Transient tumbling chaos and damping identification for parametric pendulum

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The aim of this study is to provide a simple, yet effective and generally applicable technique for determining damping for parametric pendula. The proposed model is more representative of system dynamics because the numerical results describe the qualitative features of experimentally exhibited transient tumbling chaotic motions well. The assumption made is that the system is accurately modelled by a combination of viscous and Coulomb dampings; a parameter identification procedure is developed from this basis. The results of numerical and experimental time histories of free oscillations are compared with the model produced from the parameters identified by the classic logarithmic decrement technique. The merits of the present method are discussed before the model is verified against experimental results. Finally, emphasis is placed on a close corroboration between the experimental and theoretical transient tumbling chaotic trajectories.

Keywords: dry friction; linear viscosity; parameter identification; transient tumbling chaos; parametric pendula

1. Introduction

The parametric pendulum is a paradigm model for much rich dynamical behaviour, and as such it has been given a great deal of attention for many years. Developing this, since the experimental verification of chaotic motion using a pendulum, by Leven & Koch (1981), there has been a determined focus in this area. Various analytical techniques have been reported (Clifford & Bishop 1994), which identify the locus of parameters describing the period-doubling bifurcation, approximately bounding one side of the escape zone—seeing this as the primary route to predicting the onset of chaos. Szemplińska-Stupnicka et al. (2000) seemingly turned the more classical approach ‘on its head’ when the authors resolved that the global bifurcation leading to the existence of fractal basin boundaries, which make the presence of chaotic motions possible, is not the one described by Melnikov theory. The work adopts the definition of persistent and transient tumbling chaos by Bishop & Clifford (1996). From this starting point, Szemplińska-Stupnicka et al. (2000) built a complete picture of the steady-state existence of different attractors within the parametrically excited pendulum

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system in parameter space. In another paper by Szemplińska-Stupnicka & Tyrkiel (2002), an attempt was made to compare the steady-state cross-well chaotic response of three oscillators. Similar to the findings reported in the work by Szemplińska-Stupnicka et al. (2000), the bifurcations bounding the different regimes of motion in parameter space are presented in each case. Attention is paid to the route to persistent chaos for each system. The conclusion drawn is that despite the nature of the motion being random-like, distinctly common features between each route can be observed. The findings effectively echo the results of Leven & Koch (1981). The cornerstone to this field of research must be the exploitation of chaotic regimes as useful entities and Bishop et al. (1996) developed a method similar to the OGY method proposed by Ott et al. (1990), specifically for the control of the parametrically excited pendulum.

The systems studied in the literature presented above assume that linear viscous damping is the dominant mechanical dissipation necessary to model system dynamics. The literature discussed below identifies that a combination of linear viscosity and dry friction, perhaps also accounting for secondary effects, is felt to be more appropriate for modelling the energy dissipation in a wide variety of systems.

Coulomb or dry friction damping in nonlinear systems has received a great deal of treatment in the past few decades. Den Hartog (1956) presented the fact that instabilities in many systems can be caused by dry friction. The author uses the relative velocities of a bow and string as a violin is played to introduce the concept of mechanical chatter. Shaw (1986) considered the stick–slip behaviour of a driven one-degree-of-freedom oscillator, with dry friction damping. Developing this, Shaw identifies the sticking region in the extended phase space and goes on to consider the stable fixed points for each case: stick and no stick, using a mapping. Feeny (1992) built on Shaw’s work when the author considers a forced oscillator allowed to drift via a frictional slider. Notably, Feeny constructs a template of a chaotic attractor by extrapolating successive mappings of periodic orbits onto a flow in a cylindrical space. In addition, Pavlovskaia et al. (2001) and Pavlovskaia & Wiercigroch (2003) considered dry friction in a piecewise system, where the authors study the dynamics of a model for percussive drilling, with a progression.

Parameter identification techniques based on experimental measurements for a corresponding model are very common in many fields. Panet & Jezequel (2000) considered the dynamics of actuators used in nuclear power plants by studying the damping of such structures. Moreover, Graver et al. (2003) had modelled the equilibrium equations representing steady-state glide of a buoyancy-driven underwater electric glider. The authors present a method for identifying the parameters in the equation as a combination of empirical measurement and least-squares regression carried out on actual test ‘flight’ data. Alasty & Shabani (2006) used the extended Kalman filter to identify the parameters of a model for an electromagnetic force experienced by magnetic bearings. Developing this, the method is essentially an optimization technique, iteratively comparing an estimate of the next mapping of the second-order nonlinear ordinary differential equation with the actual results from experiment until the parameters of the model reach a constant value through time.

Chang (2006) presented an evolutionary algorithm based on a minimum cost criterion to find the parameters of Roessler’s chaotic attractor. Recently Barbieri et al. (2004) reviewed a number of parameter identification methods suitable for
various two-degree-of-freedom systems. General parameter model identification is varied and wide ranging; however, for the purposes of this study it is most important that we consider the literature on damping parameter identification. Helmholtz (1885) considered the effect of damping on volumes of sound as it entered the ear and measured how many vibrations passed before the sound reached one-tenth of its original amplitude. Later in the same work, he proposed what Rayleigh (1912) presented as the logarithmic decrement: the free vibrations of a linear viscously damped dynamical system decay exponentially and the ratio of the logarithms of successive peaks is approximately constant. As a consequence of this result, the well-known logarithmic decrement method for experimentally identifying the viscous damping ratio has been developed.

Stanway et al. (1985) used a least-squares regression to determine the Coulomb friction and the linear viscous damping parameters for a simple nonlinear oscillator. Likewise, Yao et al. (1997) used a similar method when the authors determine the linear viscous and Coulomb friction damping parameters of a system comprising an electrorheological damper. In addition, Feeny & Liang (1996) considered determining Coulomb and linear viscous damping parameters. The authors propose and test a method for identifying parameters, which is applicable to a linear system with constant ratio between the damping parameters. Developing this, an experiment is described where a spring–mass–damper system is used on an air track, with an eddy current used to provide the effect of linear viscosity. Interestingly, the system is set up such that it may be used to demonstrate the effect of dry friction when there is no eddy current, and of linear viscous damping when the dampers are removed. Liang (2005) reported that it is not only possible to determine the damping parameters separately using time history of free oscillations data measured by experiment, but acceleration data can also be exploited. The procedure to allow such data to be used is outlined and close corroboration between the numerical model, with identified parameters, and experimental results, is found.

Hinrichs et al. (1998) undertook an analysis of the bifurcations of a frictional oscillator able to exhibit the stick–slip phenomena. The damping coefficients are deduced separately from the logarithmic decrement method. The friction ratio is calculated as the ratio of the measured time-dependent friction and constant normal forces. Liang & Feeny (2006) built on their earlier work, Feeny & Liang (1996), when the authors used an energy balance between the dissipation of energy and the energy input to an oscillator, to determine both the Coulomb friction and the linear viscous damping parameters.

Above, damping parameter identification has been focused on more general dry friction oscillators. Nevertheless, if we were to consider de Paula et al. (2006), for example, the authors focus on the damping parameters of a pendulum model and present the argument that the linear viscous damping and dry friction are the dissipative terms required for good corroboration with the experimental set-up used. In addition, they represent dry friction as Coulomb friction in the mathematical model and proceed with a simple parameter identification technique. Furthermore, an explanation is given that by considering where the velocity is high in the experimentally obtained time history of free oscillations, ‘linear viscous damping is preponderant’ and as time evolves, Coulomb damping becomes the most prevalent. In other words, they argue that linear viscous damping is dominant where the velocity of free oscillations is high and, as a
consequence, the decay is approximately exponential. Developing this, the classical logarithmic decrement method may be applied. Similarly, as time evolves dry friction becomes more important and eventually dominates. The authors suggest that the decay is linear over this region and the relevant damping parameter can be found from the gradient.

Water et al. (1991) sought to obtain a pendulum model which provides accurate corroboration between theory and experiment. Nevertheless, since linear viscous damping is not the only dominating dissipation as the phase space variables change, it is decided that a number of other dissipative terms must feature. This reference finds good corroboration with experiment incorporating additional terms representing air resistance, Coulomb friction and harmonic forces due to the ball bearings having an imperfect geometry. On the other hand, Trueba et al. (2003) reckoned that all dissipative nonlinearities may be satisfactorily modelled through a polynomial function of the \( n \)th degree, for a generalized pendulum model.

A model of the parametric pendulum is provided in equation (1.1) and assumes linear viscous damping as the dissipation of energy at the pivot. Before a full dynamical analysis of the bifurcation scenario of the parametric pendulum is undertaken, it is convenient to apply an approximate analytical method to construct a qualitative plot of the parameter space. First, the equation is non-dimensionalized with respect to time (1.2) and if librations with no energy dissipation are assumed and the equation is also linearized, equation (1.3) is obtained as follows:

\[
\ddot{\theta} + \frac{c}{m} \dot{\theta} + \left( \frac{g}{l} + \frac{Y^2 \cos(\Omega t)}{l} \right) \sin \theta = 0, \tag{1.1}
\]

\[
\theta'' + \gamma \theta' + (1 + p \cos(\omega_0 t)) \sin \theta = 0 \tag{1.2}
\]

and

\[
\theta'' + (1 + p \cos(\omega_0 t)) \theta = 0, \tag{1.3}
\]

where \( \theta', \theta'' \) denote non-dimensional angular velocity and acceleration, respectively; \( \omega_0 = g/l; \tau = \omega_0 t; \gamma = c/m\omega_0; p = Y\Omega^2/l\omega_0^2; \omega = \Omega/\omega_0; \theta \) is the angular displacement; \( m \) is the pendulum bob mass; \( c \) is the damping ratio; \( \omega_0 \) is the natural frequency; \( Y \) is the excitation amplitude; \( l \) is the pendular arm length; \( g \) is the acceleration due to gravity; \( \Omega \) is the excitation frequency; and \( t \) is the time.

Equation (1.3) can be rescaled to obtain the standard form of Mathieu’s equation given by

\[
\theta'' + (\alpha^2 + \epsilon \cos x) \theta = 0, \tag{1.4}
\]

where \( x = \omega_0 t; \alpha = 1/\omega; \) and \( \epsilon = p/\omega_0^2 = Y/l \) is a small number. Struble’s method (Struble 1962) or an equivalent asymptotic method is used to determine the first \( n \) critical curves bounding the resonance zones of oscillatory and rotational instabilities in the parametric pendulum’s motion. A similar procedure is used by Xu & Wiercigroch (2007) to deduce the critical curves for the nonlinear form of Mathieu’s equation,

\[
\theta'' + (1 + p \cos(\omega_0 t)) \left( \theta + \frac{1}{6} \theta^3 \right) = 0. \tag{1.5}
\]
Figure 1 shows the results from such an asymptotic technique. Different parameter ranges cause different attractors, and therefore different types of motion coexist. Xu et al. (2005) developed the study further by numerically computing the parameter space for specified damping and initial conditions. Importantly, the authors annotate the parameter space to highlight the distinct regions of motion.

The plots in Figure 2 show the time histories, phase plane diagrams and Poincaré sections for the parametric pendulum exhibiting tumbling chaotic motions. Figure 2a, b shows tumbling chaotic motions at the turning point of the primary critical curve, shown in Figure 1, for different base excitation amplitudes, $\epsilon$. Finally, figure 2c shows tumbling chaotic motion near the turning point of the secondary critical curve for oscillations, for a particular excitation amplitude, $\epsilon$. Much of the body of work on parametric pendula assumes linear viscous damping alone.

Classical theory suggests that, for a linear viscously damped system, the damping coefficient, $\gamma$, may be experimentally identified by manipulating the linearized equation of motion. Damping parameters are deduced by substituting known quantities obtained from the decay of the free oscillations of the pendulum into the resulting expression. Nevertheless, numerical modelling of tumbling chaotic motion of the pendulum, which provides close corroboration with experiment, is not well understood and the present models are insufficient.

Therefore, it is proposed in this study that the mechanical dissipation in a parametrically excited pendulum is accurately represented by assuming Coulomb friction and linear viscous damping. In addition, parameter identification is carried out and an analysis of the merits of the proposed
classical model is given. To compare the proposed model and validity of associated damping identification with classical theory, the calculated values of the respective amplitude envelopes and peak-to-peak periods are compared with those measured from experiment. To check the accuracy of the model, different stable motions have been identified at parameter settings experimentally, and these have been compared with the numerical solutions obtained at the same parameter values.

The main driver behind this study is to produce an accurate model with robust damping identification, which is sufficiently accurate to provide good corroboration between the qualitative features of numerical and experimental tumbling chaotic trajectories. Finally, experimental transient tumbling chaotic motions and the motions computed numerically at the same parameter settings are compared.

2. Proposed model: with viscous and Coulomb damping

Section 1 introduced the proposition that linear viscous damping alone was insufficient to represent mechanical dissipation in models of parametric pendula. Furthermore, it has been highlighted that various other combinations of dissipative terms have been applied to the model, and corresponding parameter identifications have been reported for both spring–mass–damper and pendula
arrangements. The method developed by de Paula et al. (2006), for example, is not suited to the experimental rig developed and used in the Centre for Applied Dynamics Research at Aberdeen, which is extensively described in Xu (2005) and Xu et al. (2007). As a consequence, a more representative model and identification procedure are required.

Feeny & Liang (1996) reported on modelling a system with a Coulomb friction and linear viscous dissipation and the related damping identification. The authors present that although the envelope of free oscillations for such a system is nonlinear, a manipulation can be done on the linearized equation of motion to find expressions for the damping parameters separately. Developing this, these expressions reduce to ratios of differences between peak amplitudes of small-angle oscillations, very similar to the expression for the logarithmic decrement. Furthermore, the authors validate the procedure and subsequent model via corroboration with an equivalent experimental rig. In addition, Feeny & Liang (1996) compared the estimates of each parameter with those obtained from the equivalent procedure assuming the model has that type of damping alone. Errors between the deduced parameters for the ‘combined decrement method’ and parameters found from assuming each dissipation alone are discussed, but overall the method produces a model which finds close corroboration with the experiment.

The present work applies the procedure documented in the work by Feeny & Liang (1996), previously for a spring–mass–damper system with frictional slider, to the parametric pendulum. The proposed form of the equation governing the pendulum model shown in figure 3 is

$$\ddot{\theta} + \frac{c}{m} \dot{\theta} + \omega_0^2 (1 + p \cos(\Omega t + \phi)) \sin \theta = -T_r \text{sign} \dot{\theta},$$

where $p = Y \Omega^2/g$ and $c$ is the damping ratio; $\omega_0$ is the undamped natural frequency of the pendulum; $T_r$ is the scaled residual torque effect due to Coulomb friction; $p$ is the scaled amplitude of excitation acceleration; $\phi$ is an arbitrary phase angle associated with the sinusoidal driving force; and $t$ is the time.

Figure 3. Illustration of a parametric pendulum: a pendulum of mass $m$ concentrated at the bob, with length $l$ vertically excited at its pivot.
Figure 4. (a) Time history of amplitude of free oscillations of the pendulum showing the ‘even’ and ‘odd’ half cycles. (b) Associated velocity history for free oscillations of the pendulum, showing the evolution of the signum function $\text{sgn} \dot{\theta}$.

Figure 4a shows a schematic of a typical free oscillatory response of the parametric pendulum and figure 4b shows the associated velocity time history and how the sign function changes with the velocity evolution. The approach adopted in this work is based upon the classical method for parameter identification, where the linearized equation of motion is algebraically solved. Furthermore, the solution is rearranged in terms of experimentally measurable quantities and the free oscillations of the pendulum are measured experimentally to obtain these necessary parameters. Finally, the unknown damping parameters are calculated. Equation (2.1) may be linearized leading to a system of piecewise linear equations. Assuming $p$ to be zero, the global solution is obtained by the local solutions

$$\theta_+(t) = e^{-\gamma \omega_d t}(A \cos \omega_d t + B \sin \omega_d t) - \theta_t; \quad \dot{\theta} > 0$$  \hspace{1cm} (2.2)

and

$$\theta_-(t) = e^{-\gamma \omega_d t}(C \cos \omega_d t + D \sin \omega_d t) + \theta_t; \quad \dot{\theta} < 0, \hspace{1cm} (2.3)$$

where $\gamma$ is the damping coefficient; $\omega_d$ is the damped natural frequency; and $\theta_t = T_t/\omega_d^2$. $\theta_t$ is the ‘opposing’ angle of rotation due to the Coulomb friction and $T_t$ is the scaled residual torque effect due to the dry friction.

The initial value problem is solved for the first half cycle, from the maxima $\theta_0$ at time $t=t_0$ to the minima $\theta_1$ at $t=T/2 = \pi/\omega_d$, as per figure 4a, and the
arbitrary constants are obtained. The end of this half cycle is taken as the initial condition for the half beginning at \( t = T/2 \) and substituting for known values, the expression representing this present half cycle is also deduced. Similarly, the procedure is repeated for further half cycles and the resulting expressions for the first, second and \( k \)th half periods are given as

\[
\begin{align*}
|\theta_1| &= -\theta_r(e^{-\hat{\delta} \pi} + 1) + |\theta_0|e^{-\hat{\delta} \pi}, \\
|\theta_2| &= -\theta_r(e^{-\hat{\delta} \pi} + 1) + |\theta_1|e^{-\hat{\delta} \pi}, \\
&\vdots \\
|\theta_k| &= -\theta_r(e^{-\hat{\delta} \pi} + 1) + |\theta_{k-1}|e^{-\hat{\delta} \pi},
\end{align*}
\]

where \( \hat{\delta} \) is the decrement parameter; \( \theta_i \) is the excursion maxima corresponding to the beginning of the \( i+1 \)th half cycle. The symmetric properties of triplets \( \theta_0, \theta_1, \theta_2 \) to \( \theta_{k-2}, \theta_{k-1}, \theta_k \) are exploited and an expression for the decrement parameter, \( \hat{\delta} \), is

\[
\hat{\delta} = \frac{1}{(k-1)\pi} \ln \left( \frac{|\theta_0| - |\theta_1|}{|\theta_{k-1}| - |\theta_k|} \right). 
\]

The magnitude of the dry friction torque is calculated using a procedure similar to that for the viscous damping coefficient above. Equations (2.4) and (2.6) can be rearranged in terms of the exponentially decaying forms

\[
\begin{align*}
e^{-\hat{\delta} \pi} &= \frac{|\theta_2| + |\theta_r|}{|\theta_1| - |\theta_r|}, \\
&\vdots \\
e^{-\hat{\delta} \pi} &= \frac{|\theta_k| + |\theta_r|}{|\theta_{k-1}| - |\theta_r|}.
\end{align*}
\]

Setting equations (2.8) and (2.9) equal, and rearranging, \( T_r \) may be expressed as follows:

\[
T_r = \omega_0^2 \frac{|\theta_1||\theta_{k-1}| - |\theta_0||\theta_k|}{|\theta_1| + |\theta_0| - |\theta_{k-1}| - |\theta_k|}. 
\]

The values obtained from the method presented above hold for all angles of displacement, large or small, exactly the same as the method of logarithmic decrement for the simple pendulum with viscous damping alone. Section 3 describes the implementation of equations (2.7) and (2.10) as part of a proposed experimental parameter identification procedure. Furthermore, the experimental results are compared with those numerically computed.

### 3. Parameter identification procedure and results

In this section a new damping parameter identification procedure is proposed. In addition, the subsequent numerically computed dynamic responses from different theoretical models are compared with experimental results, to identify which model shows best corroboration.
Initially, the experimental rig is placed on a flat, immoveable surface to mitigate against pendulum–base interactions. Furthermore, the encoder is connected through a connection box to a computer supporting a data acquisition system, which is set up to capture the voltage measurements made by the encoder, through time. Once the equipment is set up and switched on, the pendulum is released from an arbitrary initial condition close to \( \pi \) and the information of each oscillation measured by the encoder is saved to the computer terminal, until the pendulum settles on the downwards stable equilibrium position. From the encoder calibration and the scan rate used to capture the experimental data, the voltages can be converted into angular displacement readings. In other words, the time history of the decay can be constructed. The time history of free oscillations data are converted into absolute values, and the peak amplitude values and the time at which each peak occurred are found. The results for peaks of magnitude above approximately 12° are discarded, because the linearization of equation (2.1) uses a ‘small angles’ approximation. Developing this, the retained information about the peak amplitudes are applied to equations (2.7) and (2.10) to deduce the damping parameters \( \gamma \) and \( T_r \).

Finally, the calculated parameter values are substituted into equation (2.1) to allow numerical simulation of the dynamics of the proposed model to be carried out. Initially, for the linear viscous and Coulomb damping parameters, equations (2.7) and (2.10) were applied to the entire range of data associated with librations. The numerically computed time history obtained from the results for the damping parameters appeared initially to have poor correspondence with experiment. Figure 5a shows that there are large discrepancies in measured values of the decrement parameter \( |\delta| \), and as a consequence, it has been necessary to investigate the reasons for this. The main reason is found in figure 5b. The averaging which is applied in equation (2.7) relies on the fact that values for the damping parameter \( \delta \), deduced from each set of three neighbouring

![Figure 5](http://rsta.royalsocietypublishing.org/)
displacement amplitude values, are equal. Moreover, plotting the decrement parameter in favour of the absolute value of $|\dot{\delta}|$ against the experimentally measured peak amplitude values shows that $\delta$ changes sign for each set of three triplets considered. Developing this, the result stated can only occur if the envelopes described by peaks and troughs are different.

It is proposed that the procedure is sensitive to the presence of noise in the system and the choice of smoothing of the measured signal is important to achieve accuracy in the subtle differences evident between neighbouring peak values. Presently, the consequence of noise is to underestimate the downward equilibrium position. Figure 5b shows the results: ‘even’ and ‘odd’ half cycles estimate the damping parameters as they vary with peak displacement amplitude, differently. It has been found that by considering those calculations beginning on odd half cycles alone, a good representation of damping parameters is found.

The final parameter required is the natural frequency and this is found by calculating all the periods of motion for each cycle starting on each successive maxima and minima and obtaining an average. Furthermore, the average period is substituted into the equation relating the period to the natural frequency, and the required parameter is computed.

Figure 6a shows the free oscillations described by the numerical model including the parameters identified using the method presently being discussed. Figure 6b plots the free oscillations for a model with only linear viscous damping, identified from the classical logarithmic decrement method, against the experimental results obtained. It is clear that at a first glance the present ‘combined’ method is far more accurate for free oscillatory motion than the classic method.

The exponentially decaying envelope computed by using only linear viscous damping causes decay to occur far too quickly and due to the nature of this damping mode, the oscillations theoretically carry on to infinitum. On the other hand, the hybrid envelope defined by the linear viscous and Coulomb damping numerical model decays to the downward equilibrium position at the same time as for the experimental results.

The free oscillations of the pendulum can be examined by applying different time scales: one to closely examine each oscillation and another which allows the envelope to be studied. As a consequence, corroboration between the experimental envelope and the period of free oscillations and those produced theoretically from different parameter identification techniques has been sought. Figure 7a depicts the difference in the period of each free oscillation obtained theoretically, and those from experiment plotted against the time at which each peak occurs. Similarly, figure 7b shows the difference between the amplitude of numerically computed and experimentally measured free oscillations, plotted against the experimental peaks. Figure 7b shows parabolic-type relationships between the theoretical and the experimental peak amplitude differences, as they vary with experimental peak amplitude. The errors incurred for a linear viscous damping model alone, and using classic logarithmic decrement theory, increase to a maximum that is approximately half way through the motion.

De Paula et al. (2006) stated that linear viscous damping is dominant for large amplitude oscillations (near $\pi$) and also librations with Coulomb damping becoming prevalent in between, discussed above. It is concluded that this is exactly the reason why there is such a definite maxima in the error using viscous
damping alone: where Coulomb friction becomes prevalent in practice, the model does not account for this. The results for how the period of oscillations change with time for experimentally measured peaks confirm that linear viscous damping is not the most suitable for modelling mechanical dissipation in the parametric pendulum. Developing this, the farther away the pendulum moves from its initial condition, the more out of phase the motion becomes with that

Figure 6. Comparison of classical logarithmic decrement method with experimental results, for free oscillations of the pendulum; experimental results in grey and numerical results in black. (a) Parameter identification for model with dry and linear viscous damping; $\gamma=0.00208$, $T_r=0.01256$. (b) ‘Classical logarithmic decrement method’; $\gamma=0.01998$. 

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described by experiment. The free oscillations for the model and associated parameter identification proposed by de Paula et al. (2006) has been computed, since it should be more accurate. It is observed that data are not available over the entire set of experimental peak values and this is due to the fact that using this method vastly overestimates the level of linear viscous damping present in the system. The model represents a pendulum, which becomes stationary at the downward equilibrium position much before that from the experiment and this causes large errors in both the computed period and peak amplitude envelope. Finally, the free oscillations of the combined damping model with associated damping were computed and the errors shown in figure 7 take a parabolic form.

Modelling the dry friction as Coulomb friction is not wholly representative because the dry friction will change through the period of motion; however, the errors using this model are far smaller than those provided by the previously discussed models. Moreover, this corroboration culminates in the final period of motion, before the bob becomes stationary, having excellent correspondence with the experimental results. To state these findings another way, table 1 describes the error in period and peak amplitude values $T_\mu$ and $\theta_\mu$, respectively, as a percentage of the experimentally measured results for each model discussed in figure 7. In the case of the percentage error in the period $T_\mu$, the results are

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*Figure 7. (a) Plot of the errors in the period of oscillations $|\delta T_c|$ against the time at which each experimental peak occurred $t_m$. (b) Plot of the errors in the amplitude of the numerically computed free oscillation solutions $|\delta \theta_c|$ against the amplitude of the experimentally measured free decay of the rig $\omega_m$. (c) Image of the experimental pendulum rig with close-ups of the shaft and bearings configuration (lower inset), the shaft between the pendulum and the DC generator used to apply a resistive torque (centre inset) and the DC generator and the encoder (upper inset).*
Table 1. Table of percentage errors $T_m$ and $q_m$. (Table of errors of numerically computed free oscillation peaks, and peak-to-peak periods as a percentage difference $T_m$ of the corresponding experimental variables $T_m$ and $|q_m|$, respectively, for different parameter identification schemes and models.)

<table>
<thead>
<tr>
<th>$T_m$ (s)</th>
<th>‘combined’ method $T_m$ (%)</th>
<th>logarithmic decrement method $T_m$ (%)</th>
<th>de Paula’s method $T_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.26</td>
<td>7.31</td>
<td>16.95</td>
<td>64.81</td>
</tr>
<tr>
<td>1.54</td>
<td>7.18</td>
<td>17.46</td>
<td>41.58</td>
</tr>
</tbody>
</table>

| $|\theta_m|$ (rad) | ‘combined’ method $\theta_m$ (%) | logarithmic decrement method $\theta_m$ (%) | de Paula’s method $\theta_m$ (%) |
|----------------|-----------------------------|--------------------------------------|-----------------------------|
| 1.2            | 0                           | 4.35                                 | 9.17                        |
| 2.53           | 1.58                        | 6.04                                 | 13.29                       |

tabulated against the peak-to-peak time period measured experimentally, $T_m$, and the percentage error in amplitude, $q_m$, is tabulated against the corresponding experimental peak amplitude value $|\theta_m|$.

These results echo those discussed above: the ‘combined damping’ model is superior in modelling mechanical dissipation in a parametric pendulum. Section 4 compares experimental and theoretical results for pendulum dynamics with parametric forcing. Oscillations and rotational motion are exhibited and the corroboration is discussed. Moreover, attention is focused on the good corroboration found between experimental and theoretical transient tumbling chaotic trajectories.

4. Experimental verification of model

In this section a combination of numerical proposition and experimental corroboration is used to test the model with linear viscous and Coulomb damping against what is actually observed under different regimes of motion of the experimental rig. The experimental results were measured by Xu (2005) as part of a full nonlinear dynamical analysis of system behaviour and the same excitation parameters have been used in the verification of the present model. Furthermore, the packages DYNAMICS (Nusse & Yorke 1994) and DYNAMICS SOLVER (Aguirregabiria 2005) were employed for dynamical explorations.

The attractors corresponding to the motion observed experimentally have been located by plotting the basins of attraction for the system in the state space. Moreover, using initial conditions in the region close to such attractors, datasets capturing periodic motion once the solution has reached the steady state have been obtained. The aforementioned datasets have then been compared with their experimental counterparts to confirm the validity of the parameter identification method.

Convergence has been found to be slow; nevertheless, all solutions have converged after 2500 s elapsed. Figure 8a,b shows the comparison between the experimental measurements and the numerical results for steady-state solutions, which settled on period-2 oscillatory attractors, under different parameter settings. Both results show good correspondence in period and the numerical

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results overestimate the amplitude of response by only a very small magnitude. In addition, figure 8c, d shows the comparison between experiment and numerics for period-1 rotational regimes. It is observed that each provides excellent correlation with no discernable difference in phase or amplitude.

To identify transient chaotic trajectories, no pre-iterates have been taken and comparisons have been made over a neighbourhood of initial conditions very close to those for the experimental results. Figure 9a, c depicts the transient chaotic trajectories obtained through experiment and those from figure 9b, d shows their numerically computed counterparts. The shape of the wells carved out by each respective trajectory is very similar. What is most intriguing is that the angular displacement covered by each respective trajectory is similar; however, the maximum velocity described by each numerical trajectory has the ratio 3:2 with that of the experimentally determined counterpart. In spite of the differences in velocity, the numerically computed results are closely representative of how the pattern of escape and confinement from each well for the experimental chaotic trajectories emerges.

5. Conclusions

A new parameter identification technique has been proposed for parametric pendula with linear viscous and Coulomb friction damping, based on a need for a more accurate and representative model of the system. The experimental
measurements of the time history of free oscillations have been taken and peak displacement amplitude values have been used to find the unknown damping parameter values. The experimental and theoretical results were then compared for both the classical theory and the theory associated with linear viscous and Coulomb friction damping. Better corroboration was found from the proposed theory than for the classical model.

By comparing the experimental and numerical results for pendula with forcing, it has been confirmed that the present model is indeed more accurate for representing mechanical pendula. An excellent correlation between the experimental and numerical transient tumbling chaotic trajectories has been obtained.

In conclusion, the proposed parameter identification technique is easily applicable and the resulting model has shown good agreement with experiment. It is advised that the proposed technique should be tested against other experimental results before it is determined to be suitable for application to the entire family of mechanical pendula.

**References**


