By tuning a control parameter, a chaotic system can either display two or more attractors (generalized multistability) or exhibit an interior crisis, whereby a chaotic attractor suddenly expands to include the region of an unstable orbit (bursting regime).

Recently, control of multistability and bursting have been experimentally proved in a modulated class B laser by means of a feedback method. In a bistable regime, the method relies on the knowledge of the frequency components of the two attractors. Near an interior crisis, the method requires retrieval of the unstable orbit colliding with the chaotic attractor.

We also show that a suitable parameter modulation is able to control bistability in the Lorenz system. We observe that, for every given modulation frequency, the chaotic attractor is destroyed under a boundary crisis. The threshold control amplitude depends on the control frequency and the location of the operating point in the bistable regime. Beyond the boundary crisis, the system remains in the steady state even if the control is switched off, demonstrating control of bistability.

**Keywords:** generalized multistability; bursting; control of chaos; Lorenz system

### 1. Introduction

A chaotic system can either display two attractors (generalized bistability; Arecchi & Lisi 1982, 1983; Arecchi et al. 1982; Saucedo-Solorio et al. 2003) or exhibit an interior crisis (Grebogi et al. 1982; Dangoisse et al. 1986), whereby a chaotic attractor suddenly expands to include the region of an unstable orbit. In the former case, starting from an assigned initial condition, the system remains on one attractor, and only external perturbations can induce jumps between the two attractors; the
relative weight of the two attractors is given by the size ratio of the two basins of attraction. In the latter case, whatever is the initial condition, the system visits the whole phase space of the two dynamical objects, albeit for different amounts of time. Any initial condition generates the trajectory visiting the whole single attractor; however, a memory of the previously separated regions remains, and the fractional time spent on the formerly unstable orbit results in bursts of anomalous amplitude signal (crisis-induced intermittency; Grebogi et al. 1983).

In the bistable case, we show how a single attractor selection is performed by a feedback method including a frequency filter, based on the different spectral contents of the two attractors (Meucci et al. 2005). We demonstrate that the selection method is still effective beyond the crisis. In this latter case, the selection eliminates the anomalous bursts, thus regularizing the output signal. The method leaves unperturbed the chaotic features of the selected attractor; thus, it should not be confused with chaos control (Ott et al. 1990), which stabilizes one of the unstable periodic orbits within a single attractor.

2. Intermittent behaviour in a CO\textsubscript{2} laser

We apply these general considerations to a periodically driven CO\textsubscript{2} laser (Arecchi et al. 1982). The attractor selection is realized by adding to the loss
modulator of the CO\textsubscript{2} laser a feedback loop including a frequency filter, which modifies the driving signal, quenching or enhancing some specific frequency components depending on the feedback sign. The laboratory set-up is shown in figure 1.

It consists of a single-mode CO\textsubscript{2} laser with an intracavity electro-optic loss modulator (EOM). The cavity length is \(L=1.42\) m and the total transmission \(T\) is approximately 0.10 for a single pass. The intensity decay rate can be expressed as

\[
k(t) = k(1 + \alpha \sin^2(B_0 + F_{\text{mod}}(t))),
\]

where \(k = cT/L\), \(c\) being the speed of light in the vacuum, \(\alpha = (1 - 2T)/2T\); \(B_0\) is a bias voltage; and \(F_{\text{mod}}(t)\) is the modulation applied to the EOM.

We consider a sinusoidal modulation \(F_{\text{mod}}(t) = A \sin(2\pi ft)\), where \(f=100\) kHz is about twice the relaxation frequency of the laser (Arecchi \textit{et al.} 1982). The modulation signal is provided by a waveform generator (WG). The laser is pumped by a DC discharge current stabilized at 8.00 mA, while the threshold current is 6.50 mA. The notch filter (NF; figure 1b) has a notch frequency given by \(f_{\text{notch}} = 1/(2\pi\sqrt{L_2C_2})\); its input is the laser intensity and its output is amplified by \(a\).

Increasing the modulation amplitude \(A\) (see figure 2a), the system undergoes a sequence of subharmonic bifurcations leading to a small-amplitude chaotic attractor. Further increase of \(A\) (above 1.2 V) induces the occurrence of a regime
in which bursts of high-amplitude orbits of period 3 and period 4 ($P_3$ and $P_4$) are intercalated with the small-amplitude chaotic attractor (figure 2b).

Bursting events follow a statistical law typical of type I intermittency (Meucci et al. 2004). Plotting the counts of the inter-bursting times $t$, we obtain experimentally, and confirm numerically, exponential decay of the burst occurrence versus the inter-bursting time, with an experimental mean separation $\langle \tau_{\text{exp}} \rangle = (0.366 \pm 0.001) \text{ ms}$; in the numerical case, we find $\langle \tau_{\text{num}} \rangle = (1.04 \pm 0.05) \text{ ms}$. The quantitative difference is due to the fact that the numerical noise is much smaller than the laboratory noise.

The modulated CO$_2$ laser has been initially modelled by two coupled rate equations, one for the laser intensity and the other for population inversion (Arecchi et al. 1982). However, for good agreement between the model and the experimental data, one must account for long-time interactions of the resonant molecular transition with other molecular levels. It is then convenient to add

Figure 3. (a,b) Numerical bifurcation diagrams obtained when starting from two different initial conditions. (a) $x_1(0)=0.43 \times 10^{-3}$ and (b) $x_1(0)=0.43 \times 10^{-1}$. The bistable range is $A \in (0.05, 0.07)$ while the bursting regime occurs after $A=0.1$. (c) Parameter regions of stable existence of $P3$ for different initial conditions.
three further linear equations, using a model of five differential equations. The rescaled equations for the modulated laser are (Ott 1994)

\[
\begin{align*}
\dot{x}_1 &= k_0 x_1 (x_2 - (1 + \alpha \sin^2(F_{\text{mod}} + B_0))), \\
\dot{x}_2 &= -\Gamma_1 x_2 - 2k_0 x_1 x_2 + \gamma x_3 + x_4 + P_0, \\
\dot{x}_3 &= -\Gamma_1 x_3 + x_5 + \gamma x_2 + P_0, \\
\dot{x}_4 &= -\Gamma_2 x_4 + \gamma x_5 + z x_2 + z P_0 \quad \text{and} \\
\dot{x}_5 &= -\Gamma_2 x_5 + z x_3 + \gamma x_4 + z P_0.
\end{align*}
\] (2.2)

The parameter values that fit the present experimental situation are \(K_0 = 30, \alpha = 4, B_0 = 0.1794, \Gamma_1 = 10.0643, \Gamma_2 = 1.0643, P_0 = 0.01987, \gamma = 0.05 \) and \(z = 10\). The threshold for \(P_0\) is 0.0164.

We report in figure 3\(a, b\) the numerical bifurcation diagrams for two different initial conditions. Figure 3\(a\) shows the subharmonic cascade corresponding to the experimental data of figure 2; precisely, it starts at \(A = 0.05\) with \(P2\) and continues with the chaotic window at \(A = 0.065\); at \(A \approx 0.102\), the attractor displays a sudden increase that corresponds to a crisis; from there, it merges with the ‘ghost’ of the unstable \(P3\) and hence yields the bursts observed in figure 2.

Figure 4. (a) Numerical power spectrum corresponding to the attractor at \(A = 0.06\) and to the initial condition in figure 3\(b\). (b) Selection of the other attractor after the control. (c) Numerical simulation of the time dependence of the laser intensity \(x_1\) for \(A = 0.105\) (beyond crisis). The control is applied at \(t = 7\) ms. The upper trace represents the control signal for \(G = 2.1\%\), on the same amplitude scale as \(x_1\). Similar results can be obtained by applying the perturbation to the pump parameter \(P_0\).
Figure 3b shows a P3 attractor born from a saddle-node bifurcation at \( A = 0.055 \) and disappearing by boundary crisis (Grebogi et al. 1982) at \( A = 0.07 \), for an initial condition \( x_1(0) = 0.43 \times 10^{-1} \). Figure 3c shows the stability regimes of P3 for a set of different initial conditions; the fact that P3 is born and dies at different control parameters is a demonstration of (i) the nestedness of the basin boundaries between the P3 and the subharmonic attractors and (ii) the fragility of the P3 attractor with respect to the subharmonic one.

The bistable interval around \( A = 0.06 \) is characterized by two different frequency contents of the P3 and subharmonic attractor; based on this difference, we adjust the feedback filter in order to enhance or depress the \( f/3 \) component. This way, we select either one of the two attractors of the bistable regime, as shown in figure 4a,b.

Above \( A = 0.1 \), an interior crisis expands the chaotic attractor into the region of the unstable P3, thus giving rise to a new attractor region which can still be discriminated by its frequency content. If we want to quench this region, we send its main frequency components (P3 and P4) as a negative feedback signal via the frequency filter. If, on the contrary, we change the feedback sign, we enhance the role of this region against the rest of the attractor.

The frequency components corresponding to the subharmonic attractor are slightly affected by the filter (NF) whose notch frequency is selected at the first subharmonic \( f/2 \), i.e. 50 kHz. The filter is AC coupled, so that the low-frequency

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components, which control the long-time dynamics, are not affected. The notch filter used in the experiment has been introduced previously (Meucci et al. 1996; Tesi et al. 1996).

The filter output is amplified and sent to the inverting and non-inverting inputs of the differential amplifier (R). When the control is inserted, the modulation \( F_{\text{mod}}(t) \) is perturbed by \( aV_{\text{out}} \), where \( V_{\text{out}} \) and \( a \) are the output of the filter and its amplification, respectively. We define the coupling strength \( G \) as the ratio \( aV_{\text{out}}/A \) between the perturbation and the modulation amplitude \( A \). Figure 4c shows the bursting regime beyond crisis and its suppression when the perturbation is \( G \approx 2\% \).

In figure 5 we report the experimental data at \( A=1.25 \text{ V} \) for the controlled dynamics, with both negative (figure 5a,c) and positive (figure 5b,d) feedback. On the right, we show the frequency spectra and on the left the corresponding attractors, reconstructed by the embedding technique. The larger amplitude chaotic attractor is drastically reduced in figure 5a,c, providing giving evidence of burst suppression; on the contrary, it is enhanced for a positive feedback in figure 5b,d.

Quantitative evidence of suppression of the large-amplitude bursts is provided by the fractional time indicator \( \eta \), which is measured by

\[
\eta = \frac{\left( \sum_k T_{B_k} \right)}{T}, \quad (2.3)
\]

where \( T \) is the duration of the recorded time-series and \( \sum_k T_{B_k} \) is the total duration of burst events within \( T \).

Figure 6. (a) Numerical and (b) experimental evidence of the power law dependence \( \eta \) on \( G \). The straight lines have slope \(-1/2\). Modelling this transition as \( \eta \propto |G - G_c|^{-1/2} \), the critical parameter \( G_c \) is estimated from the fit as \( G_c=0.00244 \) in the numerical case and \( G_c=0.0112 \) in the experimental case.
The bursts are detected by selecting a threshold in order to distinguish the large-amplitude spikes from the small-amplitude chaotic dynamics. The behaviour of $h$ from the non-controlled to the controlled regime is reported in figure 6 as a function of the coupling strength $G$. The transition towards complete control ($h \rightarrow 0$) is characterized by a power law with an exponent $K_{1/2}$,

$$h \propto G^{K_{1/2}}.$$  

Such a behaviour, also confirmed by numerical simulations, indicates the presence of a phase transition of the type I intermittency. The estimated value for the exponential decay of $h$ as a function of $G$ is in agreement with the theoretical predictions for this transition (Manneville & Pomeau 1980; Meucci et al. 2004). A similar feature has also been reported in the transition to phase synchronization of homoclinic chaos with periodic perturbation (Boccaletti et al. 2000, 2002). Very close to complete control, numerical and experimental data deviate from the power law.

The perturbation can be applied to other control parameters instead of the cavity losses. For example, it can be added to the pump parameter $P_0$; in this case, control of bursting is achieved for a relative perturbation of the order of $G \approx 0.05\%$.

Figure 7. Experimental set-up showing the Lorenz circuit made of linear amplifiers, resistors, capacitances and three multipliers. The values of the resistors are $R_1 = R_2 = 50 \, \text{k} \Omega$, $R_3 = R_6 = 5 \, \text{k} \Omega$, $R_4 = 25 \, \text{k} \Omega$, $R_5 = 500 \, \text{k} \Omega$, $R_7 = 333 \, \text{k} \Omega$ and $R_8 = 100 \, \text{k} \Omega$. Each capacitor value is $C = 1 \, \text{nF}$. $V_4$ is the fixed bias voltage and $V_6$ is the applied sinusoidal control ($= A \sin 2\pi ft$).
3. Control of bistability in the Lorenz system

Recently, control of bursting in a modulated laser has also been achieved by means of an open-loop control method (Zambrano et al. 2006a, b), in which the relative phase between the driving and the applied sinusoidal perturbation plays the crucial role. Here, we prove control of bistability in autonomous systems by applying a suitable modulation. In this context, we chose the well-known Lorenz system (Lorenz 1963), formally equivalent to the class C laser (Haken 1975), as confirmed by experimental studies on far infrared lasers (Weiss & Klische 1984; Weiss et al. 1988). We consider the bistable region present in the Lorenz model. The dynamics is bistable in the range \( r \in (16.1, 18.7) \) when the other two control parameters \( \sigma \) and \( b \) assume the values 10 and 1.2, respectively. In this range, there are two types of attractors depending on the initial conditions, namely fixed points and a strange attractor. The bistable region is extremely sensitive to an external modulation applied to the parameter \( r \) as \( r(1+\epsilon \sin(2\pi ft)) \). Such a perturbation is able to switch the dynamics from the chaotic attractor to one of the two fixed points. Removing the perturbation, the system remains in the steady state, demonstrating controlled switching of the two attractors. This strategy is different from control of chaos, in which a periodic orbit in the chaotic attractor is stabilized, as proved in Kociuba & Heckenberg (2002).

Figure 8. (a) Chaotic attractor and stable fixed point \( C^+ \) of the Lorenz system for \( V_4 = -1.9 \) V and (b) a backward and forward bifurcation diagram showing the bistable window.
The electronic implementation of the Lorenz system is shown in figure 7. The state equations of the analogue circuit are

\[
\begin{align*}
\dot{X} &= -\frac{X}{R_1C} + \frac{Y}{R_2C}, \\
\dot{Y} &= -\frac{XZ}{V_0R_3C} - \frac{Y}{R_5C} + X\left[\frac{1}{R_4C} + \frac{1}{R_8C} \frac{(V_4 + V_0)}{V_0}\right] \quad \text{and} \\
\dot{Z} &= \frac{XY}{V_0R_6C} - \frac{Z}{R_7C},
\end{align*}
\]

where the variable DC voltage \(V_4\) accounts for adjustments of the parameter \(r\); \(V_6 = A \sin 2\pi ft\) is the sinusoidal perturbation with relative strength \(\varepsilon = A/V_4\); and \(V_0 = 2.5\) V is a characteristic voltage of the multiplier.

In figure 8a, the two coexisting attractors for \(V_4 = -1.9\) V are shown. The stable fixed point \(C^+\) is surrounded by a circle denoting its basin of attraction.

The coexistence between the two attractors clearly appears as we perform a slow linear back and forth modulation of the parameter \(V_4\) (see figure 8b).

The frequency value of the sinusoidal perturbation to be used is suggested by measuring the relaxation oscillation towards one of the fixed points. Such a value is 1.35 kHz in our configuration. The chaotic attractor present in the bistable

Figure 9. (a) Time-series showing the switch between a chaotic attractor and the stable point after insertion of the perturbation, (b) plot of \(A_{\text{min}}\) versus frequency showing two minima, one at 1.36 ± 0.05 kHz and its harmonic at 2.60 ± 0.05 kHz, and (c) plot of \(A_{\text{min}}\) versus \(V_4\).
region collapses to the fixed point $C^+$ with a perturbation amplitude $\varepsilon = 0.5\%$ as shown in figure 9a. The mechanism responsible for the stabilization of the fixed point $C^+$ is the stimulation of an unstable limit cycle around the vicinity of the basin of attraction of $C^+$. At the tangency, the limit cycle spirals towards the fixed point. By varying the frequency of the external perturbation, we observe the expected resonant behaviour in correspondence of $f_{\text{min}} = (1.36 \pm 0.05)$ kHz and its harmonic (figure 9b).

4. Conclusions

The switching between two coexisting attractors and the bursting behaviour at the onset of an interior crisis, which play a crucial role in different fields of nonlinear dynamics, can be controlled by means of a feedback method selecting the frequency components of two attractors or by a suitable parameter modulation (non-feedback control method). The applicability of these methods has been proved on two general systems, i.e., a modulated class B laser and the Lorenz system, formally equivalent to the class C laser.

References


