Water waves and integrability

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Euler’s equations describe the motion of inviscid fluid. In the case of shallow water, when a perturbative asymptotic expansion of Euler’s equations is considered (to a certain order of smallness of the scale parameters), relations to certain integrable equations emerge. Some recent results concerning the use of integrable equation in modelling the motion of shallow water waves are reviewed in this contribution.

Keywords: Euler’s equations; integrability; Camassa–Holm equation; Degasperis–Procesi equation; Korteweg–de Vries equation

1. Governing equations for the inviscid fluid motion

The motion of inviscid fluid with a constant density \( \rho \) is described by Euler’s equations

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \mathbf{g},
\]

\[
\nabla \cdot \mathbf{v} = 0,
\]

where \( \mathbf{v}(x, y, z, t) \) is the velocity of the fluid at the point \( (x, y, z) \) in time \( t \), \( P \) is the pressure of the fluid and \( \mathbf{g} = (0, 0, -g) \) is the constant acceleration due to gravity of Earth.

Consider now the motion of shallow water over a flat surface, which is located at \( z = 0 \). We assume that the motion is in the \( x \)-direction and the physical variables do not depend on \( y \). Let \( h \) be the mean level of water and \( \eta(x, t) \) be the shape of the water surface, i.e. the deviation from the average level. The pressure is

\[
P = P_A + \rho g (h - z) + p(x, z, t),
\]

where \( P_A \) is the constant atmospheric pressure and \( p \) is a pressure variable, measuring the deviation from the hydrostatic pressure distribution. On the surface \( z = h + \eta \), \( P = P_A \) and therefore \( p = \eta \rho g \). Taking \( \mathbf{v} = (u, 0, w) \), we can write the kinematic condition on the surface as (e.g. Johnson 1997)

\[
w = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \quad \text{on} \quad z = h + \eta.
\]

Finally, there is no horizontal velocity at the bottom, thus

\[
w = 0 \quad \text{on} \quad z = 0.
\]
Equations (1.1)–(1.5) give the system

\[ u_t + uu_x + wu_z = -\frac{1}{\rho} p_x, \quad w_t + uw_x + ww_z = -\frac{1}{\rho} p_z, \quad u_x + w_z = 0, \]
\[ w = \eta_t + u\eta_x, \quad p = \eta \rho g, \quad \text{on} \quad z = h + \eta, \]
\[ w = 0 \quad \text{on} \quad z = 0. \tag{1.6} \]

Let us now introduce the dimensionless parameters \( \varepsilon = a/h \) and \( \delta = h/\lambda \), where \( a \) is the typical amplitude of the wave and \( \lambda \) is the typical wavelength of the wave, and then the dimensionless quantities according to the magnitude of the physical quantities (see Johnson 1997, 2002 for details)

\[ x \to \lambda x, \quad z \to zh, \quad t \to \frac{\lambda}{\sqrt{gh}} t, \quad \eta \to a\eta, \quad u \to \varepsilon \sqrt{gh} u, \]
\[ w \to \varepsilon \delta \sqrt{gh} w, \quad p \to \varepsilon \rho gh. \]

This scaling is due to the observation that both \( w \) and \( p \) are proportional to \( \varepsilon \), i.e. the wave amplitude, since at undisturbed surface (\( \varepsilon = 0 \)) both \( w = 0 \) and \( p = 0 \). Substituting the new dimensionless variables in system (1.6) gives

\[ u_t + \varepsilon(uu_x + wu_z) = -p_x, \quad \delta^2 (w_t + \varepsilon(uw_x + ww_z)) = -p_z, \quad u_x + w_z = 0, \]
\[ w = \eta_t + \varepsilon u\eta_x, \quad p = \eta, \quad \text{on} \quad z = 1 + \varepsilon \eta, \]
\[ w = 0 \quad \text{on} \quad z = 0. \tag{1.7} \]

For the right-moving waves, one can introduce the so-called far-field quantities (Johnson 1997, 2002, 2003a)

\[ \zeta = \sqrt{\varepsilon} (x - t), \quad \tau = \varepsilon^{3/2} t, \quad w = \sqrt{\varepsilon} W, \tag{1.8} \]

and the system (1.7) acquires the form

\[ \varepsilon u_\zeta - u_\zeta + \varepsilon(uu_\zeta + Wu_z) = -p_\zeta, \tag{1.9} \]
\[ \varepsilon \delta^2 (\varepsilon W_\tau - W_\tau + \varepsilon(uW_\zeta + WW_z)) = -p_z, \tag{1.10} \]
\[ u_\zeta + W_z = 0, \tag{1.11} \]
\[ W = \varepsilon \eta_\tau - \eta_\zeta + \varepsilon u\eta_\zeta, \quad p = \eta, \quad \text{on} \quad z = 1 + \varepsilon \eta, \tag{1.12} \]
\[ W = 0 \quad \text{on} \quad z = 0. \tag{1.13} \]

2. Asymptotic expansion of the variables

Following the idea of Johnson (2002), we can express the variables \( u, W \) and \( p \) as double asymptotic expansion (in \( \varepsilon \) and \( \delta \)) with terms depending only on \( \eta(x, t) \) and explicitly on \( z \). As a result, a single nonlinear equation for \( \eta \) will be obtained, and thus all the variables will be expressed by the solution of this equation.
From equation (1.10), it is evident that \( p_z = O(\varepsilon \delta^2) \), and thus in the leading order \( p \) does not depend on \( z \), i.e.

\[ p = \eta. \]  

Substitution of equation (2.1) into (1.9) and (1.11) gives the leading orders

\[ u = \eta, \quad W = -z \eta_z. \]  

Consider the next terms (first corrections) in the expansion of \( u \) and \( W \), denoted by \( E(u) \) and \( E(W) \), which possibly contain terms of orders \( \varepsilon \) and \( \delta^2 \). Writing \( u = \eta + E(u) \) and \( W = -z \eta_z + E(W) \), from equation (1.9) it follows that

\[ E_\zeta(u) = \varepsilon (\eta_\tau + \eta \eta_z), \]  

and from equations (1.11), (1.13) and (2.3),

\[ E(W) = - \int E_\zeta(u) dz = -\varepsilon z (\eta_\tau + \eta \eta_z), \]  

\[ W = -z (\eta_z + \varepsilon \eta_\tau + \varepsilon \eta \eta_z). \]  

Now substitution of equation (2.4) into (1.12) gives the leading-order equation for \( \eta \)

\[ \eta_\tau = -\frac{3}{2} \eta \eta_z + O(\varepsilon, \delta^2). \]  

From equations (2.5) and (2.3), we obtain \( E(u) = -\varepsilon \eta^2/4 \), i.e. no \( \delta^2 \) term is present, and finally using equations (2.4) and (2.5)

\[ u = \eta - \frac{\varepsilon}{4} \eta^2, \quad W = -z \left( \eta_z - \frac{\varepsilon}{2} \eta \eta_z \right). \]  

Substituting equation (2.2) into (1.10), we obtain \( p_z = -\varepsilon \delta^2 z \eta_{\xi \xi} \). This can be integrated using the condition on the surface (1.12), giving the next order approximation for \( p \):

\[ p = \eta - \varepsilon \delta^2 \frac{1 - z^2}{2} \eta_{\xi \xi}. \]  

We accomplished the first step, i.e. starting from the leading order (2.1) and (2.2), we obtained the first corrections (2.6) and (2.7) and an equation for \( \eta \) (2.5). The next step can be performed in a similar manner to give

\[ u = \eta - \frac{\varepsilon}{4} \eta^2 + \frac{\varepsilon^2}{8} \eta^3 + \varepsilon \delta^2 \left( \frac{1}{3} - \frac{z^2}{2} \right) \eta_{\xi \xi \xi}, \]  

\[ W = -z \left( \eta_z - \frac{\varepsilon}{2} \eta \eta_z + \frac{3 \varepsilon^2}{8} \eta^2 \eta_z \right) + \varepsilon \delta^2 \left( -\frac{z}{3} + \frac{z^3}{6} \right) \eta_{\xi \xi \xi}, \]  

\[ p = \eta - \varepsilon \delta^2 \frac{1 - z^2}{2} \eta_{\xi \xi} + \varepsilon^2 \delta^2 (\eta \eta_{\xi \xi} + (1 - z^2) \eta_z^2), \]  

where \( \eta(x, t) \) satisfies the equation

\[ \eta_\tau = -\frac{3}{2} \eta \eta_z + \frac{3}{8} \varepsilon \eta^2 \eta_z - \frac{1}{6} \delta^2 \eta_{\xi \xi \xi} + O(\varepsilon^2, \delta^4, \varepsilon \delta^2). \]
We observe that at the end of each step, the equation for $\eta(x, t)$ contains terms of smaller order than those that appear in the expressions for $u$ and $W$. Since we need an equation containing terms of order $O(\varepsilon^2, \delta^4, \varepsilon^5)$, we need to perform several intermediate substeps (like equations (2.3) and (2.4)) of the next step, which leads to the desired equation

$$
\eta_\tau = -\frac{3}{2} \eta_\zeta + \frac{3}{8} \varepsilon \eta_\zeta - \frac{3}{16} \varepsilon^2 \eta_\zeta - \frac{1}{6} \delta^2 \eta_{\zeta\zeta} - \frac{1}{24} \varepsilon \delta^2 (23 \eta_{\zeta\zeta} + 10 \eta_{\zeta\zeta\zeta}) + O(\varepsilon^3, \delta^6, \varepsilon^2 \delta^2, \varepsilon^4). \tag{2.12}
$$

Now we can invert equation (2.8) by specifying $u$ at a specific depth, $z_0$ ($0 \leq z_0 \leq 1$); defining $\tilde{u} = u(\zeta, \tau, z_0)$, we obtain

$$
\eta = \tilde{u} + \frac{\varepsilon}{4} \tilde{u}^2 - \varepsilon \delta^2 \lambda \tilde{u}_{\zeta\zeta} + O(\varepsilon^3, \delta^6, \varepsilon^4, \varepsilon^2 \delta^2), \tag{2.13}
$$

where

$$
\lambda \equiv \frac{1}{3} - \frac{z_0^2}{2}, \quad -\frac{1}{6} \leq \lambda \leq \frac{1}{3}. \tag{14.14}
$$

Note that in equation (2.13), there is no term of order $\varepsilon^2$. The substitution of equation (2.13) into (2.12) yields

$$
\hat{u}_\tau = -\frac{3}{2} \hat{u}_\zeta - \frac{1}{6} \delta^2 \hat{u}_{\zeta\zeta\zeta} - \frac{1}{2} \varepsilon \delta^2 \left[ \frac{29}{12} + 6\lambda \right] \hat{u}_\zeta \hat{u}_{\zeta\zeta} + \frac{5}{6} \hat{u}_{\zeta\zeta\zeta} + O(\varepsilon^3, \delta^6, \varepsilon^2 \delta^2, \varepsilon^4). \tag{2.15}
$$

Next, we go back to the original variables introducing $T \equiv \sqrt{\varepsilon} t$ and $X \equiv \sqrt{\varepsilon} x$ (equation (1.8)) keeping only the scaling with $\varepsilon$

$$
T = \frac{1}{\varepsilon} \tau, \quad X = \frac{1}{\varepsilon} \tau + \zeta, \tag{16.16}
$$

or $\partial_\zeta = \partial_X$ and $\varepsilon \partial_\tau = \partial_T + \partial_X$. Thus, equation (2.15) yields

$$
\hat{u}_T = -\hat{u}_X - \frac{3}{2} \varepsilon \hat{u}_X - \frac{1}{6} \varepsilon \delta^2 \hat{u}_{XX} - \frac{1}{2} \varepsilon^2 \delta^2 \left[ \frac{29}{12} + 6\lambda \right] \hat{u}_X \hat{u}_{XX} + \frac{5}{6} \hat{u}_{XXX} + O(\varepsilon^4, \varepsilon^6, \varepsilon^3 \delta^2, \varepsilon^2 \delta^4). \tag{2.17}
$$

Further, we add formally $(\varepsilon \delta^2 \mu \hat{u}_{XXX} - \varepsilon \delta^2 \mu \hat{u}_{XXXT})/2$ to the left-hand side of equation (2.17), where $\mu$ is an arbitrary real parameter. In the first term, we substitute $\hat{u}_T = -\hat{u}_X - (3/2) \varepsilon \hat{u}_X$, according to equation (2.17)

$$
\left( \hat{u} - \frac{1}{2} \varepsilon \delta^2 \mu \hat{u}_XX \right)_T = -\hat{u}_X - \frac{3}{2} \varepsilon \hat{u}_X + \varepsilon \delta^2 \left( \frac{1}{2} \mu - \frac{1}{6} \right) \hat{u}_{XXX} - \frac{1}{2} \varepsilon^2 \delta^2 \left[ \frac{29}{12} + 6\lambda - \frac{3}{2} \mu \right] \hat{u}_X \hat{u}_{XX} + \left( \frac{5}{6} - \frac{3}{2} \mu \right) \hat{u}_{XXX} + O(\varepsilon^4, \varepsilon^6, \varepsilon^3 \delta^2, \varepsilon^2 \delta^4). \tag{2.18}
$$
We observe that equations (2.17) and (2.18) do not contain terms of orders $\varepsilon$ and $\varepsilon^2$. Thus, the set-up from Johnson (2002) naturally leads to the conclusion that equations containing nonlinearities, as those appearing in the equations of Fokas & Fuchssteiner (1981), Camassa & Holm (1993; referred to as CH from now on), Degasperis & Procesi (1999) and Degasperis et al. (2002; referred to as DP from now on), are generalizations of the Korteweg–de Vries (KdV) equation, containing the next order term ($\varepsilon^2 \delta^2$) in the expansion with respect to the small parameters $\varepsilon$ and $\delta$.

3. Integrable nonlinear equations

In this section, we start from a known integrable equation and try to write it in a form in which it matches equation (2.18) or (2.17). For another approach of matching between water-wave and integrable equations see Dullin et al. (2003, 2004). The CH and DP equations can be written as

$$\left(U-U_{XX}\right)_T = \omega U_X-(b+1)UU_X + bU_XU_{XX} + UU_{XXX},$$

(3.1)

where $U=U(X, T)$, $\omega$ is an arbitrary constant, $b=2$ for CH and $b=3$ for DP. Furthermore, one can show (Ivanov 2005a) that the nonlinear equation

$$\left(U-U_{XX}\right)_T = \omega U_X + \alpha_1 UU_X + \alpha_2 U_XU_{XX} + \alpha_3 UU_{XXX}$$

is integrable if and only if the choice of the constants $\alpha_1$, $\alpha_2$, $\alpha_3$ is such that this equation coincides with either the CH or the DP equation. Let us change the variables in equation (3.1) according to

$$X \to X-vT, \quad T \to T, \quad U \to U + C,$$

(3.2)

where $v$ and $C$ are arbitrary constants. Then equation (3.1) acquires the form

$$\left(U-U_{XX}\right)_T = [\omega-(b+1)C+v]U_X + (C-v)U_{XXX}-(b+1)UU_X + bU_XU_{XX} + UU_{XXX}.$$  

(3.3)

It is now clear that through the transforms (3.2), one can achieve arbitrary coefficients for the linear terms $U_X$ and $U_{XXX}$. Let us now consider the following scaling of the variables:

$$X \to \frac{1}{\alpha}X, \quad T \to \beta T, \quad U \to \gamma U.$$

(3.4)

Then equation (3.3) can be written as

$$\left(U-U_{XX}\right)_T = \Gamma_1 U_X + \Gamma_2 U_{XXX} - \alpha \beta \gamma (b+1)UU_X + \alpha^2 \beta \gamma (bU_XU_{XX} + UU_{XXX}),$$

(3.5)

where $\Gamma_1$ and $\Gamma_2$ are arbitrary constants. In order to match equation (3.5) with (2.18) up to the given order, we need to make the following identifications:

$$\alpha^2 = \frac{1}{2} \varepsilon \delta^2 \mu, \quad \alpha \beta \gamma (b+1) = \frac{3}{2} \varepsilon, \quad \alpha^3 \beta \gamma = -\frac{1}{2} \left(\frac{5}{6} \lambda - \frac{3}{2} \mu \right) \varepsilon \delta^2,$$

$$\frac{29}{12} + 6 \lambda = b \left(\frac{5}{6} \lambda - \frac{3}{2} \mu \right),$$

(3.6)
which are compatible if
\[
\mu = \frac{5(b+1)}{9b}, \quad \lambda = \frac{30-9b}{72b}.
\] (3.7)

Thus, equations (3.7) and (2.14) show that equation (3.5) describes water waves at depth
\[
z_0 = \sqrt{\frac{11b-10}{12b}},
\] (3.8)
i.e. CH \((b=2)\) corresponds to \(z_0 = (1/\sqrt{2}) \approx 0.71\) and DP \((b=3)\) corresponds to \(z_0 = \sqrt{23/36} \approx 0.80\). The scaling coefficients from equation (3.6) are
\[
\alpha = \sqrt{\frac{\mu}{2}} \varepsilon^{1/2}, \quad \beta \gamma = \frac{3}{\sqrt{2\mu(b+1)}} \varepsilon^{1/2} \delta^{-1},
\] (3.9)
and apparently only the product \(\beta \gamma\) is determined, i.e. there is additional freedom in the choice of \(\beta\) and \(\gamma\), one can take, for simplicity, just \(\gamma = 1\) and then finally
\[
\alpha = \sqrt{\frac{\mu}{2}} \varepsilon^{1/2}, \quad \beta = \frac{3}{\sqrt{2\mu(b+1)}} \varepsilon^{1/2} \delta^{-1}, \quad \gamma = 1.
\] (3.10)

Another equation which passes the integrability check developed by Sanders & Wang (1998), Olver & Wang (2000) and Mikhailov & Novikov (2002), and is presumably integrable (although we do not have a proof of this fact—the test provides only a necessary condition for integrability) is
\[
(U - U_{XX} + U_{XXXX})_T = \Gamma_1 U_X - \Gamma_2 U_{XXX} + \Gamma_2 U_{XXXXX} - UU_X + U_X U_{XX}
\]
\[- U_X U_{XXXX},
\] (3.11)
where \(\Gamma_1\) and \(\Gamma_2\) are arbitrary constants. This equation contains nonlinearities, similar to those appearing in the non-integrable equations studied by Holm & Hone (2003).

The scaling equation (3.4) gives
\[
(U - \varepsilon^2 U_{XX} + \varepsilon^4 U_{4X})_T = \alpha \beta \Gamma_1 U_X - \alpha^3 \beta \Gamma_2 U_{XXX} + \alpha^5 \beta \Gamma_2 U_{5X}
\]
\[- \alpha \beta \gamma U U_X + \alpha^3 \beta \gamma U_X U_{XX} - \alpha^5 \beta \gamma U_X U_{4X}.
\] (3.12)
The matching between equations (3.12) and (2.18) leads to the following identifications:
\[
\alpha^2 = \frac{1}{2} \varepsilon \delta^2, \quad \alpha \beta \gamma = \frac{3}{2} \varepsilon, \quad \alpha^3 \beta \gamma = -\frac{1}{2} \left(6\lambda - \frac{9}{2} \mu + \frac{29}{12} \mu \right) \varepsilon \delta^2, \quad \mu = \frac{5}{9}.
\] (3.13)

In a similar way, from equation (3.12), we find (assuming again \(\gamma = 1\))
\[
\lambda = -\frac{1}{8}, \quad z_0 = \sqrt{\frac{11}{12}} \approx 0.96,
\] (3.14)
\[ \alpha = \sqrt{\frac{\mu}{2}} \varepsilon^{1/2} \delta, \quad \beta = \frac{3}{\sqrt{2\mu}} \varepsilon^{1/2} \delta^{-1}, \quad (\gamma = 1). \] (3.15)

The terms with fourth and fifth derivative in equation (3.12) are of orders
\[ \alpha^4 = \frac{\mu^2}{4} \varepsilon^2 \delta^4, \quad \alpha^5 \beta = \frac{3\mu^2}{8} \varepsilon^3 \delta^4, \quad \alpha^5 \beta \gamma = \frac{3\mu^2}{8} \varepsilon^3 \delta^4, \] (3.16)
and therefore are small in comparison with the other terms.

Another set of integrable equations is of the type
\[ U_T + U_{XXXX} + 2(6b + 1) U_X U_{XX} + 4(b + 1) U U_{XX} + 20bU^2 U_X = 0, \] (3.17)
where one can recover the Caudrey–Dodd–Gibbon equation (Caudrey et al. 1976) for \( b=1/4 \), the Sawada–Kotera equation (Sawada & Kotera 1974) for \( b=3/2 \) and the Kaup–Kuperschmidt equation (Kaup 1980) for \( b=4 \).

It is a natural question to ask whether equation (3.17) can match (2.17). Applying the transformation (3.2) to (3.17), it can be written in the form
\[ U_T = \Gamma U_X - 4(b + 1) C U_{XXX} - 5U_X - 40bCUU_X - 2(6b + 1) U_X U_{XX} \]
\[ - 4(b + 1)UU_{XX} - 20bU^2 U_X, \] (3.18)
where \( \Gamma = v + 20bC^2 \) can apparently be arranged to be an arbitrary constant with the help of the free parameter \( v \). The scaling (3.4) applied to equation (3.18) gives
\[ U_T = \alpha \beta \Gamma U_X - 4(b + 1) C \alpha^3 \beta U_{XXX} - \alpha^5 U_{XX} - 40bC\alpha \beta \gamma UU_X \]
\[ - \alpha^3 \beta \gamma [2(6b + 1) U_X U_{XX} + 4(b + 1)UU_{XXX}] - 20b \alpha \beta \gamma^2 U^2 U_X. \] (3.19)

Apparently, we need to make the following identifications:
\[ 4(b + 1) C \alpha^3 \beta = \frac{1}{6} \varepsilon \delta^2, \quad 40bC\alpha \beta \gamma = \frac{3}{2} \varepsilon, \quad 4(b + 1) \alpha^3 \beta \gamma = \frac{5}{12} \varepsilon^2 \delta^2, \] (3.20)
giving
\[ \alpha \beta = \frac{3}{200bC^2}, \quad \gamma = \frac{5}{2} \varepsilon. \] (3.21)

The order of the term \( U^2 U_X \) is \( 20b \alpha \beta \gamma^2 = (3/16)\varepsilon^2 \); it is not small in comparison with the other terms, and therefore cannot be neglected. Thus, there is no direct match between equations (3.17) and (2.17); however, there is a more complicated transformation given by Fokas & Liu (1996; based on the Kodama transform (Kodama 1985)), providing the link between the water-wave equations and the integrable systems (3.17).

### 4. Water waves moving over a shear flow

So far we have only considered waves in the absence of shear. Now let us note that there is an exact solution of the governing equations (1.6) of the form \( u = \tilde{U}(z), \) \( 0 \leq z \leq h, \) \( w \equiv 0, \) \( p \equiv 0, \) \( \eta \equiv 0. \) This solution is nothing but an arbitrary underlying
‘shear’ flow. Waves of small amplitude (of the order $\epsilon$) propagating over this underlying flow are studied by many authors and here we will partially follow Burns (1953) and Johnson (2003a). The scaling for such solution is clearly

$$u \mapsto \sqrt{gh} (\tilde{U}(z) + \epsilon u),$$

and the scaling for the other variables is the same as before. Thus, from equation (1.6) instead of equation (1.7) in this case, we have

$$u_t + \tilde{U} u_x + w \tilde{U}' + \epsilon (uw_x + wz) = -p_x, \quad \delta^2 (w_t + \tilde{U} w_x + \epsilon (uw_x + wz)) = -p_z, \quad u_x + w_z = 0, \quad w = \eta_t + (\tilde{U} + \epsilon u) \eta_x, \quad p = \eta, \quad \text{on } z = 1 + \epsilon \eta, \quad w = 0 \quad \text{on } z = 0.$$ (4.1)

The prime denotes the derivative with respect to $z$.

In what follows, we need the propagation speed $c$ of the waves in the linear approximation, i.e. in the case when in equation (4.1) it is taken $\epsilon = \delta = 0$. This velocity is now not independent on $\tilde{U}$. Since $p_z = 0$, in the linear approximation $p = \eta$. Let us introduce a stream function $\psi$, such that $u = \psi_x$ and $w = -\psi_x$. In the case of linear waves, we can assume that $\psi = \phi(z)e^{ik(x-ct)}$ and $\eta = \eta_0 e^{ik(x-ct)}$, where $k$ is the wavenumber and $\eta_0$ is a constant. From equation (4.1) with $\epsilon = \delta = 0$, we now easily find a relation between $\tilde{U}(z)$ and $\phi(z)$

$$\phi' (\tilde{U} - c) - \tilde{U}' \phi + \eta_0 = 0, \quad \phi(1) = -(U(1) - c) \eta_0, \quad \phi(0) = 0.$$ (4.2)

The first equation in (4.2) can be written as

$$\frac{d}{dz} \frac{\phi}{\tilde{U} - c} = -\frac{\eta_0}{(\tilde{U} - c)^2},$$

and can be integrated directly. Imposing the boundary conditions from equation (4.2), we finally obtain the following relation for the speed of propagation $c$ (the so-called Burns condition):

$$\int_0^1 \frac{dz}{[\tilde{U}(z) - c]^2} = 1.$$ (4.3)

For a non-decreasing function $\tilde{U}(z)$, such that $\tilde{U}(0) \leq \tilde{U}(z) \leq \tilde{U}(1)$, there are always two solutions: $c > \tilde{U}(1)$ and $c < \tilde{U}(0)$. In the absence of flow, $\tilde{U} \equiv 0$, these two solutions are simply $c = \pm 1$. The presentation in the previous sections corresponds to the choice $c = 1$.

Again, we introduce the far-field variables, cf. (1.8)

$$\zeta = \sqrt{\epsilon} (x - ct), \quad \tau = \epsilon^{3/2} t, \quad w = \sqrt{\epsilon} W,$$ (4.4)

and the system (4.1) acquires the form

$$\begin{align*}
\epsilon u_\tau + (\tilde{U} - c) u_\zeta + W \tilde{U}' + \epsilon (uw_\zeta + Wz) &= -p_\zeta, \\
\epsilon \delta^2 (\epsilon W_\tau + (\tilde{U} - c) W_\zeta + \epsilon (uW_\zeta + Wz)) &= -p_z, \\
u_\zeta + W_z &= 0, \\
W &= \epsilon \eta_\tau + (\tilde{U} - c) \eta_\zeta + \epsilon u\eta_z, \quad p = \eta, \quad \text{on } z = 1 + \epsilon \eta, \\
W &= 0 \quad \text{on } z = 0.
\end{align*}$$ (4.5)
Now let us concentrate on the simplest non-trivial case, a linear shear 
\( \hat{U}(z) = Az \), where \( A \) is a constant. We choose \( A > 0 \), so that the underlying flow is propagating in the positive direction of the \( x \)-coordinate. The condition (4.3) gives the following expression for \( c \):
\[
c = \frac{1}{2} \left( A \pm \sqrt{4 + A^2} \right). \tag{4.6}
\]
If there is no shear (\( A = 0 \)), then \( c = \pm 1 \).
Of course, a parabolic distribution \( \hat{U}(z) = \hat{U}(1)(2z - z^2) \) would be more realistic, but then the solution of equation (4.3) is not so simple (cf. Burns 1953).

The solution of the system (4.5) can be obtained as a series in \( \varepsilon \) and \( \delta \) following the method explained in §3. Here we present the final result obtained by Johnson (2003a). The equation for \( \eta \) is
\[
\eta_t = -\frac{c^4 + c^2 + 1}{c(c^2 + 1)} \eta \eta_z + \frac{c(c^4 + 4c^2 + 1)}{2(c^2 + 1)^3} \varepsilon \eta^2 \eta_z - \frac{1}{3c(c^2 + 1)} \delta^2 \eta \eta_z \eta_z - \frac{1}{3c(c^2 + 1)} \varepsilon \delta^2 [(2c^6 + 4c^4 + 11c^2 + 6) \eta \eta_z \eta_z + (c^4 + 6c^2 + 3) \eta \eta \eta_z] + O(\varepsilon^2, \delta^4). \tag{4.7}
\]
In the no-shear case and right-moving waves (\( c = 1 \)), one can recover (2.12); \( c = -1 \) corresponds to left-moving waves. The horizontal velocity to this order is
\[
u = \frac{1}{c} \eta - \varepsilon \frac{c}{2(c^2 + 1)} \eta^2 + \varepsilon \delta^2 \left[ \frac{c^2 + 3}{6c(c^2 + 1)} - \frac{z^2}{2c} \right] \eta \eta_z. \tag{4.8}
\]
Again in order to invert equation (4.7), we have to specify \( u \) at a specific depth, \( z_0 (0 \leq z_0 \leq 1) \): \( \hat{u} = u(\zeta, \tau, z_0) \). The result is
\[
\eta = c \hat{u} + \varepsilon \frac{c^4}{2(c^2 + 1)} \hat{u} z^2 + \varepsilon \delta^2 c^2 \hat{u} \hat{u}_z, \quad \text{where} \quad A = \frac{3 + c^2}{6c(1 + c^2)} - \frac{z_0^2}{2c}. \tag{4.9}
\]
It is convenient to introduce a new dependent variable
\[
V = \hat{u} + \varepsilon \sigma \hat{u}^2 + O(\varepsilon^2, \delta^4), \quad \sigma = \frac{c^5(c^4 + c^2 - 2)}{2(c^4 + c^2 + 1)(c^2 + 1)^2} \tag{4.10}
\]
(which is not the Kodama transform) for which the equation is
\[
V_t = -\frac{c^4 + c^2 + 1}{(c^2 + 1)}VV_z - \delta^2 \frac{1}{3c(c^2 + 1)} V_{zzz} - \varepsilon \delta^2 \left\{ \frac{2cA}{c^2 + 1} + \frac{c \delta^2}{3(c^2 + 1)^3} \right\} V_z V_z + \frac{c^4 + 6c^2 + 3}{3(c^2 + 1)^3} VV_{zzz} + O(\varepsilon^2, \delta^4). \tag{4.11}
\]
Next, we return to the original variables in equation (4.11), up to a scaling: \( T = \sqrt{\varepsilon t}, \ X = \sqrt{\varepsilon x}, \) see equation (4.4): \( T = \varepsilon^{-1} r, \ X = \zeta + \varepsilon^{-1} c \tau, \) then \( \partial_z = \partial_X; \varepsilon \partial_t = \partial_r + c \partial_X \).
In conjunction, we add \( (\varepsilon \delta^2 \mu V_{XXX} - \varepsilon \delta^2 \mu V_{XXX})/(1 + c^2) \) where \( \mu \) is an arbitrary real parameter. In the first term, we substitute the leading order of
\[
V_T \sim -V_X - \varepsilon \frac{c^4 + c^2 + 1}{c^2 + 1} VV_X, \tag{4.12}
\]
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and obtain
\[
V_T = -V_X - \frac{c^4 + c^2 + 1}{c^2 + 1} VV_X - \epsilon^2 \delta^2 \frac{1-3\mu c}{3(c^2+1)} V_{XXX} + \epsilon^2 \delta^2 \frac{\mu}{c^2+1} V_{XXT} \\
- \epsilon^2 \delta^2 \left[ \frac{2cA(c^2+1) - 3\mu(c^4 + c^2 + 1)}{(c^2+1)^2} + \frac{2c^6 + 7c^4 + 14c^2 + 6}{3(c^2+1)^3} \right] VX V_{XX} \\
- \epsilon^2 \delta^2 \left[ \frac{c^4 + 6c^2 + 3 - \mu c^4 + c^2 + 1}{3(c^2+1)^3} \right] VV_{XXX} + O(\epsilon^3, \epsilon \delta^4).
\] (4.12)

The comparison between equations (4.12) and (3.5) gives the following possibilities for the parameters, cf. (3.6)–(3.8):
\[
\mu = \frac{5(b + 1)(c^4 + 6c^2 + 3)}{3b(c^2+1)(c^4 + c^2 + 1)},
\]
\[
A = \frac{-2bc^{10} + (3-4b)c^8 + (21-6b)c^6 + (30-13b)c^4 + (27-2b)c^2 + 9}{6bc(c^2+1)^2(c^4 + c^2 + 1)^2}.
\]

Therefore, according to the relation between \(z_0\) and \(A\) in equation (4.9), for a given propagation speed \(c\), equation (3.5) describes water waves at depth
\[
z_0^2 = \frac{bc^{12} + 8bc^{10} + (18b-3)c^8 + (26b-21)c^6 + (31b-30)c^4 + (12b-27)c^2 + 3b - 9}{3b(c^2+1)^2(c^4 + c^2 + 1)^2}.
\] (4.13)

In the case of right-moving wave without underlying flow (\(c = 1\)), we recover equation (3.8). For the CH equation (\(b = 2\)), this gives (cf. Johnson 2003a)
\[
z_0 = \frac{\sqrt{2c^{12} + 16c^{10} + 33c^8 + 31c^6 + 32c^4 - 3c^2 - 3}}{\sqrt{6(c^2+1)(c^4 + c^2 + 1)}},
\] (4.13)

and for the DP equation (\(b = 3\))
\[
z_0 = \frac{|c|\sqrt{c^{10} + 8c^8 + 17c^6 + 19c^4 + 21c^2 + 3}}{\sqrt{3(c^2+1)(c^4 + c^2 + 1)}},
\] (4.14)

The comparison between equations (4.12) and (3.12) gives (cf. equations (3.13)–(3.16))
\[
\mu = \frac{c^4 + 6c^2 + 3}{3(c^2+1)(c^4 + c^2 + 1)}, \quad A = \frac{-c^2(c^2+2)(2c^6 + 6c^2 + 1)}{6c(c^2+1)^2(c^4 + c^2 + 1)^2},
\]
\[
z_0 = \frac{\sqrt{c^{12} + 8c^{10} + 18c^8 + 26c^6 + 31c^4 + 12c^2 + 3}}{\sqrt{3(c^2+1)(c^4 + c^2 + 1)}}.
\] (4.15)

As expected, \(c = 1\) in equation (4.15) gives the result from equation (3.14), \(z_0 \approx 0.96\).

Without loss of generality, we can assume that the underlying flow is propagating in the positive direction of the \(x\)-coordinate, i.e. \(A > 0\). Then equation (4.6) leads to the following restriction to the possible values of \(c\): \(c \geq 1\) (for the waves moving in the direction of the flow downstream) or \(-1 \leq c < 0\) (for the waves propagating upstream). The plot of the dependence of \(z_0\) on \(c\),

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according to equations (4.13)–(4.15), is given in figures 1 and 2. From figure 2, we note that there is a region where the function (4.13) is not real: \( c_0 \% c \% 0 \), \( c_0 = 0.544 \), and therefore the CH equation (in this setting) is not a relevant model for these values of \( c \). Although the graph of equation (4.15) has a maximum \( z_0 = 0.762 \) at \( c = 1.416 \). The DP graph has a maximum \( z_0 = 0.806 \) at \( c = 1.149 \).

5. Conclusions

In conclusion, equations CH, DP and (3.11) describe in a direct way (without the use of the Kodama transform) the velocity (and consequently all related variables) of shallow water waves at depths \( 0.71h \), \( 0.80h \) and \( 0.96h \), respectively (in the absence of a shear flow), where \( h \) is the depth of the undisturbed water. From a modelling point of view, the advantage of the CH and DP equations over KdV equation consists also in the fact that they capture the wave-breaking phenomenon (cf. Constantin 1997, 2000; Constantin & Escher 1998a,b, 2000; Zhou 2004). These equations can also be used as water-wave models in the presence of an arbitrary shear flow. It is always more convenient to work with integrable equations, since their solutions are explicitly known or can be, in principle, explicitly constructed. For CH, the solitons are stable patterns and thus physically recognizable, e.g. see the papers by Constantin & Strauss (2000, 2002) and Constantin & Molinet (2001). The \( N \)-soliton solution for CH is explicitly obtained by the inverse scattering method of Constantin et al. (2006; see also the earlier works on the CH spectral problem by Constantin (1998, 2001) and Constantin & McKeen (1999)). In parametric form, the \( N \)-soliton solution
for CH is obtained by Johnson (2003b), Li & Zhang (2004), Parker (2004, 2005a,b), Li (2005) and Matsuno (2005b). Other types of explicitly known CH solutions are multi-peakons (Beals et al. 2003), periodic solutions (Gesztesy & Holden 2003) and travelling waves (Lenells 2005a; Parkes & Vakhnenko 2005). The construction of multi-soliton and multi-positon solutions for the associated CH equation using the Darboux–Bäcklund transform is presented in Schiff (1998), Hone (1999) and Ivanov (2005b). The N-soliton solution for the DP equation was recently derived by Matsuno (2005a), the multi-peakon solutions for DP are obtained by Lundmark & Szmigielski (2003, 2005) and the travelling wave solution by Parkes & Vakhnenko (2004) and Lenells (2005b). There are similarities between the CH and DP equations, in a sense that they both are integrable and have a hydrodynamic derivation. However, it is interesting to note that only CH has a geometric interpretation as a geodesic flow (cf. Constantin & Kolev 2003; Kolev 2004).

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References


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Figure 2. Upstream propagation: plot of the dependence of $z_0$ on $c$. Solid line, equation (4.13) for CH; dashed line, equation (4.14) for DP; dash-dotted line, equation (4.15) for (3.12). The function (4.13) is not real for $c_0 \leq c \leq 0$ and $c_0 = -0.544$. The graph of equation (4.15) has a maximum $z_0 \approx 1.0237$ at $c \approx -0.600$. 

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