Bi-Hamiltonian systems on the dual of the Lie algebra of vector fields of the circle and periodic shallow water equations

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This paper is a survey article on bi-Hamiltonian systems on the dual of the Lie algebra of vector fields on the circle. Here, we investigate the special case where one of the structures is the canonical Lie–Poisson structure and the second one is constant. These structures, called affine or modified Lie–Poisson structures, are involved in the integrability of certain Euler equations that arise as models for shallow water waves.

Keywords: bi-Hamiltonian formalism; diffeomorphisms group of the circle; Lenard scheme; Camassa–Holm equation

1. Introduction

In the last 40 years or so, the Korteweg–de Vries equation (KdV) (Korteweg & de Vries 1895) has received much attention in the mathematical physics literature. Some significant contributions were made in particular by Gardner, Green, Kruskal and Miura—see Praught & Smirnov (2005) for a complete bibliography and a historical review. It is through these studies that the theory of solitons as well as the inverse scattering method emerged.

One remarkable property of the KdV equation, highlighted at this occasion, is the existence of an infinite number of first integrals. The mechanism by which these conserved quantities were generated is at the origin of an algorithm called the Lenard recursion scheme or bi-Hamiltonian formalism (Magri 1978; Gel’fand & Dorfman 1979). This is representative of infinite-dimensional systems known as formally integrable, in reminiscence of finite-dimensional, classical integrable systems (in the sense of Liouville). Other examples of bi-Hamiltonian systems are the Camassa–Holm equation (Fokas & Fuchssteiner 1981; Camassa & Holm 1993; Constantin 1998; Constantin & McKean 1999; Gesztesy & Holden 2003) and the Burgers equation (Burgers 1948).

One common feature of all these systems is that they can be described as the geodesic flow of some right-invariant metric on the diffeomorphism group of the circle or on a central real extension of it, the Virasoro group. Each left (or right)-invariant metric on a Lie group induces, by a reduction process, a canonical flow on the dual of its Lie algebra. The corresponding evolution equation, known as the Euler equation, is Hamiltonian relatively to some canonical Poisson

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structure. It generalizes the Euler equation of the free motion of a rigid body.\(^1\) In a famous article (Arnold 1966), Arnold pointed out that this formalism could be applied to the group of volume-preserving diffeomorphisms to describe the motion of an ideal fluid.\(^2\) Thereafter, it became clear that many equations from mathematical physics could be interpreted in the same way.

Gel’fand & Dorfman (1981) showed that the KdV equation can be obtained as the geodesic equation, on the Virasoro group, of the right-invariant metric defined on the Lie algebra by the \(L^2\) inner product (see also Ovsienko & Khesin 1987). Misiolek (1998) has shown that the Camassa–Holm equation, which is a one-dimensional model for shallow water waves, can also be obtained as the geodesic flow on the Virasoro group for the \(H^1\) metric.

While both the KdV and Camassa–Holm equations have a geometric derivation and are models for the propagation of shallow water waves, the two equations have quite different structural properties. For example, while all smooth periodic initial data for the KdV equation develop into periodic waves that exist for all times (Tao 2002), smooth periodic initial data for the Camassa–Holm equation develop either into global solutions or into breaking waves—see the papers by Constantin (1997, 2000), Constantin & Escher (1998a, b, 2000) and McKean (2004).

In this paper, we study the case of right-invariant metrics on the diffeomorphism group of the circle, \(\text{Diff}(S^1)\). However, note that a similar theory is probable without the periodicity condition, in which case some weighted spaces express how close the diffeomorphisms of the line are to the identity (Constantin 2000).

Each right-invariant metric on \(\text{Diff}(S^1)\) is defined by an inner product \(a\) on the Lie algebra of the group, \(\text{Vect}(S^1) = C^\infty(S^1)\). If this inner product is local, it is given by the expression

\[
a(u, v) = \int_{S^1} u A(v) dx \quad u, v \in C^\infty(S^1),
\]

where \(A\) is an invertible symmetric linear differential operator. To this inner product on \(\text{Vect}(S^1)\) corresponds a quadratic functional (the energy functional)

\[
H_A(m) = \frac{1}{2} \int_{S^1} m A^{-1}(m),
\]

on the (regular) dual \(\text{Vect}^*(S^1)\). Its corresponding Hamiltonian vector field \(X_A\) generates the Euler equation

\[
\frac{dm}{dt} = X_A(m).
\]

Among the Euler equations of this kind, we have the well-known inviscid Burgers equation

\[
u_t + 3uu_x = 0,
\]

and the Camassa–Holm shallow water equation (Fokas & Fuchssteiner 1981; Camassa & Holm 1993)

\[
u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1} \left( u^2 + \frac{1}{2} u_x^2 \right) = 0.
\]

\(^1\)In this case, the group is just the rotation group, \(SO(3)\).

\(^2\)However, this formalism seems to have been extended to hydrodynamics before Arnold by Moreau (1959).

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Indeed, the inviscid Burgers equation corresponds to $A = I \ (L^2 \text{ inner product})$, whereas the Camassa–Holm equation corresponds to $A = 1 - D^2 \ (H^1 \text{ inner product})$—see Constantin & Kolev (2002, 2003).

The Burgers, KdV and Camassa–Holm equations are precisely bi-Hamiltonian relative to some second affine (Souriau 1997) compatible Poisson structure\(^3\) (cf. McKean 1979; Constantin & McKean 1999; Lenells 2004). Since these equations are special cases of Euler equations induced by $H^k$ metric, it is natural to ask whether, in general, these equations have similar properties for any value of $k$. Constantin & Kolev (2006) have shown that this was not the case. There are no affine structures on $\text{Vect}^*(\mathbb{S}^1)$ which makes the Eulerian vector field $X_k$, generated by the $H^k$ metric, a bi-Hamiltonian system, unless $k = 0$ (Burgers) or 1 (Camassa–Holm). One similar result for the Virasoro algebra was given by Constantin et al. (2006). Here, we investigate the problem of finding a modified Lie–Poisson structure for which the vector field $X_A$ is bi-Hamiltonian. We show, in particular, that for an operator $A$ with constant coefficients, this is possible only if $A = aI + bD^2$, where $a, b \in \mathbb{R}$.

In §2, we recall the definition of Hamiltonian and bi-Hamiltonian manifolds and the basic materials on bi-Hamiltonian vector fields. Section 3 contains a description of Poisson structures on the dual of the Lie algebra of a Lie group. Section 4 is devoted to the study of bi-Hamiltonian Euler equations on $\text{Vect}^*(\mathbb{S}^1)$; the main results are stated and proved.

For the description of modified affine Poisson structures, we rely on Gelfand–Fuks cohomology. Since the handling of this cohomology theory is not obvious, we derive in appendix A an elementary ‘hands-on’ computation of the first two Gelfand–Fuks cohomological groups of $\text{Vect}(\mathbb{S}^1)$.

### 2. Hamiltonian and bi-Hamiltonian manifolds

In this section, we recall the definitions and the well-known results on finite-dimensional smooth Poisson manifolds.

\[\text{(a) Poisson manifolds}\]

**Definition 2.1.** A symplectic manifold is a pair $(M, \omega)$, where $M$ is a manifold and $\omega$ is a closed non-degenerate 2-form on $M$, i.e. $d\omega = 0$, and for each $m \in M$, $\omega_m$ is a non-degenerate skew-symmetric bilinear map on $T_m M$.

Since a symplectic form $\omega$ is non-degenerate, it induces an isomorphism

\[TM \rightarrow T^* M, \quad X \mapsto i_X \omega, \quad (2.1)\]

defined via $i_x \omega(Y) = \omega(X, Y)$. For example, this allows the definition of the symplectic gradient $X_f$ of a function $f$ by the relation $i_{X_f} \omega = -df$. The inverse of the isomorphism (2.1) defines a skew-symmetric bilinear form $P$ on the cotangent space $T^* M$. This bilinear form $P$ induces itself a bilinear mapping on $C^\infty(M)$, the space of smooth functions $f : M \rightarrow \mathbb{R}$, given by

\[[f, g] = P(df, dg) = \omega(X_f, X_g), \quad f, g \in C^\infty(M), \quad (2.2)\]

and called the Poisson bracket of the functions $f$ and $g$.

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\(^3\)The affine structure on the Virasoro algebra, which makes the KdV equation a bi-Hamiltonian system, seems to have been first discovered by Gardner (1971) and for this reason, some authors call it the Gardner bracket, see also Faddeev & Zakharov (1971).
The observation that a bracket like (2.2) could be introduced on \( C^\infty(M) \) for a smooth manifold \( M \), without the use of a symplectic form, leads to the general notion of a Poisson structure (Lichnerowicz 1977).

**Definition 2.2.** A Poisson (or Hamiltonian\(^4\)) structure on a \( C^\infty \) manifold \( M \) is a skew-symmetric bilinear mapping \( (f, g) \mapsto \{f, g\} \) on the space \( C^\infty(M) \), which satisfies the Jacobi identity
\[
\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \tag{2.3}
\]
as well as the Leibnitz identity
\[
\{f, gh\} = \{f, g\}h + g\{f, h\}. \tag{2.4}
\]

When the Poisson structure is induced by a symplectic structure \( \omega \), the Leibnitz identity is a direct consequence of equation (2.2), whereas the Jacobi identity (2.3) corresponds to condition \( \mathrm{d}\omega = 0 \) satisfied by the symplectic form \( \omega \). In the general case, the fact that the mapping \( g \mapsto \{f, g\} \) satisfies equation (2.4) means that it is a derivation of \( C^\infty(M) \).

Each derivation on \( C^\infty(M) \) corresponds to a smooth vector field, i.e. to each \( f \in C^\infty(M) \) is associated a vector field \( X_f : M \to TM \), called the Hamiltonian vector field of \( f \), such that
\[
\{f, g\} = X_f \cdot g = L_{X_f} g, \tag{2.5}
\]
where \( L_{X_f} g \) is the Lie derivative of \( g \) along \( X_f \).

Jost (1964) pointed out that, just like a derivation on \( C^\infty(M) \) corresponds to a vector field, a bilinear bracket \( \{f, g\} \) satisfying the Leibnitz rule (2.4) corresponds to a field of bivectors, i.e. there exists a \( C^\infty \) tensor field \( P \in \Gamma(\wedge^2TM) \), called the Poisson bivector of \( (M, \{\cdot, \cdot\}) \), such that
\[
\{f, g\} = P(df, dg), \tag{2.6}
\]
for all \( f, g \in C^\infty(M) \).

**Proposition 2.1.** A bivector field \( P \in \Gamma(\wedge^2TM) \) is the Poisson bivector of a Poisson structure on \( M \) if, and only if, one of the following equivalent conditions holds:

(i) \( [P, P] = 0 \), where \( [\ , \ ] \) is the Schouten–Nijenhuis bracket\(^5\),

(ii) the bracket \( (f, g) = P(df, dg) \) satisfies the Jacobi identity, and

(iii) \( [X_f, X_g] = X_{\{f, g\}} \) for all \( f, g \in C^\infty(M) \).

**Proof.** By the definition of the Schouten–Nijenhuis bracket (Vaisman 1994), we have
\[
-\frac{1}{2}[P, P](df, dg, dh) = \bigcirc P(dQ(df, dg), dh)
= \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}
= X_{\{f, g\}} \cdot h - X_f \cdot X_g \cdot h + X_g \cdot X_f \cdot h,
\]
for all \( f, g, h \in C^\infty(M) \), where \( \bigcirc \) indicates the sum over circular permutations of \( f, g, h \). Hence, all these expressions vanish together. \( \blacksquare \)

\(^4\)The expression ‘Hamiltonian manifold’ is often used for the generalization of Poisson structure in the case of infinite-dimension manifolds.

\(^5\)The Schouten–Nijenhuis bracket is an extension of the Lie bracket of vector fields to skew-symmetric multivector fields (see Vaisman 1994).
Remark 2.1. The notion of a Poisson manifold is more general than that of a symplectic manifold. Symplectic structures correspond to non-degenerate Poisson structure. In this case, the Poisson bracket satisfies the additional property that $\{f, g\} = 0$ for all $g \in C^\infty(M)$ only if $f \in C^\infty(M)$ is a constant, whereas for Poisson manifolds such non-constant functions $f$ might exist, in which case they are called Casimir functions. Such functions are constants of motion for all vector fields $X_g$, where $g \in C^\infty(M)$.

On a Poisson manifold $(M, P)$, a vector field $X: M \to TM$ is said to be Hamiltonian if there exists a function $f$ such that $X = X_f$. On a symplectic manifold $(M, \omega)$, the necessary condition for a vector field $X$ to be Hamiltonian is that

$$L_X \omega = 0.$$ 

A similar criterion exists for a Poisson manifold $(M, P)$ (Vaisman 1994). The necessary condition for a vector field $X$ to be Hamiltonian is

$$L_X P = 0.$$ 

(b) Integrability

An integrable system on a symplectic manifold $M$ of dimension $2n$ is a set of $n$ functionally independent $f_1, \ldots, f_n$ which are in involution, such that $\forall j, k \{f_j, f_k\} = 0$.

A Hamiltonian vector field $X_H$ is said to be (completely) integrable if the Hamiltonian function $H$ belongs to an integrable system. In other words, $X_H$ is integrable if there exists $n$ first integrals $X_{f_1} = H, f_2, \ldots, f_n$ which commute together.

Remark. 2.2. At any point $x$ where the functions $f_1, \ldots, f_n$ are functionally independent, the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ generate a maximal isotropic subspace $L_x$ of $T_x M$. When $x$ varies, the subspaces generate what one calls a Lagrangian distribution, i.e. a sub-bundle $L$ of $TM$ whose fibres are maximal isotropic subspaces. In our case, this distribution is integrable (in the sense of Frobenius). The leaves of $L$ are defined by the equations $f_1 = \text{constant}, \ldots, f_n = \text{constant}$.

A Lagrangian distribution which is integrable (in the sense of Frobenius) is called a real polarization and is a key notion in Geometric Quantization.

In the study of dynamical systems, the importance of integrable Hamiltonian vector fields is emphasized by the Arnold–Liouville theorem (Arnold 1997), which asserts that each compact leaf is actually diffeomorphic to an $n$-dimensional torus

$$T^n = \{(\varphi^1, \ldots, \varphi^n), \quad \varphi^k \in \mathbb{R}/2\pi\mathbb{Z}\},$$

on which the flow of $X_H$ defines a linear quasi-periodic motion, i.e. in angular coordinates $\varphi^1, \ldots, \varphi^n$

$$\frac{d\varphi^k}{dt} = \omega^k, \quad k = 0, \ldots, n,$$

where $(\omega^1, \ldots, \omega^n)$ is a constant vector.

6 This means that the corresponding Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are independent on an open dense subset of $M$.

7 A first integral is a function which is constant on the trajectories of the vector field.
Remark 2.3. In the case of a Poisson manifold, it can be confusing to define an integrable system. However, we can use the symplectic definition on each symplectic leaf of the Poisson manifold.

(c) Bi-Hamiltonian manifolds

Two Poisson brackets \{ , \}_P and \{ , \}_Q are compatible if any linear combination

\[ \{f, g\}_\lambda,\mu = \lambda\{f, g\}_P + \mu\{f, g\}_Q, \quad \lambda, \mu \in \mathbb{R}, \]

is also a Poisson bracket. A bi-Hamiltonian manifold \((M, P, Q)\) is a manifold equipped with two Poisson structures \(P\) and \(Q\) which are compatible.

Proposition 2.2. Let \(P\) and \(Q\) be two Poisson structures on \(M\). Then, \(P\) and \(Q\) are compatible if, and only if, one of the following equivalent conditions holds:

(i) \([P, Q] = 0\), where \([, , ]\) is the Schouten–Nijenhuis bracket,
(ii) \(\bigcirc \{\{g, h\}_P, f\}_Q + \{\{g, h\}_Q, f\}_P = 0\), where \(\bigcirc\) is the sum over circular permutations of \(f, g, h\), and
(iii) \([X^P_f, X^Q_g] + [X^Q_f, X^P_g] = X^P_{\{f,g\}_Q} + X^Q_{\{f,g\}_P}\) for all \(f, g \in C^\infty(M)\).

Proof. By definition of the Schouten–Nijenhuis bracket (Vaisman 1994), we have

\[ -[P, Q](df, dg, dh) = \bigcirc P(dQ(df, dg, dh)) + Q(dP(df, dg, dh)) \]
\[ = \bigcirc \{\{g, h\}_P, f\}_Q + \{\{g, h\}_Q, f\}_P \]
\[ = -[X^P_f, X^Q_g] \cdot h - [X^Q_f, X^P_g] \cdot h + X^P_{\{f,g\}_Q} \cdot h + X^Q_{\{f,g\}_P} \cdot h, \]

for all \(f, g, h \in C^\infty(M)\). Hence, all these expressions vanish together. \(\blacksquare\)

(d) Lenard recursion relations

On a bi-Hamiltonian manifold \(M\), equipped with two compatible Poisson structures \(P\) and \(Q\), we say that a vector field \(X\) is (formally) integrable\(^8\) or bi-Hamiltonian if it is Hamiltonian for both structures. The reason for this terminology is that for such a vector field, there exists under certain conditions a hierarchy of first integrals in involution that may lead in certain cases to complete integrability, in the sense of Liouville. A useful concept for obtaining such a hierarchy of first integrals is the so-called Lenard scheme (McKean 1993).

Definition 2.3. On a manifold \(M\) equipped with two Poisson structures \(P\) and \(Q\), we say that a sequence \((H_k)_{k \in \mathbb{N}^*}\) of smooth functions satisfies the Lenard recursion relation if

\[ PdH_k = QdH_{k+1}, \quad (2.7) \]

for all \(k \in \mathbb{N}^*\).

Proposition 2.3. Let \(P\) and \(Q\) be Poisson structures on a manifold \(M\) and let \((H_k)_{k \in \mathbb{N}^*}\) be a sequence of smooth functions on \(M\) that satisfies the Lenard recursion relation. Then the functions, \(H_k\), are pairwise in involution with respect to both the brackets \(P\) and \(Q\).

\(^8\)This terminology is used for the evolution equations in infinite dimension.
Proof. Using skew symmetry of $P$ and $Q$ and relation (2.7), we get
\[ P(dH_k, dH_{k+p}) = Q(dH_{k+1}, dH_{k+p}) = P(dH_{k+1}, dH_{k+p-1}), \]
for all $k, p \in \mathbb{N}^*$. From this we deduce, by induction on $p$, that
\[ \{H_k, H_{k+p}\}_P = 0, \]
for all $k, p \in \mathbb{N}^*$. It is then an immediate consequence that
\[ \{H_k, H_l\}_Q = 0, \]
for all $k, l \in \mathbb{N}^*$.

Remark 2.4. Note that in the proof of proposition 2.2, the compatibility of $P$ and $Q$ is not needed.

Suppose now that $(M, P, Q)$ is a bi-Hamiltonian manifold and that at least one of the two Poisson brackets, say $Q$, is invertible. In this case, we can define a $(1, 1)$-tensor field
\[ R = PQ^{-1}, \]
which is called the recursion operator of the bi-Hamiltonian structure. Kosmann-Schwarzbach & Magri (1990, 1996) have shown that, as a consequence of the compatibility of $P$ and $Q$, the Nijenhuis torsion of $R$, defined by
\[ T(R)(X, Y) = [RX, RY] - R([RX, Y] + [X, RY]) + R^2[X, Y] \]
vanishes. In this situation, the family of Hamiltonians
\[ H_k = \frac{1}{k} \text{tr} R^k, \quad (k \in \mathbb{N}^*), \]
satisfies the Lenard recursion relation (2.7). Indeed, this results from the fact that
\[ L_X \text{tr}(T) = \text{tr}(L_X T), \]
for every vector field $X$ and every $(1, 1)$-tensor field $T$ on $M$ and that the vanishing of the Nijenhuis torsion of $R$ can be rewritten as
\[ L_{RX} R = RL_X R, \]
for all vector field $X$.

Remark 2.5. This construction has to be compared with Lax isospectral equation associated to an evolution equation
\[ \frac{du}{dt} = F(u). \quad (2.8) \]
The idea is to associate to equation (2.8), a pair of matrices (or operators in the infinite-dimensional case) $(L, B)$, called a Lax pair, whose coefficients are functions of $u$ and in such a way that when $u(t)$ varies according to equation (2.8), $L(t) = L(u(t))$ varies according to
\[ \frac{dL}{dt} = [L, B]. \]
This equation has been formulated by Lax (1968) in order to obtain a hierarchy of first integrals of the evolution equation as eigenvalues or traces of the operator $L$. This analogy between $R$ and $L$ is not casual and has been studied by Kosmann-Schwarzbach & Magri (1996). Many evolution equations which admit a Lax pair also appear to be bi-Hamiltonian systems generated by a recursion operator $R = PQ^{-1}$. 

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In practice, we may be confronted to the following problem. We start with an evolution equation represented by a vector field \( X \) on a manifold \( M \). We find two compatible Poisson structures \( P \) and \( Q \) on \( M \) that make \( X \) a bi-Hamiltonian vector field; but both \( P \) and \( Q \) are non-invertible. In this case, it is however still possible to find a Lenard hierarchy if the following algorithm works.

**Step 1.** Let \( H_1 \) be the Hamiltonian of \( X \) for the Poisson structure \( P \) and let \( X_1 = X \). The vector field \( X_1 \) is Hamiltonian for the Poisson structure \( Q \) by assumption; this defines Hamiltonian function \( H_2 \). We define \( X_2 \) to be the Hamiltonian vector field generated by \( H_2 \) for the Poisson structure \( P \).

**Step 2.** Inductively, having defined the Hamiltonian function \( H_k \) and letting \( X_k \) be the Hamiltonian vector field generated by \( H_k \) for the Poisson structure \( P \), we check if \( X_k \) is Hamiltonian for the Poisson structure \( Q \). If the answer is yes, then we define \( H_{k+1} \) to be the Hamiltonian of \( X_k \) for the Poisson structure \( Q \).

### 3. Poisson structures on the dual of a Lie algebra

**a) Lie–Poisson structure**

The fundamental example of a non-symplectic Poisson structure is the Lie–Poisson structure on the dual \( g^* \) of a Lie algebra \( g \).

**Definition 3.1.** On the dual space \( g^* \) of a Lie algebra \( g \) of a Lie group \( G \), there is a Poisson structure defined by

\[
\{f, g\}(m) = m([d_m f, d_m g]), \tag{3.1}
\]

for \( m \in g^* \) and \( f, g \in C^\infty(g^*) \), called the canonical Lie–Poisson structure.\(^9\)

**Remark 3.1.** The canonical Lie–Poisson structure has the remarkable property to be linear, that is the bracket of the two linear functionals is itself a linear functional. Given a basis of \( g \), the components\(^10\) of the Poisson bivector \( P \) associated to equation (3.1) are

\[
P_{ij} = c_{ij}^k x_k, \tag{3.2}
\]

where \( c_{ij}^k \) are the structure components of the Lie algebra \( g \).

**b) Modified Lie–Poisson structures**

Under the general name of modified Lie–Poisson structures, we mean an affine\(^11\) perturbation of the canonical Lie–Poisson structure on \( g^* \). In other words, it is represented by a bivector

\[
P + Q,
\]

where \( P \) is the canonical Poisson bivector defined by equation (3.2) and \( Q = (Q_{ij}) \) is a constant bivector on \( g^* \). Such a \( Q \in \wedge^2 g^* \) is itself a Poisson bivector. Indeed, 9Here, \( d_m f \), the differential of a function \( f \in C^\infty(g) \) at \( m \in g^* \), is to be understood as an element of the Lie algebra \( g \).

\(^10\)In what follows, the convention for lower or upper indices may be confusing since we shall deal with tensors on both \( g \) and \( g^* \). Therefore, we emphasize that the convention we use in this paper is the following: upper indices correspond to contravariant tensors on \( g \) and therefore covariant tensors on \( g^* \), whereas lower indices correspond to covariant tensors on \( g \) and therefore contravariant tensors on \( g^* \).

\(^11\)A Poisson structure on a linear space is affine if the bracket of two linear functionals is an affine functional.
the Schouten–Nijenhuis bracket

\[ [Q, Q] = 0, \]

since \( Q \) is a constant tensor field on \( \mathfrak{g}^* \).

The fact that \( P + Q \) is a Poisson bivector, or equivalently that \( Q \) is compatible with the canonical Lie–Poisson structure, is expressed using proposition 2.2 by the condition

\[ Q([u, v], w) + Q([v, w], u) + Q([w, u], v) = 0, \]

for all \( u, v, w \in \mathfrak{g} \).

(c) Lie algebra cohomology

In this section, we deal with left-invariant forms but, of course, everything we say may be applied equally to right-invariant forms up to a sign in the definition of the coboundary operator. On a Lie group \( G \), a left-invariant \( p \)-form \( \omega \) is completely defined by its value at the unit element \( e \), and hence by an element of \( \bigwedge^p \mathfrak{g}^* \). In other words, there is a natural isomorphism between the space of left-invariant \( p \)-forms on \( G \) and \( \bigwedge^p \mathfrak{g}^* \). Moreover, since the exterior differential \( \mathrm{d} \) commutes with left translations, it induces a linear operator \( \partial: \bigwedge^p \mathfrak{g}^* \to \bigwedge^{p+1} \mathfrak{g}^* \) defined by

\[ \partial \gamma(u_0, \ldots, u_p) = \sum_{i<j} (-1)^{i+j} \gamma([u_i, u_j], u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_p), \]

where the hat means that the corresponding element should not appear in the list. \( \gamma \) is said to be a cocycle if \( \partial \gamma = 0 \). It is a coboundary if it is of the form \( \gamma = \partial \mu \) for some cochain \( \mu \) in the dimension \( p-1 \). Every coboundary is a cocycle, i.e. \( \partial \circ \partial = 0 \).

Example 3.1. For every \( \gamma \in \bigwedge^0 \mathfrak{g}^* = \mathbb{R} \), we have \( \partial \gamma = 0 \). For \( \gamma \in \bigwedge^1 \mathfrak{g}^* = \mathfrak{g}^* \), we have

\[ \partial \gamma(u, v) = -\gamma([u, v]), \]

where \( u, v \in \mathfrak{g} \). For \( \gamma \in \bigwedge^2 \mathfrak{g}^* \), we have

\[ \partial \gamma(u, v, w) = -\gamma([u, v], w) - \gamma([v, w], u) - \gamma([w, u], v), \]

where \( u, v, w \in \mathfrak{g} \).

The kernel \( Z^p_\mathfrak{g}(\partial) \) of \( \partial: \bigwedge^p (\mathfrak{g}^*) \to \bigwedge^{p+1}(\mathfrak{g}^*) \) is the space of \( p \)-cocycles and the range \( B^p_\mathfrak{g}(\partial) \) of \( \partial: \bigwedge^{p-1}(\mathfrak{g}^*) \to \bigwedge^p(\mathfrak{g}^*) \) is the spaces of \( p \)-coboundaries. The quotient space \( H^p_{CE}(\mathfrak{g}) = Z^p_\mathfrak{g}(\partial)/B^p_\mathfrak{g}(\partial) \) is the \( p \)th Lie algebra cohomology or Chevalley–Eilenberg cohomology group of \( \mathfrak{g} \). Note that in general the Lie algebra cohomology is different from the de Rham cohomology \( H^1_{DR}(\mathbb{R}) = \mathbb{R} \), but \( H^1_{CE}(\mathbb{R}) = 0 \).

Remark 3.2. Each 2-cocycle \( \gamma \) defines a modified Lie–Poisson structure on \( \mathfrak{g}^* \). The compatibility condition (3.3) can be recast as \( \partial \gamma = 0 \). Note that the Hamiltonian vector field \( X_f \) of a function \( f \in C^\infty (\mathfrak{g}^*) \) computed with respect to the Poisson structure defined by the 2-cocycle \( \gamma \) is

\[ X_f(m) = \gamma(\mathrm{d}_m f, \cdot). \]

Example 3.2. A special case of modified Lie–Poisson structure is given by a 2-cocycle \( \gamma \) which is a coboundary. If \( \gamma = \partial m_0 \) for some \( m_0 \in \mathfrak{g}^* \), the expression

\[ \{f, g\}_0(m) = m_0([\mathrm{d}_m f, \mathrm{d}_m g]), \]

looks like if the Lie–Poisson bracket had been ‘frozen’ at a point \( m_0 \in \mathfrak{g}^* \) and for this reason some authors call it a freezing structure.
4. Bi-Hamiltonian vector fields on $\text{Vect}^*(S^1)$

(a) The Lie algebra $\text{Vect}(S^1)$

The group $\mathfrak{D}$ of smooth orientation-preserving diffeomorphisms of the circle $S^1$ is endowed with a smooth manifold structure based on the Fréchet space $C^\infty(S^1)$. The composition and the inverse are both smooth maps $\mathfrak{D} \times \mathfrak{D} \to \mathfrak{D}$, respectively, so that $\mathfrak{D}$ is a Lie group (Milnor 1984). Its Lie algebra $\mathfrak{g}$ is the space $\text{Vect}(S^1)$ of smooth vector fields on $S^1$, which is itself isomorphic to the space $C^\infty(S^1)$ of periodic functions. The Lie bracket on $\mathfrak{g}$ is given by

$$[u, v] = u_x v - uv_x.$$

Lemma 4.1. The Lie algebra $\text{Vect}(S^1)$ is equal to its commutator algebra, i.e.

$$[\text{Vect}(S^1), \text{Vect}(S^1)] = \text{Vect}(S^1).$$

Proof. Any real periodic function $u$ can be written uniquely as the sum

$$u = w + c,$$

where $w$ is a periodic function of total integral zero and $c$ is a constant. To be of total integral zero is the necessary and sufficient condition for a periodic function $w$ to have a periodic primitive $W$. Hence, we have $[W, 1] = w$. Moreover, since $[\sin, \cos] = 1$, we have proved that every periodic function $u$ can be written as the sum of two commutators.

(b) The regular dual $\text{Vect}^*(S^1)$

Since the topological dual of the Fréchet space $\text{Vect}(S^1)$ is too big and not tractable for our purpose, being isomorphic to the space of distributions on the circle, we restrict our attention in the following to the regular dual $\mathfrak{g}^*$, the subspace of $\text{Vect}(S^1)^*$ defined by linear functionals of the form

$$u \mapsto \int_{S^1} m u \, dx,$$

for some function $m \in C^\infty(S^1)$. The regular dual $\mathfrak{g}^*$ is therefore isomorphic to $C^\infty(S^1)$ by means of the $L^2$ inner product

$$\langle u, v \rangle = \int_{S^1} u v \, dx.$$

With these definitions, the coadjoint action of the Lie algebra $\text{Vect}(S^1)$ on the regular dual $\text{Vect}^*(S^1)$ is given by

$$\text{ad}_u^* m = -(mu_x + (mu)_x) = -(2mu_x + m_x u).$$

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12 This corresponds to the opposite of the usual Lie bracket of vector fields.

13 In the sequel, we use the notation $u, v, \ldots$ for elements of $\mathfrak{g}$ and $m, n, \ldots$ for elements of $\mathfrak{g}^*$ to distinguish them, although they all belong to $C^\infty(S^1)$.

14 The coadjoint action of a Lie algebra $\mathfrak{g}$ on its dual is defined as

$$(\text{ad}_u^* m, v) = -(m, \text{ad}_u v) = -(m, [u, v]),$$

where $u, v \in \mathfrak{g}$, $m \in \mathfrak{g}^*$, and the pairing is the standard one between $\mathfrak{g}$ and $\mathfrak{g}^*$. 

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Let $F$ be a smooth real-valued function on $C^\infty(S^1)$. Its Fréchet derivative $dF(m)$ is a linear functional on $C^\infty(S^1)$. We say that $F$ is a regular function if there exists a smooth map $\delta F:C^\infty(S^1)\to C^\infty(S^1)$ such that

$$dF(m)M = \int_{S^1} M \cdot \delta F(m) dx, \quad m, M \in C^\infty(S^1).$$

That is, the Fréchet derivative $dF(m)$ belongs to the regular dual $q^*$ and the mapping $m\mapsto \delta F(m)$ is smooth. The map $\delta F$ is a vector field on $C^\infty(S^1)$, called the gradient of $F$ for the $L^2$ metric. In other words, a regular function is a smooth function on $C^\infty(S^1)$, which has a smooth $L^2$ gradient.

**Example 4.1.** Typical examples of regular functions on the space $C^\infty(S^1)$ are linear functionals

$$F(m) = \int_{S^1} um dx,$$

where $u \in C^\infty(S^1)$. In this case, $\delta F(m) = u$. Other examples are nonlinear polynomial functionals

$$F(m) = \int_{S^1} Q(m) dx,$$

where $Q$ is a polynomial in derivatives of $m$ up to a certain order $r$. In this case, the gradient of $F$ is just the Eulerian derivative

$$\delta F(m) = \sum_{k=0}^r (-1)^k \frac{d^k}{dx^k} \left( \frac{\partial Q}{\partial X_k}(m) \right).$$

Note that the smooth function $F_\theta:C^\infty(S^1)\to \mathbb{R}$ defined by $F_\theta(m) = m(\theta)$ for some fixed $\theta \in S^1$ is not regular since $dF_\theta$ is the Dirac measure at $\theta$.

A smooth vector field $X$ on $q^*$ is called a gradient if there exists a regular function $F$ on $q^*$ such that $X(m) = \delta F(m)$ for all $m \in q^*$. Observe that if $F$ is a smooth real-valued function on $C^\infty(S^1)$, then its second Fréchet derivative is symmetric (Hamilton 1982), i.e.

$$d^2F(m)(M,N) = d^2F(m)(N,M), \quad m, M, N \in C^\infty(S^1).$$

For a regular function, this property can be rewritten as

$$\int_{S^1} (\delta F'(m)M) N dx = \int_{S^1} (\delta F'(m)N) M dx,$$

for all $m, M, N \in C^\infty(S^1)$, i.e. the linear operator $\delta F'(m)$ is symmetric for the $L^2$-inner product on $C^\infty(S^1)$ for each $m \in C^\infty(S^1)$. Conversely, a smooth vector field $X$ on $q^*$, whose Fréchet derivative $X'(m)$ is a symmetric linear operator, is the gradient of the function

$$F(m) = \int_0^1 <X(tm), m> dt.$$  \hfill (4.2)

This can be checked directly, using the symmetry of $X'(m)$ and an integration by parts. We will resume this fact in lemma 4.2.
Lemma 4.2. On the Fréchet space $C^\infty(S^1)$ equipped with the (weak) $L^2$ inner product, a necessary and sufficient condition for a smooth vector field $X$ to be a gradient is that its Fréchet derivative $X'(m)$ is a symmetric linear operator.

(c) Hamiltonian structures on $\text{Vect}^*(S^1)$

To define a Poisson bracket on the space of regular functions on $g^*$, we consider a one-parameter family of linear operators $P_m$ ($m \in C^\infty(S^1)$) and set

$$\{F, G\}(m) = \int_{S^1} \delta F(m) P_m \delta G(m) \, dx. \quad (4.3)$$

The operators $P_m$ must satisfy certain conditions in order for equation (4.3) to be a valid Poisson structure on the regular dual $g^*$.

Definition 4.1. A family of linear operators $P_m$ on $g^*$ defines a Poisson structure on $g^*$ if equation (4.3) satisfies the following conditions:

(i) $\{F, G\}$ is regular if $F$ and $G$ are regular,
(ii) $\{G, F\} = -\{F, G\}$, and
(iii) $\{\{F, G\}, h\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$.

Note that the second condition simply means that $P_m$ is a skew-symmetric operator for each $m$.

Example 4.2. The canonical Lie–Poisson structure on $g^*$ given by

$$\{F, G\}(m) = m([\delta F, \delta G]) = -\int_{S^1} \delta F(m)(mD + Dm)\delta G(m) \, dx,$$

is represented by the one-parameter family of skew-symmetric operators

$$P_m = -(mD + Dm), \quad (4.4)$$

where $D=\partial_x$. It can be checked that all the three required properties are satisfied. In particular, we have

$$\delta\{F, G\} = \delta F'(P_m \delta G) - \delta G'(P_m \delta F) - \delta FD\delta G + \delta GD\delta F.$$

Definition 4.2. The Hamiltonian of a regular function $F$ for a Poisson structure defined by $P$ is defined as the vector field

$$X_F(m) = P\delta F(m).$$

Proposition 4.1. A necessary condition for a smooth vector field $X$ on $g^*$ to be Hamiltonian with respect to the Poisson structure defined by a constant linear operator $Q$ is the symmetry of the operator $X'(m)Q$ for each $m \in g^*$.

Proof. If $X$ is Hamiltonian, we can find a regular function $F$ such that

$$X(m) = Q\delta F(m).$$

Moreover, since $Q$ is a constant linear operator, we have

$$X'(m) = Q\delta F'(m).$$
and therefore, we get

\[ X'(m)Q = Q\delta F'(m)Q, \]

which is a symmetric operator since \( Q \) is skew symmetric and \( \delta F'(m) \) is symmetric. □

(d) Hamiltonian vector fields generated by right-invariant metrics

A right-invariant metric on the diffeomorphism group \( \text{Diff}(S^1) \) is uniquely defined by its restriction to the tangent space to the group at the unity, hence by a non-degenerate continuous inner product \( a \) on \( \text{Vect}(S^1) \). If this inner product \( a \) is local, then according to Peetre (1959), there exists a linear differential operator

\[ A = \sum_{j=0}^{N} a_j \frac{d^j}{dx^j}, \tag{4.5} \]

where \( a_j \in C^\infty(S^1) \) for \( j=0, \ldots, N \), such that

\[ a(u, v) = \int_{S^1} A(u)v dx = \int_{S^1} A(v)u dx, \]

for all \( u, v \in \text{Vect}(S^1) \). The condition for \( a \) to be non-degenerate is equivalent for \( A \) to be a continuous linear isomorphism of \( C^\infty(S^1) \).

Remark 4.1. In the special case where \( A \) has constant coefficients, the symmetry is traduced by the fact that \( A \) contains only even derivatives and the non-degeneracy by the fact that the symbol of \( A \),

\[ s_A(\xi) = e^{i\xi} A(e^{-i\xi}) = \sum_{j=0}^{N} a_{2j}(-i\xi)^{2j}, \]

has no root in \( \mathbb{Z} \).

The right-invariant metric on \( \text{Diff}(S^1) \) induced by a continuous linear invertible operator \( A \) gives rise to an Euler equation\(^{15} \) on \( \text{Vect}(S^1)^* \)

\[ \frac{dm}{dt} = -z(2mu_x + m_x u), \tag{4.6} \]

where \( m = Au \). This equation is Hamiltonian with respect to the Lie–Poisson structure on \( \text{Vect}(S^1)^* \) with Hamiltonian function on \( \text{Vect}(S^1)^* \) given by

\[ H_2(m) = \frac{1}{2} \int_{S^1} m u dx. \]

\(^{15}\) The second order geodesic equation corresponding to a one-sided invariant metric on a Lie group can always be reduced to a first-order quadratic equation on the dual of the Lie algebra of the group: the Euler equation (see Arnold & Khesin (1998) or Kolev (2004)). The generality of this reduction was first revealed by Poincaré (1901) and applied to hydrodynamics by Arnold (1966).
The corresponding Hamiltonian vector field $X_A$ is given by

$$X_A(m) = -(mD + Dm)(A^{-1}m) = -(2mu_x + m_x u).$$

**Remark 4.2.** The family of operators

$$A_k = 1 - \frac{d^2}{dx^2} + \cdots + (-1)^k \frac{d^{2k}}{dx^{2k}},$$

corresponding respectively to the Sobolev $H^k$ inner product, have been studied by Constantin & Kolev (2002, 2003). The Riemannian exponential map of the corresponding geodesic flow has been shown to be a local diffeomorphism, except for $k=0$. This latter case corresponds to the $L^2$ metric on $\text{Diff}(S^1)$ and happens to be singular.

**Remark 4.3.** A non-invertible inertia operator $A$ may induce in some cases a weak Riemannian metric on a homogenous space. This is the way to interpret Hunter–Saxton and Harry Dym equations as Euler equations (see Khesin & Misiolek 2003).

Theorem 4.1 is a generalization of theorem 3.7 in Constantin & Kolev (2006).

**Theorem 4.1.** The only continuous linear invertible operators

$$A : \text{Vect}(S^1) \to \text{Vect}(S^1)^*,$$

with constant coefficients, whose corresponding Euler vector field $X_A$ is bi-Hamiltonian relative to some modified Lie–Poisson structure, are

$$A = aI + bD^2,$$

where $a, b \in \mathbb{R}$ satisfy $a - bn^2 \neq 0, \forall n \in \mathbb{Z}$. The second Hamiltonian structure is induced by the operator

$$Q = -DA = -aD - bD^3,$$

where $D=d/dx$ and the Hamiltonian function is

$$H_3(m) = \frac{1}{2} \int_{S^1} (au^3 - bu(u_x)^2) dx,$$

where $m = Au$.

**Remark 4.4.** We insist on the fact that the proof we give applies for an operator with constant coefficients. It would be interesting to study the case of a continuous linear invertible operator whose coefficients are not constant. Is there such an operator $A$ with bi-Hamiltonian Euler vector field $X_A$ relative to some modified Lie–Poisson structure? In this case, for which modified Lie–Poisson structures $Q$ is there an Euler vector field $X_A$ which is bi-Hamiltonian relatively to $Q$?

**Proof.** The proof is essentially the same as the one given by Constantin & Kolev (2006). A direct computation shows that

$$X_A(m) = -(aD + bD^3)\delta H_3(m),$$
where

\[ H_3(m) = \frac{1}{2} \int_{S^1} (au^3 - bu(u_x)^2) \, dx, \]

and

\[ A = aI + bD^2, \]

where \( a, b \in \mathbb{R} \).

Each modified Lie–Poisson structure on \( \text{Vect}^*(S^1) \) is given by a local 2-cocycle of \( \text{Vect}(S^1) \). According to proposition A.2 (see appendix A), such a cocycle is represented by a differential operator

\[ Q = -(m_0 D + Dm_0) - \beta D^3, \quad (4.7) \]

where \( m_0 \in C^\infty(S^1) \) and \( \beta \in \mathbb{R} \). We will now show that there is no such cocycle for which \( X_A \) is Hamiltonian if the order of

\[ A = \sum_{j=0}^{N} a_{2j} D^{2j}, \]

is strictly greater than 2.

By virtue of proposition 4.1, a necessary condition for \( X_A \) to be Hamiltonian with respect to the cocycle represented by \( Q \) is that

\[ K(m) = X_A'(m)Q, \]

is a symmetric operator. We have

\[ X_A'(m) = -(2u_x I + uD + 2mA^{-1} + m_x A^{-1}), \]

and in particular, for \( m=1 \),

\[ X_A'(1) = -D - 2DA^{-1}. \]

Hence,

\[ K(1) = (D + 2DA^{-1}) \circ (m_0 D + Dm_0) + \beta D^4(1 + 2A^{-1}), \]

whereas

\[ K(1)^* = (m_0 D + Dm_0) \circ (D + 2DA^{-1}) + \beta D^4(1 + 2A^{-1}). \]

Therefore, letting \( m_0' = (m_0 / dx) \), we get

\[ K(1) - K(1)^* = (m_0' D + Dm_0') + 2(A^{-1} Dm_0 D - Dm_0 DA^{-1}) + 2(A^{-1} D^2 m_0 - m_0 D^2 A^{-1}), \]

and this operator vanishes if and only if

\[ A(K(1) - K(1)^*)A = 0. \quad (4.8) \]
But $A(K(1) - K(1)^*)A$ is the sum of two linear differential operators,

$$2(Dm_0DA - ADm_0D) + 2(D^2m_0A - Am_0D^2),$$

which is of the order of less than $2N+2$, and

$$A(m_0'D + Dm_0')A,$$

which is of the order of $4N+1$ unless $m_0'=0$, which must be the case if equation (4.8) holds. Therefore, $m_0$ has to be a constant. Let $\alpha = 2m_0 \in \mathbb{R}$. Then

$$K(m) = \alpha \{ 2u_x D + u D^2 + 2m D^2 A^{-1} + m_x DA^{-1} \}
+ \beta \{ 2u_x D^3 + u D^4 + 2m D^4 A^{-1} + m_x D^3 A^{-1} \},$$

because $D$ and $A$ commute. The symmetry of the operator $K(m)$ means

$$\int_{S^1} NK(m) M dx = \int_{S^1} MK(m) N dx,$$

(4.9)

for all $m, M, N \in C^\infty(S^1)$. Since this last expression is trilinear in the variables $m$, $M$ and $N$, the equality can be checked for complex periodic functions $m$, $M$ and $N$. Let $m = Au$, $u = e^{-ipx}$, $M = e^{-iqx}$ and $N = e^{-irx}$ with $p, q, r \in \mathbb{Z}$. We have

$$\int_{S^1} NK(m) M dx = [(2pq^3 + q^4)\beta - (2pq + q^2)\alpha] + \int_{S^1} \frac{s_A(p)}{s_A(q)} \int_{S^1} e^{-i(p+q+r)x} dx,$$

whereas

$$\int_{S^1} MK(m) N dx = [(2pr^3 + r^4)\beta - (2pr + r^2)\alpha] + \int_{S^1} \frac{s_A(p)}{s_A(r)} \int_{S^1} e^{-i(p+q+r)x} dx.$$

Now, we set $p = n$, $q = -2n$, $r = n$ and must have

$$(24n^4\beta - 6n^2\alpha)s_A(n) = (6n^4\beta - 6n^2\alpha)s_A(2n),$$

(4.10)

if $K(m)$ is symmetric.

If $\beta \neq 0$, the leading term in the left-hand side of equation (4.10) is $24(-1)^Na_{2N}\beta n^{2N+4}$, whereas the leading term of the right-hand side is $6(-1)^Na_{2N}\beta n^{2N+4}$. Hence, unless $N=1$, we must have $\beta = 0$.

On the other hand, if $\beta = 0$, we must have $\alpha s_A(n) = \alpha s_A(2n)$ for all $n \in \mathbb{N}^*$. Thus, $\alpha = 0$ unless $N=0$. This completes the proof.

(e) **Hierarchy of first integrals**

In view of theorem 4.1, the next step is to find a hierarchy of first integrals in involution for the vector field $X_A$, where

$$A = aI + bD^2,$$

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and $a, b \in \mathbb{R}$ satisfy $a - bn^2 \neq 0, \forall n \in \mathbb{Z}$. The vector field

$$X_A(m) = -2mu - um_x,$$

is bi-Hamiltonian. It can be written as

$$X_A(m) = P_m \delta H_2(m),$$

where

$$H_2(m) = \frac{1}{2} \int_{S^1} um \, dx,$$

and $P_m = -(mD + Dm)$, or as

$$X_A(m) = Q \delta H_3(m),$$

where

$$H_3(m) = \frac{1}{3} \int_{S^1} u(um + q(u)) \, dx,$$

$$q(u) = 1/2(au^2 + bu^2) \quad \text{and} \quad Q = -DA = -aD - bD^3.$$

The problem we get when we try to apply the Lenard scheme to obtain a hierarchy of conserved integrals is that both Poisson operators $P_m$ and $Q$ are non-invertible. However, $Q$ is composed of two commuting operators, $A$ which is invertible and $D$ which is not. The image of $D$ is the codimension 1 subspace, $C^\infty_0(S^1)$, of smooth periodic functions with zero integral. The restriction of $D$ to this subspace is invertible with inverse $D^{-1}$, the linear operator which associates to a smooth function with zero integral its unique primitive with zero integral. Following Lax (1976), we are able to prove the following result.

**Theorem 4.2.** There exists a sequence $(H_k)_{k \in \mathbb{N}}$ of functionals, whose gradients $G_k$ are polynomial expressions of $u = A^{-1}m$ and its derivatives, which satisfies the Lenard recursion scheme

$$P_m G_k = Q G_{k+1}.$$  

**Remark 4.5.** It is worth noting that contrary to the result given by Lax (1976), for the KdV equation, the operators $G_k$ are polynomials in $u = A^{-1}m$ and not in $m$. In particular, there are non-local operators$^{16}$, if $A \neq aI$, for some $a \in \mathbb{R}$.

Before giving a sketch of proof of this theorem, let us illustrate the explicit computation of the first Hamiltonians of the hierarchy. We start with

$$H_1(m) = \int_{S^1} m \, dx, \quad G_1(m) = 1.$$

We define $X_1$ to be the Hamiltonian vector field of $H_1$ for the Lie–Poisson structure $P_m$,

$$X_1(m) = P_m G_1(m) = -m_x.$$

$X_1(m)$ is in the image of $D$ for all $m$ and we can define

$$G_2(m) = Q^{-1}X_1(m) = A^{-1}D^{-1}(m_x) = A^{-1}(m) = u,$$

$^{16}$Note that our $m$ corresponds to $u$ in the notations of Lax (1976).
which is the gradient of the second Hamiltonian of the hierarchy

\[ H_2(m) = \frac{1}{2} \int_{s^{1}} m u \, dx. \]

We then compute \( X_2 \), the Hamiltonian vector field of \( H_2 \) for \( P_m \),

\[ X_2(m) = P_m G_2(m) = -2m u_x - m_x u = -(mu + q(u))_x, \]

where \( q(u) = 1/2(au^2 + bu_x^2) \). \( X_2(m) \) is in the image of \( D \) for all \( m \) and we can define

\[ G_3(m) = Q^{-1} X_2(m) = A^{-1}(mu + q(u)), \]

which is the gradient of the third Hamiltonian of the hierarchy

\[ H_3(m) = \frac{1}{3} \int_{s^{1}} u(mu + q(u)) \, dx. \]

So far, we obtain in this way a hierarchy of Hamiltonians \( (H_k)_{k \in \mathbb{N}} \) satisfying

the Lenard recursion relations for the Euler equation associated to the operator \( A \).

**Example 4.3.** (Burgers Hierarchy). For \( A = I \), we obtain explicitly the whole Burgers hierarchy

\[ H_{k+1}(m) = \frac{(2k!)^{2}}{2^{k}(k!)^{2}(k + 1)} \int_{s^{1}} m^{k+1} \, dx, \quad (k \in \mathbb{N}). \]

**Example 4.4.** (Camassa–Holm Hierarchy) For \( A = I - D^2 \), we obtain the Camassa–Holm hierarchy. The first members of the family are

\[ H_1(m) = \int_{s^{1}} m \, dx = \int_{s^{1}} u \, dx, \quad H_2(m) = \frac{1}{2} \int_{s^{1}} m u \, dx = \frac{1}{2} \int_{s^{1}} (u^2 + u_x^2) \, dx, \]

\[ H_3(m) = \frac{1}{2} \int_{s^{1}} u(u^2 + u_x^2) \, dx. \]

The next integrals of the hierarchy are much harder to compute explicitly. One may consider Lenells (2005) and Loubet (2005) for further studies on the subject.

**Sketch of proof of theorem 4.2.** The proof is divided into two steps. We refer to Lax (1976) for the details.

**Step 1.** We show by induction that there exists a sequence of vector fields \( G_k \), which is a polynomial expression of \( u = A^{-1} m \) and its derivatives and satisfies

\[ G_1 = 1, \quad PG_k = QG_{k+1}, \quad \forall k \in \mathbb{N}. \quad (4.11) \]

**Step 2.** We show that \( G_k \) is, for all \( k \), the gradient of a function \( H_k \).

To prove Step 1, we suppose that \( G_1, \ldots, G_n \) have been constructed satisfying equation (4.11) and we use lemmas 4.3 and 4.4 to show that \( G_{n+1} \) exists.

The proof of lemma 4.3 can be found in Olver (1993), while the proof of lemma 4.4 can be found in Lax (1976).
Lemma 4.3. Suppose that $Q$ is a polynomial in derivatives of $u$ up to order $r$ such that
\[ \int_{S^1} Q(u) \, dx = 0, \]
for all $u \in C^\infty(S^1)$. Then, there exists a polynomial $G$ in derivatives of $u$ up to order $r-1$ such that $Q = DG$.

Lemma 4.4. We have
\[ \int_{S^1} PG \, dx = 0, \]
for all $n \in N^*$. To prove Step 2, it is enough to show that $G^k$ is a symmetric operator for all $k$, by virtue of lemma 4.2. We suppose that $G_1, \ldots, G_n$ are gradients and show first the following result.

Lemma 4.5. The operator
\[ QG'_{n+1}(m)Q, \]
is symmetric for all $m \in C^\infty(S^1)$.

We conclude then, like in Lax (1976), that $G_{n+1}^'(m)$ itself is symmetric. We will give here the details of the proof of lemma 4.5, since the proof of the corresponding result for KdV in Lax (1976) is just a direct, hand-waving computation and does not apply in our more general case.

Proof of lemma 4.5. First, we differentiate the recurrence formula (4.11) and we obtain
\[ QG'_{n+1}(m) = ad^*_{G_n} + P_m G'_{n}(m), \tag{4.12} \]
and
\[ QG'_{n}(m) = ad^*_{G_{n-1}} + P_m G'_{n-1}(m). \tag{4.13} \]
Multiplying equation (4.12) by $Q$ on the right, (4.13) by $P$ on the right, and subtracting equation (4.13) from (4.12), we get
\[ QG'_{n+1}(m)Q = QG_{n}(m)P_m + P_m G'_{n}(m)Q + ad^*_{G_n} Q - ad^*_{G_{n-1}} P_m - P_m G'_{n-1}(m)P_m. \]
Using the fact that
\[ (ad^*_{u})^* = -ad_u, \]
we finally get
\[ (QG'_{n+1}(m)Q)^* - QG'_{n+1}(m)Q = Qad_{G_n} - P_m ad_{G_{n-1}} - ad^*_{G_n} Q + ad^*_{G_{n-1}} P_m. \]
Using the fact that $Q$ satisfies the cocycle condition
\[ Q([u, v]) = ad^*_{u} Q(v) - ad^*_{v} Q(u), \]
which can be rewritten as
\[ Qad_u = ad^*_{u} Q - PQ(u), \]
we get
\[ (QG'_{n+1}(m)Q)^* - QG'_{n+1}(m)Q = -PQ(G_n) - P_m ad_{G_{n-1}} + ad^*_{G_{n-1}} P_m. \]

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But this last expression is zero because

\[ P_m \text{ad}_v = \text{ad}_v^* P_m - P_{P_m(v)}, \]

and \( Q(G_n) = P_m G_{n-1}. \)

**Remark 4.6.** In the special case where the cocycle \( \gamma \) is a coboundary, i.e. when the second structure is a freezing structure, the algorithm used to generate a hierarchy of first integrals is known as the translation argument principle (Arnold & Khesin 1998; Khesin & Misiolek 2003). Let \( H_\lambda \) be a function on \( g^* \), which is a Casimir function of the Poisson structure

\[ \{ \cdot, \cdot \}_\lambda = \{ \cdot, \cdot \}_0 + \lambda \{ \cdot, \cdot \}_{LP}. \]

That is, for every function \( F \) one has

\[ \{ H_\lambda, F \}_\lambda = 0. \]

Suppose that \( H_\lambda \) can be expressed as a series

\[ H_\lambda = H_0 + \lambda H_1 + \lambda^2 H_2 + \cdots. \]

Then, one can check that \( H_0 \) is a Casimir function of \( \{ \cdot, \cdot \}_0 \) and that for all \( k \), the Hamiltonian vector field of \( H_{k+1} \) with respect to \( \{ \cdot, \cdot \}_0 \) coincides with the Hamiltonian vector field of \( H_k \) with respect to \( \{ \cdot, \cdot \}_{LP} \). Furthermore, all the Hamiltonians \( H_k \) are in involution with respect to both the Poisson structures and the corresponding Hamiltonian vector fields commute with each other. In practice, to obtain such a Casimir function \( H_\lambda \), one chooses a Casimir function \( H \) of the Poisson structure \( \{ \cdot, \cdot \}_{LP} \) and then translates the argument

\[ H_\lambda(m) = H(m_0 + \lambda m). \]

The above method has been successfully applied to the KdV equation viewed as a Hamiltonian field on the dual of the Virasoro algebra.

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**Appendix A. The Gelfand–Fuks cohomology**

Gel’fand & Fuks (Gel’fand & Fuks 1968; Guieu & Roger 2005) have developed a systematic method to compute the cohomology of the Lie algebra of vector fields on a smooth manifold. This theory is quite sophisticated. The aim of this section is to present a computation of the first two cohomological groups of \( \text{Vect}(S^1) \), using only elementary arguments.

The first difficulty when we deal with infinite-dimensional Lie algebras like \( \text{Vect}(S^1) \) is to define what we call a cochain, since a linear or a multilinear map on \( \text{Vect}(S^1) \) may be too vague as already stated.
**Definition A.1.** A $p$-cochain $\gamma$ on $\text{Vect}(S^1)$ with values in $\mathbb{R}$ is called \textit{local} if it has the expression

$$\gamma(u_1, \ldots, u_p) = \int_{S^1} P(u_1, \ldots, u_p) dx,$$

where $P$ is a $p$-linear differential operator.

It is easy to check that if $\gamma$ is local then $\partial \gamma$ is also local. In the sequel, a cochain on $\text{Vect}(S^1)$ will always mean a \textit{local cochain}\footnote{Using a theorem of Peetre (1959), a local cochain can be characterized by the condition $\bigcap_{i=1}^p \text{Supp}(u_i) = \emptyset \Rightarrow \gamma(u_1 \cdots u_p) = 0$.}. The associated cohomology is called the \textit{Gelfand–Fuks cohomology}.

(a) \textit{The first cohomology group}

A \textit{local} 1-cochain $\gamma$ on $\text{Vect}(S^1)$ has the following expression:

$$\gamma(u) = \int_{S^1} P(u) dx,$$

where $P$ is a linear differential operator. Integrating by parts, we can write it as

$$\gamma(u) = \int_{S^1} mu dx,$$

where $m \in C^\infty(S^1)$ is uniquely defined by $\gamma$.

**Proposition A.1.**

$$H^1_{GF}(\text{Vect}(S^1); \mathbb{R}) = \{0\}.$$

**Proof.** If $\gamma$ is a 1-cocycle, it satisfies the condition

$$\gamma([u, v]) = 0,$$

for all $u, v$ in $\text{Vect}(S^1)$. It a very general result that a Lie algebra which is equal to its commutator algebra has a trivial one-dimensional cohomology group. Indeed, a linear functional which vanishes on commutators vanishes everywhere. The proposition is therefore a corollary of lemma 4.1. \hfill \blacksquare

(b) \textit{The second cohomology group}

A local 2-cochain $\gamma$ on $\text{Vect}(S^1)$ has the expression

$$\gamma(u, v) = \int_{S^1} P(u, v) dx,$$

where $P$ is a quadratic differential operator. Integrating by parts, we can write it as

$$\gamma(u, v) = \int_{S^1} uK(v) dx,$$

where $K:C^\infty(S^1) \rightarrow C^\infty(S^1)$ is a linear differential operator

$$K = \sum_{k=0}^n a_k(x) D^k,$$
which is skew symmetric relatively to the $L^2$-inner product. This operator is uniquely defined by $\gamma$. Moreover, if $\gamma$ is a 2-coboundary, there exists $m \in \mathfrak{g}^*$ such that $\gamma = \partial m$, i.e.

$$ \gamma(u, v) = -\int_{S^1} m[u, v]dx = -\int_{S^1} u(\text{ad}_v^* m)dx, $$

where $u, v \in \mathfrak{g}$. We will therefore introduce the following operator

$$ m(v) = -\text{ad}_v^* m = mv_x + (mv)_x = 2mv_x + m_x v, \quad (A\ 1) $$

to represent the coboundary of the 1-cochain $m \in \mathfrak{g}^*$.

**Proposition A.2.** The cohomology group $H^2_G(F(Vect(S^1); \mathbb{R})$ is one dimensional. It is generated by the Virasoro cocycle

$$ \text{vir}(u, v) = \int_{S^1} (u'v'' - v'u'')dx. $$

**Proof.** Let $\gamma$ be a 2-cocycle and $K$ the corresponding linear differential operator. The cocycle condition $\partial \gamma = 0$ leads to the following condition on $K$

$$ K([u, v]) = \text{ad}_v^* K(v) - \text{ad}_u^* K(u), \quad (A\ 2) $$

for all $u, v \in \mathcal{C}^\infty(S^1)$. Let $w \in \mathcal{C}^\infty(S^1)$ with zero integral and $W \in \mathcal{C}^\infty(S^1)$ a primitive of $w$, then we have $w = [1, W]$ and hence

$$ K(w) = K([W, 1]) = \text{ad}_W^* K(1) - \text{ad}_1^* K(W) = K(W)' - (2a_0 W' + a_0' W) $$

$$ = \left( a_1' w + a_2' w' + \ldots + a_n' w^{(n-1)} \right) + K(w) - 2a_0 w. $$

Therefore, we have

$$ (a_1' - 2a_0) w + a_2' w' + \ldots + a_n' w^{(n-1)} = 0, $$

for all periodic function $w$ with zero integral which leads to $2a_0 = a_1'$ and $a_k =$ constant, for $2 \leq k \leq n$, i.e. any linear differential operator $K$ which satisfies equation $(A\ 2)$ can be written

$$ K = \partial m + \sum_{k=2}^{N} \lambda_k D^k, $$

where $m$ is a smooth periodic function$^{19}$ and the $\lambda_k$ are real numbers. Using again equation $(A\ 2)$, we get for all periodic functions $u, v$,

$$ \sum_{k=2}^{N} \lambda_k (uv' - vu')^{(k)} = 2 \sum_{k=2}^{n} \lambda_k (v^{(k)} u' - u^{(k)} v') + \sum_{k=2}^{n} \lambda_k (v^{(k+1)} u - u^{(k+1)} v), $$

$^{19}$Recall that $\partial m$ is the linear differential operator defined by

$$ \partial m(u) = \text{ad}_u^* m = mu' + (mu)' = 2mu' + m'u.
which can be rewritten using Leibnitz rule as
\[
\sum_{k=2}^{n} \lambda_k \left\{ \sum_{p=1}^{k-1} C_k^p (u^{(p)} v^{(k+1-p)} - v^{(p)} u^{(k+1-p)}) + 3(u^{(k)} v' - v^{(k)} u') \right\} = 0.
\]
If we fix \(v\) and consider this expression as a linear differential equation in \(u\), all the coefficients of that operator must be zero, and in particular for the coefficient of \(u'\), we have
\[
\sum_{k=2}^{N} \lambda_k (k-3) v^{(k)} = 0.
\]
Therefore, we have \(\lambda_k = 0\) for \(k \neq 3\). Since \(D^3\) is easily seen to verify equation (5.2), we can conclude that every cocycle operator \(K\) is of the form
\[
K = \lambda D^3 + \partial m,
\]
for some \(\lambda \in \mathbb{R}\) and \(m\) in \(C^\infty(S^1)\). Since every coboundary operator \(\partial m\) is a linear differential operator of order 1, \(D^3\) represents a non-trivial cohomology class which ends the proof. \(\blacksquare\)

References


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