to the symmetry of periodic water waves. In contrast, Okamoto & Shoji (2001) and Constantin & Escher (2004a,b) use the Poisson equation in an unknown domain in which the vorticity function is the inhomogeneous term.

The same technique, as adapted to a nonlinear elliptic boundary value problem in an infinite strip, leads to symmetry and monotonicity results for supercritical solitary waves with arbitrary vorticity (Hur 2006). The proof for the solitary-wave problem uses the asymptotic description of solutions at infinity, which asserts the positivity property analogous to (3.11) at infinity. In case of periodic waves, the monotonicity of the wave profile near the trough is not a conclusion, as for solitary waves, but rather it is a hypothesis.

Sub-harmonic bifurcations of Stokes waves (steady periodic water waves of infinite depth in an irrotational flow) near the wave of the greatest height (Buffoni et al. 2000) suggest that periodic waves of oscillation (infinitely many troughs and crests) may be asymmetric. It would be nice to construct asymmetric periodic water waves with oscillatory trough or to find a sufficient condition for symmetry with general distribution of vorticity.

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References


