Lamé polynomials, hyperelliptic reductions and Lamé band structure

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The band structure of the Lamé equation, viewed as a one-dimensional Schrödinger equation with a periodic potential, is studied. At integer values of the degree parameter $\ell$, the dispersion relation is reduced to the $\ell=1$ dispersion relation, and a previously published $\ell=2$ dispersion relation is shown to be partly incorrect. The Hermite–Krichever Ansatz, which expresses Lamé equation solutions in terms of $\ell=1$ solutions, is the chief tool. It is based on a projection from a genus-$\ell$ hyperelliptic curve, which parametrizes solutions, to an elliptic curve. A general formula for this covering is derived, and is used to reduce certain hyperelliptic integrals to elliptic ones. Degeneracies between band edges, which can occur if the Lamé equation parameters take complex values, are investigated. If the Lamé equation is viewed as a differential equation on an elliptic curve, a formula is conjectured for the number of points in elliptic moduli space (elliptic curve parameter space) at which degeneracies occur. Tables of spectral polynomials and Lamé polynomials, i.e. band-edge solutions, are given. A table in the earlier literature is corrected.

Keywords: Lamé equation; Lamé polynomial; dispersion relation; band structure; hyperelliptic reduction; Hermite–Krichever Ansatz

1. Introduction

(a) Background

The term 'Lamé equation' refers to any of several closely related second-order ordinary differential equations (Whittaker & Watson 1927; Erdélyi 1953–1955; Arscott 1964). The first such equation was obtained by Lamé by applying the method of separation of variables to Laplace's equation in an ellipsoidal coordinate system. Lamé equations arise elsewhere in theoretical physics. Recent application areas include: (i) the analysis of preheating after inflation, arising from parametric amplification (Boyanovsky et al. 1996; Greene et al. 1997; Kaiser 1998; Ivanov 2001), (ii) the stability analysis of critical droplets in bounded spatial domains (Maier & Stein 2001), (iii) the stability analysis of static configurations in Josephson junctions (Caputo et al. 2000), (iv) the computation of the distance–redshift relation in inhomogeneous cosmologies.

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(Kantowski & Thomas 2001) and (v) magnetostatic problems in triaxial ellipsoids (Dobner & Ritter 1998).

In some versions of the Lamé equation, elliptic functions appear explicitly, and in others (the algebraic versions) they appear implicitly. The version that appears most often in the physics literature is the Jacobi one

\[ -\frac{d^2}{d\alpha^2} + \ell(\ell + 1)m \text{sn}^2(\alpha|m) \Psi = E\Psi, \]

(1.1)

which is a one-dimensional Schrödinger equation with a doubly periodic potential, parametrized by \( m \) and \( \ell \). Here, \( \text{sn}(\cdot|m) \) is the Jacobi elliptic function with modular parameter \( m \). The parameter \( m \) is often restricted to \((0, 1)\), though in general \( m \in \mathbb{C} \setminus \{0,1\} \) is allowed. When \( m \in (0, 1) \), the function \( \text{sn}^2(\cdot|m) \) has real period \( 2K := 2K(m) \) and imaginary period \( 2iK' := 2iK(1-m) \), with \( K(m) \) being the first complete elliptic integral.

If \( \alpha \) is restricted to the real axis and \( m \) and \( \ell \) are real, (1.1) becomes a real-domain Schrödinger equation with a periodic potential, i.e. a Hill’s equation. Standard results on Hill’s equation apply (Magnus & Winkler 1979; McKean & van Moerbeke 1979). Equipping (1.1) with a quasi-periodic boundary condition

\[ \Psi(\alpha + 2K) = e^{iK(2K)} \Psi(\alpha) := \xi \Psi(\alpha), \]

(1.2)

where \( k \in \mathbb{R} \) is fixed, defines a self-adjoint boundary value problem. For any real \( k \) (i.e. for any Floquet multiplier \( \xi \) with \( |\xi| = 1 \)), there will be an infinite discrete set of energies \( E \in \mathbb{R} \) for which this problem has a solution, called a Bloch solution with crystal momentum \( k \). Each such \( E \) will lie in one of the allowed zones, which are intervals delimited by energies corresponding to \( \xi = \pm 1 \), i.e. to periodic and anti-periodic Bloch solutions. These form a sequence \( E_0 < E_1 \leq E_2 < E_3 \leq E_4 < \cdots \), where \( E_0 \) is a ‘periodic’ eigenvalue, followed by alternating pairs of anti-periodic and periodic eigenvalues (each pair may be coincident). The allowed zones are the intervals \( [E_{2j}, E_{2j+1}] \). The complementary intervals \( (E_{2j+1}, E_{2j+2}) \) are forbidden zones or lacunae. Any solution of Hill’s equation with energy in a lacuna is unstable: its multiplier \( \xi \) will not satisfy \( |\xi| = 1 \), and its crystal momentum \( k \) will not be real.

It is a celebrated result of Ince (1940b) that if the degree \( \ell \) is an integer, which without loss of generality may be chosen to be non-negative, the Lamé equation (1.1) will have only a finite number of non-empty lacunae. A converse to this statement holds as well (Gesztesy & Weikard 1995a). If \( \ell \) is an integer, the Bloch spectrum consists of the \( \ell + 1 \) bands \( [E_0, E_1], [E_2, E_3], \ldots, [E_{2\ell}, \infty) \), and \( \ell(\ell + 1)m \text{sn}^2(\cdot|m) \) is said to be a finite-band or algebro-geometric potential. The \( 2\ell + 1 \) band edges \( E_0, \ldots, E_{2\ell} \) are algebraic functions of the parameter \( m \). In other words, they are the roots of a certain polynomial, the coefficients of which are polynomial in \( m \). The corresponding periodic and anti-periodic Bloch solutions are called Lamé polynomials: they are polynomials in the Jacobi elliptic functions \( \text{sn}(\alpha|m), \text{cn}(\alpha|m) \) and \( \text{dn}(\alpha|m) \). The double eigenvalues embedded in the topmost band \( [E_{2\ell}, \infty) \) (the ‘conduction’ band), namely \( E_{2j} = E_{2j+1}, j > \ell \), are loosely called transcendental eigenvalues. For \( \ell = 1 \) at least, they are known to be transcendental functions of \( m \) (Chudnovskys & Chudnovsky 1980).
There has been much work on algebraizing the integer-ℓ Lamé equation, to facilitate the computation of the band edges and the coefficients of the Lamé polynomials (Alhassid et al. 1983; Turbiner 1989; Li & Kusnezov 1999; Finkel et al. 2000; Li et al. 2000). Such schemes have been extended to the case when ℓ is a half-odd-integer, in which there are an infinite number of lacunae. In this case, certain ‘mid-band’ Bloch functions, namely ones with ξ = ±i and real period 8K, are algebraic functions of sn(α|m). Certain rational values of ℓ with 2ℓ ∉ ℤ also yield algebraic Bloch functions, provided the parameters m and E are chosen appropriately (Maier 2004).

An algebraic understanding of band edges is useful, but it is also desirable to have a closed-form expression for the dispersion relation: k as a function of E. The value of k is not unique, since it can be negated (equivalently ξ → 1/ξ) and any integer multiple of π/K can be added. However, each branch has the property that k ∼ E^{1/2} or k ∼ −E^{1/2} to leading order as E → +∞. Also k ∈ ℜ in each band.

The goal of this paper is the efficient computation of the dispersion relation when ℓ is an integer. The following are illustrations of why this is of importance in theoretical physics. In application (i) above (preheating after inflation), particle production is due to parametric amplification: a solution having a multiplier ξ with |ξ| > 1. This corresponds to the energy E not being at a band edge, or even in a band, but in a lacuna. In application (ii) (the stability analysis of a critical droplet), the analysis includes an imposition of Dirichlet rather than quasi-periodic boundary conditions on an ℓ = 2 Lamé equation (Maier & Stein 2001). The resulting Bloch solution is not a Lamé polynomial, but rather a mid-band solution.

When ℓ is an integer, the Lamé equation is integrable, and the general integral of (1.1) can be expressed in terms of Jacobi theta functions. The dispersion relations k = k_ℓ(E|m), ℓ ≥ 1, can in principle be computed in terms of elliptic integrals. The case ℓ = 1 is by far the easiest. If ℓ = 1, the solution space of (1.1), except when E is at a band edge, will be spanned by the pair of Hermite–Halphen solutions

\[
\Phi(α; ±α_0|m) := \frac{H(α ± α_0|m)}{\Theta(α|m)} \exp[αZ(±α_0|m)].
\]

Here H, Θ, Z are the Jacobi eta, theta and zeta functions with periods 4K, 2K, 2K, respectively, and α_0 ∈ ℂ is defined up to sign by dn^2(α_0|m) = E − m. Hence

\[
k_1(E|m) = −iZ(α_0|m) + π/2K(m), \quad dn^2(α_0|m) = E − m,
\]

up to multivaluedness. This is a parametric dispersion relation. It has been exploited in a study of Wannier–Stark resonances with non-real E and k (Grecchi & Sacchetti 1997; Sacchetti 1997). However, the extension to ℓ > 1 is numerically non-trivial. k_ℓ(E|m) turns out to equal \( \sum_{r=0}^{ℓ−1}[−iZ(α_r|m) + π/2K(m)] \), where \( \{α_r\}_{r=0}^{ℓ−1} \) satisfy coupled transcendental equations involving E and m (Whittaker & Watson 1927, §23.71). Li, Kusnezov & Iachello calculated and graphed k = k_2(E|m) as well as k = k_1(E|m) in the ‘lemniscatic’ case m = 1/2 (Li & Kusnezov 1999; Li et al. 2000). Unfortunately, their graph of k = k_2(E|1/2) is incorrect, as will be shown.

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(b) Overview of results

When $\ell > 1$, we abandon the traditional Hermite–Halphen solutions, and examine instead the implications for the Lamé dispersion relations of what is now called the Hermite–Krichever Ansatz. This is an alternative way of generating closed-form solutions of the Lamé solution at arbitrary energy $E$; for small integer values of $\ell$, at least. Until the 1980s, the only reference for the Ansatz was the classic work of Halphen (1888, ch. XII), who applied it to the cases $\ell = 2, 3, 4$, and in part to $\ell = 5$. Krichever (1980) revived it as an aid in the construction of elliptic solutions of the Korteweg–de Vries and other integrable evolution equations. Belokolos et al. (1986) and Belokolos & Enol’skii (2000) summarize early and recent developments.

The Hermite–Krichever Ansatz is easy to explain, even in the context of the Jacobi form of the Lamé equation, which is not the most convenient for symbolic manipulation. It asserts that for any integer $\ell \geq 1$, fundamental solutions of (1.1) can be constructed from the $\ell = 1$ solutions $\Phi(\alpha; \pm \alpha_0|m)$ as finite series of the form

$$\sum_{j=0}^{N_\ell} C_{j}^{(\ell)} \frac{d^j}{d\alpha^j} \Phi(\alpha; \pm \alpha_0|m) \exp(\pm \kappa_\ell \alpha),$$

(1.5)

where the parameter $\alpha_0$ is now computed from a reduced energy $E_\ell$ by the formula $dn^2(\alpha_0|m) = E_\ell - m$. The reduced energy $E_\ell$, the exponent $\kappa_\ell$ and the coefficients $\{C_{j}^{(\ell)}\}$ will depend on $E$ and $m$. $E_\ell(E|m)$ may be chosen to be rational in $E$ and $m$, and $\pm \kappa_\ell$ to be of the form $\hat{\kappa}_\ell(E|m) \times \pm i \sqrt{\hat{L}_\ell(E|m)}$, where $\hat{\kappa}_\ell(E|m)$ is also rational and $\hat{L}_\ell(E|m)$ is the spectral polynomial $\prod_{i=0}^{2\ell} |E - \lambda_i(m)|$, a degree-$(2\ell + 1)$ polynomial in $E$ the coefficients of which, as noted, are rational in $m$.

If the Lamé equation can be integrated in the framework of the Hermite–Krichever Ansatz, it follows from (1.5) that up to sign, etc.,

$$k_\ell(E|m) = k_1(E_\ell(E|m)|m) + \hat{\kappa}_\ell(E|m) \sqrt{\hat{L}_\ell(E|m)}.$$

(1.6)

The dispersion relation for any integer $\ell$ can be expressed in terms of the $\ell = 1$ relation. To compute $k_\ell$, only one transcendental function (i.e. $k_1$) needs to be evaluated, since the other functions in (1.6) are elementary. The only difficult matter is choosing the relative sign of the two terms, since each is defined only up to sign. The functions $E_\ell(E|m)$, $\hat{\kappa}_\ell(E|m)$ are rational with integer coefficients, but working them out when $\ell$ is large is a lengthy task. In principle, one can write down a recurrence relation for the coefficients $\{C_{j}\}$ and work out $E_\ell(E|m)$, $\hat{\kappa}_\ell(E|m)$ from the condition that the series terminate. However, their numerator and denominator degrees grow quadratically as $\ell$ increases. This explains why Halphen’s treatment of the $\ell = 5$ case was only partial. In a series of papers, Kostov, Enol’skii and collaborators used computer algebra systems to perform a full analysis of the cases $\ell = 2, 3, 4, 5$ (Gerdt & Kostov 1989; Kostov & Enol’skii 1993; Eilbeck & Enol’skii 1994; Enol’skii & Kostov 1994). When $\ell = 5$, using Mathematica to compute the integer coefficients of rational functions equivalent to $E_\ell(E|m)$, $\hat{\kappa}_\ell(E|m)$ required 7 h of time on a Sparc-1, a Unix workstation of that era (Eilbeck & Enol’skii 1994). Until now, their analysis has not been extended to higher $\ell$.

While performing extensive symbolic computations, we recently made a discovery, which is formalized in the central result of this paper, theorem 4.1.
below. For all integer $\ell \geq 2$, the degree-$\ell$ Lamé equation can be integrated in the framework of the Hermite–Krichever Ansatz, and the rational functions that perform the reduction to the $\ell = 1$ case can be computed by simple formulae from certain spectral polynomials of the degree-$\ell$ equation, which are relatively easy to work out. These are the usual spectral polynomial $\prod_{s=0}^{2\ell}[E - E_s(m)]$ associated with the band-edge solutions, and the spectral polynomials associated with two other types of closed-form solution that have not previously been studied in the literature. We call them twisted and theta-twisted Lamé polynomials. In the context of the Jacobi form, the former are polynomials in $\mathsf{sn}(\alpha|m)$, $\mathsf{cn}(\alpha|m)$ and $\mathsf{dn}(\alpha|m)$, multiplied by a factor $\exp(\kappa \alpha)$. (If $\kappa \in \mathbb{R}$, ‘canted’ would be better than ‘twisted’). The latter contain a factor resembling (1.3).

Theorem 4.1 follows from modern finite-band integration theory: specifically, from the Baker–Akhiezer uniformization of the relation between the energy and the crystal momentum. This uniformization is closely tied to classical work on the Lamé equation (the parametrized Baker–Akhiezer solutions of the integer-$\ell$ Lamé equation are in fact equivalent to the Hermite–Halphen solutions). Theorem 4.1 greatly simplifies the computation of higher $\ell$ dispersion relations. It also has implications for the theory of hyperelliptic reduction: the reduction of hyperelliptic integrals to elliptic ones (Belokolos et al. 1986). This is on account of the following. For any $\ell \geq 1$ and $m$, the solutions of the Lamé equation, both the Hermite–Halphen solutions and those derived from the Hermite–Krichever Ansatz, are single-valued functions not of $E$, but rather of a point $(E, \tilde{v})$ on the $\ell$th Lamé spectral curve: a hyperelliptic curve comprising all $(E, \tilde{v})$ satisfying

$$\tilde{v}^2 = \prod_{s=0}^{2\ell}[E - E_s(m)].$$

The fact that $\tilde{v} = \tilde{v}(E)$ is two-valued (except at a band edge) is responsible for the two-valuedness of, e.g. the parameter $\alpha_0 = \alpha_0(E)$ of (1.3), and in general, for the uncertainty in the sign of $k$. The $\ell$th spectral curve generically has genus $\ell$ and may be denoted $\tilde{\Gamma}_\ell := \tilde{\Gamma}_\ell(m)$. For any integer $\ell \geq 2$, the $E \mapsto \mathcal{E}_\ell(E|m)$ reduction map of the Hermite–Krichever Ansatz induces a covering $\pi_\ell : \tilde{\Gamma}_\ell \rightarrow \tilde{\Gamma}_1$. The first known covering of an elliptic curve by a higher genus hyperelliptic curve was constructed by Legendre and generalized by Jacobi (Belokolos et al. 1986, §2). However, it is difficult to enumerate such coverings or even work out explicit examples. Those generated by the Ansatz applied to the Lamé equation are a welcome exception.

The integral with respect to $E$ of any rational function of $E$ and $\sqrt{\Pi(E)}$, where $\Pi$ is a polynomial, is a line integral on the algebraic curve defined by $\tilde{v}^2 = \Pi(E)$. The covering $\pi_\ell : \tilde{\Gamma}_\ell \rightarrow \tilde{\Gamma}_1$, a formula for which is provided by theorem 4.1, reduces certain such hyperelliptic integrals to elliptic ones. In modern language, the theorem specifies how certain holomorphic differentials on hyperelliptic curves can arise as pullbacks of holomorphic differentials on elliptic curves.

As part of our analysis, we investigate the degeneracies of the band edges $\{E_s(m)\}_{s=0}^{2\ell}$ that occur when the modular parameter $m$ is non-real. Such level crossings were first considered by Cohn (1888) in a dissertation that seems not to have been followed up, though it was later cited by Whittaker & Watson (1927, §23.41). We conjecture a formula for the $\ell$-dependence of the number of values of $m \in \mathbb{C} \setminus \{0, 1\}$, or equivalently the number of values of the Klein invariant $J \in \mathbb{C}$, at which two band edges coincide. When this occurs, the genus of the hyperelliptic spectral curve $\tilde{\Gamma}_\ell$ is reduced from $\ell$ to $\ell - 1$, though its arithmetic genus remains

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equal to \( \ell \). Band-edge degeneracies are responsible for a fact discovered by Turbiner (1989): if \( \ell \geq 2 \), the complex curve comprising all points \( \{(m, E_s(m))\}_{s=0}^{2\ell} \) in \( \mathbb{C} \setminus \{0,1\} \times \mathbb{C} \) has only four, rather than \( 2\ell + 1 \), connected components.

This paper is organized as follows. Section 2 introduces the Lamé equation in its elliptic-curve form and relates the Hermite–Halphen solutions to the Baker–Akhiezer function. In §3, Lamé polynomials in the context of the elliptic-curve form are classified. In §4, the Hermite–Krichever Ansatz is introduced, and our key result, theorem 4.1, is stated and proved. The application to hyperelliptic reduction is covered in §5. In §§6 and 7, dispersion relations are worked out and the previously mentioned dispersion relation for the case \( \ell = 2 \) is corrected. The \( \ell = 3 \) dispersion relation is graphed as well. Finally, an area for future investigation is mentioned in §8.

2. The elliptic-curve algebraic form

In §§3–6, we use exclusively what we call the elliptic-curve algebraic form of the Lamé equation, which is the most convenient for symbolic computation. In this section, we derive it and also define a fundamental multivalued function \( \Phi \), which appears in the elliptic-curve version of both the Hermite–Halphen solutions and the Hermite–Krichever Ansatz.

(a) An elliptic-curve Schrödinger equation

Many algebraic forms can be obtained from (1.1) by changing to new independent variables that are elliptic functions of \( \alpha \), such as \( \text{sn}(\alpha|m) \) (e.g. Arscott & Khabaza 1962, §1.1; Arscott 1964, pp. 192–3). A form in which the domain of definition is explicitly a cubic algebraic curve of genus 1, i.e. a cubic elliptic curve, can be obtained as follows. First, the Lamé equation is restated in terms of the Weierstrassian function \( \wp = \wp(u; g_2, g_3) \). This is the canonical elliptic function with a double pole at \( u = 0 \), satisfying \( (\wp')^2 = f(\wp) \) where
\[
f(x) := 4x^3 - g_2x - g_3 = 4 \prod_{\gamma=1}^{3}(x - e_\gamma).
\]
For ellipticity, the roots \( \{e_\gamma\}_{\gamma=1}^{3} \) must be distinct, which is equivalent to the condition that the modular discriminant \( \Delta := g_3^3 - 27g_2^2 \) be non-zero. Either of \( g_2, g_3 \in \mathbb{C} \) may equal zero, but not both.

The relation between the Jacobi and Weierstrassian elliptic functions is well known (Abramowitz & Stegun 1965, §18.9). Choose \( \{e_\gamma\}_{\gamma=1}^{3} \) according to
\[
(e_1, e_2, e_3) = A^2 \left( \frac{2 - m}{3}, \frac{2m - 1}{3}, \frac{-(m + 1)}{3} \right),
\]
where \( A \in \mathbb{C} \setminus \{0\} \) is any convenient proportionality constant. Then
\[
g_2 = A^4 \frac{4(m^2 - m + 1)}{3}, \quad g_3 = A^6 \frac{4(m - 2)(2m - 1)(m + 1)}{27},
\]
and the dimensionless (\( A \)-independent) Klein invariant \( J := g_2^3/\Delta \) will be given by
\[
J = \frac{4}{27} \frac{(m^2 - m + 1)^3}{m^2(1-m)^2}.
\]
The two sorts of elliptic function will be related by, e.g.
\[ \text{sn}^2(Az|m) = \frac{e_1 - e_3}{\wp(z) - e_3}, \quad \text{ns}^2(Az|m) = \frac{\wp(z) - e_3}{e_1 - e_3}, \quad \text{(2.4)} \]
and the periods of \( \wp \), denoted \( 2\omega, 2\omega' \), will be related to those of \( \text{sn}^2 \) by
\[ 2\omega = 2K/A, \quad 2\omega' = 2iK'/A. \quad \text{(2.5)} \]
The case when \( 2K, 2K' \) are real, or equivalently \( \omega \in \mathbb{R}, \omega' \in i\mathbb{R} \) (we assume \( A \in \mathbb{R} \)), is the case when \( g_2, g_3 \in \mathbb{R} \) and \( \Delta > 0 \) (Abramowitz & Stegun 1965, §18.1).

Choosing for simplicity \( A = 1 \), so that \( e_1 - e_3 = A^2 = 1 \), and rewriting the Lamé equation (1.1) with the aid of (2.4), yields the Weierstrassian form
\[ \left\{ \frac{d^2}{du^2} - \left[ \ell(\ell + 1)\wp(u; g_2, g_3) + B \right] \right\}\psi = 0, \quad \text{(2.6)} \]
where \( u := \alpha + iK' \). (The translation of (1.1) by \( iK' \) replaces \( m\text{sn}^2 \) by \( \text{ns}^2 \).) Here, \( B := -E(e_1 - e_3) - \ell(\ell + 1)e_3 \), i.e.
\[ B := -E + \frac{1}{3}\ell(\ell + 1)(m + 1), \quad \text{(2.7)} \]
is a transformed energy parameter. Changing to the new independent variable \( x := \wp(u; g_2, g_3) \) converts (2.6) to the commonly encountered algebraic form
\[ \left\{ \frac{d^2}{dx^2} + \frac{1}{2} \sum_{\gamma=1}^{3} \frac{1}{x - e_\gamma} \frac{d}{dx} - \frac{\ell(\ell + 1)x + B}{4\prod_{\gamma=1}^{3}(x - e_\gamma)} \right\}\psi = 0. \quad \text{(2.8)} \]
This is a differential equation on the Riemann sphere \( \mathbb{P}^1 := \mathbb{C} \cup \{\infty\} \) with regular singular points at \( x = e_1, e_2, e_3, \infty \). Any solution of the original Lamé equation (1.1) or the Weierstrassian form (2.6), which is quasi-periodic in the sense that it is multiplied by \( \xi, \xi' \in \mathbb{C} \setminus \{0\} \) when \( \alpha \leftarrow \alpha + 2K \) or \( \alpha \leftarrow \alpha + 2iK' \), respectively (equivalently when \( u \leftarrow u + 2\omega \) or \( u \leftarrow u + 2\omega' \)), will be a path-multiplicative function of \( x \). In other words, it will be multiplied by \( \xi, \xi' \) when it is analytically continued around a cut joining the pair of points \( x = e_2, e_3 \) or \( x = e_1, e_2 \), respectively. In the context of the algebraic form, the dispersion relation is still a relation between the energy and a multiplier \( \xi \), but the multiplier is interpreted as specifying not quasi-periodicity on \( \mathbb{C} \), but rather multivaluedness on \( \mathbb{P}^1 \).

The algebraic form (2.8) of the Lamé equation lifts naturally to the complex elliptic curve \( E_{g_2, g_3} := \{(x, y) \in \mathbb{C}^2|y^2 = f(x)\} \cup \{\infty, \infty\} \) over \( \mathbb{P}^1 \), parametrized by \( (x, y) := (\wp(u), \wp'(u)) \). One may rewrite (2.8) in the elliptic-curve algebraic form
\[ \left\{ \left( \frac{d}{dx} \right)^2 - \left[ \ell(\ell + 1)x + B \right] \right\}\psi = 0. \quad \text{(2.9)} \]
This is an elliptic-curve Schrödinger equation of the form
\[ \left[ -\left( \frac{d}{dx} \right)^2 + q(x) \right]\psi = -B\psi, \quad \text{(2.10)} \]
with the (rational) potential function \( q(x) \) taken to equal \( \ell(\ell + 1)x \). Equation (2.9) follows directly from (2.6), since \( d/du = \varphi' \ d/d\varphi = y \ d/dx \). It is a differential equation on \( E_{g_2,g_3} \) with a single singular point: a regular one at \( (x, y) = (\infty, \infty) \). Note that the two-to-one covering map \( \pi : E_{g_2,g_3} \to \mathbb{P}^1 \) defined by \( \pi(x, y) = x \) has \( \{(e_\gamma, 0)\}_{\gamma=1}^3 \) and \( (\infty, \infty) \) as simple critical points. One reason why (2.9) is more fundamental than (2.8) is that the singular points of (2.8) at \( x = e_1, e_2, e_3 \) can be regarded as artefacts: consequences of \( \{(e_\gamma, 0)\}_{\gamma=1}^3 \) being critical points of \( \pi \).

The complex analytic differential geometry of the elliptic curve \( E_{g_2,g_3} \) takes a bit of getting used to. Both \( x \) and \( y \) are meromorphic \( \mathbb{P}^1 \)-valued functions on \( E_{g_2,g_3} \), and the only pole that either has on \( E_{g_2,g_3} \) is at the point \( O := (\infty, \infty) \). In a neighbourhood of any generic point \( (x, y) \) other than \( O \) and the three points \( (e_\gamma, 0) \), either \( x \) or \( y \) will serve as a local coordinate. However, near each \( (e_\gamma, 0) \) only \( y \) will be a good local coordinate, since \( dy/dx \) diverges at \( x = e_\gamma \). In addition, \( x \) has a double and \( y \) has a triple pole at \( O \), so the appropriate local coordinate near \( O \) is the quotient \( x/y \). The 1-form \( dx/y \) is not merely meromorphic but holomorphic, with no poles on \( E_{g_2,g_3} \). Its dual is the vector field (or directional derivative) \( y \ d/dx \).

Elliptic functions, i.e. doubly periodic functions, of the original variable \( u \in \mathbb{C} \) correspond to single-valued functions on \( E_{g_2,g_3} \). These are rational functions of \( x \) and \( y \) and may be written as \( R_0(x) + R_1(x)y \), i.e. \( R_0(\varphi(u)) + R_1(\varphi(u)) \varphi'(u) \). The formula \( (y \ d/dx)y = 6x^2 - (1/2)g_2 \) allows such functions to be differentiated algebraically. In a similar way, quasi-doubly periodic functions of \( u \) (sometimes called elliptic functions of the second kind), which are multiplied by \( \xi \) when \( u \leftarrow u + 2\omega \) and by \( \xi' \) when \( u \leftarrow u + 2\omega' \), correspond to multiplicatively multivalued functions on \( E_{g_2,g_3} \).

\( E_{g_2,g_3} \) has genus 1 and is topologically a torus. A fundamental pair of loops that cannot be shrunk to a point may be chosen to be a loop that extends between \( (e_2, 0) \) and \( (e_3, 0) \), and one that extends between \( (e_1, 0) \) and \( (e_2, 0) \), with (if \( g_2, g_3 \) are real, at least) half of each loop passing through positive values of \( y \), and the other half through negative values. One way of constructing an elliptic function of the second kind is to anti-differentiate a rational function \( R(x, y) \). Here, ‘anti-differentiate’ means to compute \( \int R(x, y) \ dx/\ y \), its indefinite integral against the holomorphic 1-form \( dx/\ y \). The resulting function will typically have a non-zero modulus of periodicity associated with each loop. If so, exponentiating it will yield a multiplicatively multivalued function on \( E_{g_2,g_3} \), with non-unit multipliers \( \xi, \xi' \).

(b) The Hermite–Halphen solutions

The fundamental multivalued function \( \Phi \) on the elliptic curve will now be defined. It is an elliptic-curve version of Halphen’s l’élément simple (Halphen 1888).

**Definition 2.1.** On the elliptic curve \( E_{g_2,g_3} \), the multivalued meromorphic function \( \Phi \), parametrized by \( (x_0, y_0) \in E_{g_2,g_3} \setminus \{(\infty, \infty)\} \), is defined up to a constant factor by a formula containing an indefinite elliptic integral,

\[
\Phi(x, y; x_0, y_0) = \exp \left[ \frac{1}{2} \int \left( \frac{y + y_0}{x - x_0} \right) \frac{dx}{y} \right].
\]  

(2.11)

Its multivaluedness, which is multiplicative, arises from the path of integration winding around \( E_{g_2,g_3} \) in any combination of the two directions. Each branch of \( \Phi \) has a simple zero at \( (x, y) = (x_0, y_0) \) and a simple pole at \( (x, y) = (\infty, \infty) \).
To motivate the definition of $\Phi$, a brief sketch will now be given of the construction of the Hermite–Halphen solutions of the elliptic-curve algebraic Lamé equation (2.9) for integer $\ell \geq 1$. The standard published exposition is not fully algebraic, being framed largely in the context of the Weierstrassian form (Whittaker & Watson 1927, §23.7). This sketch will relate the Hermite–Halphen solutions to modern finite-band integration theory and the concept of a Baker–Akhiezer function (Treibich 2001; Gesztesy & Holden 2003). The starting point is the differential equation

$$\left\{ \left( y \frac{d}{dx} \right)^3 - 4[q(x) + B]\left( y \frac{d}{dx} \right) - 2 \left[ \left( y \frac{d}{dx} \right) q(x) \right] \right\} \mathcal{F} = 0. \quad (2.12)$$

The differential operator in (2.12) is the ‘symmetric square’ of the elliptic-curve Schrödinger operator of (2.10), so the solutions of (2.12) include the product of any pair of solutions of (2.10). If the potential function $q(x)$ is rational, it is known that the solution space of (2.12) contains a function $\mathcal{F}(x; B)$ which is (i) meromorphic in $x$ (the only poles being at the poles of $q(x)$), and (ii) monic polynomial in $B$, if and only if (2.10) is a finite-band Schrödinger equation. If the potential function $q(x)$ is finite-band, let $\mathcal{F}(x; B; g_2, g_3)$ be denoted the $\ell$th Hermite–Halphen polynomial. It may be written in both $x$ and $B$ (table 1).

In the case of a general rational potential $q(x)$ that is finite-band, let $\mathcal{F}(x; B)$ denote the specified solution of (2.12), and let its degree in $B$ be denoted $\ell$. It follows from manipulations parallel to those of Whittaker & Watson that the function $\Psi$ on $E_{g_2, g_3}$ defined by a formula containing an indefinite integral,

$$\Psi(x, y; B, \nu) := \exp \int \frac{\left[ \frac{1}{2} \mathcal{F}^2(x; B)y - \nu \right]}{\mathcal{F}(x; B)} \frac{dx}{y}, \quad (2.13)$$

will be a solution of the Schrödinger equation (2.10). Here, $B \in \mathbb{C}$ and $\nu$ is a $B$-dependent but position-independent quantity, determined only up to sign, that is computed by what Whittaker & Watson call an ‘interesting formula’,

$$v^2 = -\frac{1}{2} \mathcal{F} \left( y \frac{d}{dx} \right)^2 \mathcal{F} + \left[ \frac{1}{2} \left( y \frac{d}{dx} \right) \mathcal{F} \right]^2 + [q(x) + B] \mathcal{F}_\ell^2. \quad (2.14)$$

Table 1. Hermite–Halphen polynomials (van der Waall 2002, table A.2).
(It is not obvious that the right-hand side is independent of the point \((x, y) \in E_{g_2, g_3}\).) It is widely known (Smirnov 2002) that \(a\) is identical to the coordinate \(n\) on the spectral curve \(\Gamma_{\ell}\) defined by \(n^2 = \prod_{k=0}^{2\ell}(B - B_k)\), where \(\{B_k\}_{k=0}^{2\ell}\) are the band edges of the Schrödinger operator; though no really simple proof of this fact seems to have been published. A consequence of this is that the formula (2.13) parametrizes solutions of the elliptic-curve Schrödinger equation (2.10) by \((B, \nu) \in \Gamma_{\ell} \setminus \{(-\infty, \infty)\}\). As defined, \(\Psi\) is called a Baker–Akhiezer function (Krichever 1990).

Consider now the special case of the integer-\(\ell\) Lamé equation (2.9). In this case, the function \(\Psi\) computed by (2.13) from the Hermite–Halphen polynomial \(\mathcal{F} = \mathcal{F}_{\ell}(x; B; g_2, g_3)\) is in fact an Hermite–Halphen solution of the Lamé equation, re-expressed in terms of the elliptic curve coordinates \((x, y)\). One can write \(\Psi = \Psi_{\ell}^{\pm}(x, y; B; g_2, g_3)\), where the superscript ‘\(\pm\)’ refers to the ambiguity in the sign of \(\nu = \nu(B)\). If \(\nu \neq 0\), the two solutions \(\Psi_{\ell}^{\pm}\) are distinct. They are path-multiplicative, since they are exponentials of anti-derivatives of rational functions on \(E_{g_2, g_3}\).

It should be noted that the Hermite–Halphen polynomials are not merely a tool for generating the solutions \(\Psi_{\ell}^{\pm}\) of the Lamé equation. They are algebraically interesting in their own right. Klein (1892, figs 1 and 2) supplies a sketch of the real portion of the curve \(\mathcal{F}_{\ell}(x; B) = 0\) when \(\ell = 5, 6\), showing how when \(\ell \geq 4\), \(B\) is a band edge only if \(\mathcal{F}_{\ell}(x; B)\), regarded as a polynomial in \(x\), has a double root.

The relevance of the fundamental multivalued function \(\Phi\) can now be explained. It follows from (2.13), and the fact that \(\mathcal{F}_1 = B - x\) (table 1), that

\[
\Psi_{1}^{\pm}(x, y; B; g_2, g_3) = \Phi\left(x, y, B; \pm \sqrt{4B^3 - g_2 B - g_3}\right). \tag{2.15}
\]

In other words, if \((x_0, y_0) \in E_{g_2, g_3}\) is ‘above’ \(x_0 = B\), then \(\Phi(\cdot, \cdot; x_0, y_0)\) will be a solution of the \(\ell = 1\) Lamé equation in the form (2.9). There are two such points, related by \(y_0\) being negated, unless \(4B^3 - g_2 B - g_3 = 0\), i.e. unless \(B = e_1, e_2, e_3\), in which case \(y_0 = 0\) is the only possibility. These are the three band-edge values of \(B\) for \(\ell = 1\).

It is not difficult to show that the \(\ell = 1\) Hermite–Halphen solutions (2.15) are identical to the solutions (1.3), though they are expressed as functions of the variable \((x, y) \in E_{g_2, g_3}\) rather than the original independent variable \(\alpha \in \mathbb{C}\). The parametrizing point \((x_0, \pm y_0) \in E_{g_2, g_3}\) corresponds to the parameter \(\pm \alpha_0 \in \mathbb{C}\) of (1.3). These solutions are clearly easier to formulate in the elliptic curve context.

For any integer \(\ell\), the Lamé dispersion relation can be computed numerically from (2.13) by calculating the multiplier arising from the path of integration winding around \(E_{g_2, g_3}\). However, (2.13) is not adapted to symbolic computation. By expanding the integrand in partial fractions, one can derive the remarkable formula

\[
\Psi_{\ell}^{\pm}(x, y; B; g_2, g_3) = \prod_{r=1}^{\ell} \Phi\left(x, y, x_r, y_r^{\pm}\right), \tag{2.16}
\]

where \(\{(x_r, y_r^{\pm})\}_{r=1}^{\ell}\) are points on \(E_{g_2, g_3}\) above \(\{x_r\}_{r=1}^{\ell}\), the \(B\)-dependent roots of the degree-\(\ell\) polynomial \(\mathcal{F}_{\ell}(x; B; g_2, g_3)\) (cf. Whittaker & Watson 1927, §23.7). Unfortunately, when \(\ell \geq 5\), the roots \(\{x_r\}_{r=1}^{\ell}\) cannot be computed in terms of radicals. This reduction to degree-1 solutions is less computationally tractable than the one that will be provided by the Hermite–Krichever Ansatz.

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3. Finite families of Lamé equation solutions

The solutions of the integer-\( \ell \) Lamé equation include the Lamé polynomials, which are the traditional band-edge solutions. In the Jacobi-form context, they are periodic or anti-periodic functions on \([0, 2K]\), with Floquet multiplier \( \xi = \pm 1 \), respectively. There are exactly \( 2\ell + 1 \) values of the spectral parameter \( B \in \mathbb{C} \), i.e. of the energy \( E \), for which a Lamé polynomial may be constructed, the counting being up to multiplicity. By definition, these are the roots of the spectral polynomial \( L_\ell(B; g_2, g_3) \).

As functions on the curve \( E_{g_2,g_3} \), the Lamé polynomials are single or double-valued and are essentially polynomials in the coordinates \( x, y \). (In the Weierstrassian context, \( \wp, \wp' \) substitute for \( x, y \).) However, no fully satisfactory table of the Lamé polynomials or the Lamé spectral polynomials has yet been published. Whittaker & Watson (1927, §23.42) refer to a list of Guerritore (1909) that covers \( \ell \leq 10 \). Sadly, although he produced it as a dissertazione di laurea at the University of Naples, most of his results on \( \ell \geq 5 \) are incorrect. This has long been known (Strutt 1967), but his paper is still occasionally cited for completeness (Gesztesy & Holden 2003). Arscott (1964, §9.3.2) gives a brief table of the Jacobi-form Lamé polynomials, covering only \( \ell = 1, 2, 3 \). His table is correct, with a single misprint (Fernández C. et al. 2000; Finkel et al. 2000). However, its brevity has been misinterpreted. An erroneous belief has arisen that when \( \ell \geq 4 \), the Lamé polynomial coefficients and band-edge energies cannot be expressed in terms of radicals. This sets in only when \( \ell \geq 8 \).

Owing to these confusions, we tabulate the Lamé polynomials and the spectral polynomials \( L_\ell(B; g_2, g_3) \) in this section. Both are computed from coefficient recurrence relations. We supply such relations and tables of spectral polynomials for the twisted and theta-twisted Lamé polynomials as well. The number of values of \( B \in \mathbb{C} \) for which the latter two sorts of solution exist, i.e. the degrees of their spectral polynomials, will be given. All three sorts of solution will play a role in our key result, theorem 4.1. In fact, all will be special cases of the solutions constructed for arbitrary \( B \) by the Hermite–Krichever Ansatz.

When \( \ell \geq 2 \), many of the spectral polynomials will have degenerate roots if \( g_2, g_3 \in \mathbb{C} \) are appropriately chosen. This means that, for example, a pair of the \( 2\ell + 1 \) band-edge energies can be made to coincide by moving the modular parameter \( m \in \mathbb{C}\backslash\{0, 1\} \) to one of a finite set of complex values. We indicate how to calculate these, or more precisely the corresponding values of Klein’s absolute invariant \( J = g_2^3/(g_2^3 - 27g_3^2) \in \mathbb{C} \).

\( J \) is the more fundamental parameter, in algebraic geometry at least, since two elliptic curves are isomorphic (birationally equivalent) if and only if they have the same value of \( J \). The \( m \mapsto J \) correspondence (2.3) maps \( \mathbb{C}\backslash\{0, 1\} \) onto \( \mathbb{C} \), and it also maps \( m \in \mathbb{R}\backslash\{0, 1\} \) (in fact \( m \in (0, 1/2] \)) onto \( J \in [1, \infty) \). Formally it is six-to-one. Each value of \( J \) corresponds to six values of \( m \), with the exception of \( J = 0 \) (i.e. \( g_2 = 0 \)), which corresponds to \( m = (1/2) \pm (\sqrt{3}/2)i \), and \( J = 1 \) (i.e. \( g_3 = 0 \)), which corresponds to \( m = -1, 1/2, 2 \). Elliptic curves with \( J = 0, 1 \) are called equianharmonic and lemniscatic, respectively (Abramowitz & Stegun 1965, §§18.13 and 18.15). Any equianharmonic curve has a triangular period lattice, with \( \omega'/\omega = e^{\pm 2\pi i/3} \), and any lemniscatic curve has a square period lattice, with \( \omega'/\omega = \pm i \).

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(a) Lamé polynomials

The Lamé polynomials are classified into species 1, 2, 3, 4 (Whittaker & Watson 1927, §23.2). This is appropriate for some forms of the Lamé equation, but for the elliptic-curve algebraic form, a more structured classification scheme is better.

Definition 3.1. A solution of the Lamé equation (2.9) on the elliptic curve $E_{q_1,q_2}$ is said to be a Lamé polynomial of Type I if it is single-valued and of the form $C(x)$ or $D(x)\gamma$, where $C, D$ are polynomials. A solution is said to be a Lamé polynomial of Type II, associated with the branch point $e$, of the curve $(\gamma = 1, 2, 3)$, if it is double-valued and of the form $E(x)\sqrt{x-e}$ or $F(x)\gamma/\sqrt{x-e}$, where $E, F$ are polynomials. The subtypes of Types I and II are species 1, 4 and 2, 3, respectively.

To determine necessary conditions on $\ell$ and $B$ for there to be a non-zero Lamé polynomial of each subtype, one may substitute the corresponding expression $(C(x)$, etc.) into the Lamé equation (2.9), and work out a recurrence for the polynomial coefficients. This is similar to the approach of expanding in integer or half-integer powers of $x-\epsilon$. For the Type I solutions at least, the present approach seems more natural, since they are not associated with any singular point $\epsilon$.

If $C(x) = \sum_j c_j x^j$, $D(x) = \sum_j d_j x^j$, $E(x) = \sum_j e_j x^j$ and $F(x) = \sum_j f_j x^j$, substituting the expression for each species of solution into (2.9) and equating the coefficients of powers of $x$ leads to the recurrence relations

$$ (2j - \ell)(2j + \ell + 1)c_j - Bc_{j+1} - (j + 2)\left(j + \frac{3}{2}\right)g_2 c_{j+2} $$
$$ - (j + 2)(j + 3)g_3 c_{j+3} = 0, \quad (3.1) $$

$$ (2j - \ell + 3)(2j + \ell + 4)d_j - Bd_{j+1} - (j + 2)\left(j + \frac{5}{2}\right)g_2 d_{j+2} $$
$$ - (j + 2)(j + 3)g_3 d_{j+3} = 0, \quad (3.2) $$

$$ (2j - \ell + 1)(2j + \ell + 2)e_j + [(4j + 5)e_\gamma - B]e_{j+1} $$
$$ + \left[-\left(j + \frac{5}{2}\right)g_2 + 4e_\gamma^2\right](j + 2)e_{j+2} - (j + 2)(j + 3)g_3 e_{j+3} = 0, \quad (3.3) $$

$$ (2j - \ell + 2)(2j + \ell + 3)f_j + [-(4j + 7)e_\gamma - B]f_{j+1} $$
$$ + \left[-\left(j + \frac{3}{2}\right)g_2 - 4e_\gamma^2\right](j + 2)f_{j+2} - (j + 2)(j + 3)g_3 f_{j+3} = 0. \quad (3.4) $$

It is easy to determine the integers $\ell$ for which $C, D, E, F$ may be a polynomial.

Proposition 3.1. If $\ell \geq 1$ is odd, non-zero Type I Lamé polynomials of the fourth species and Type II ones of the second species can in principle be constructed from these recurrence relations with $\deg D = (\ell - 3)/2$ and $\deg E = (\ell - 1)/2$, respectively. (The former assumes $\ell \geq 3$.) If $\ell \geq 2$ is even, non-zero Type I Lamé polynomials of the first species and Type II ones of the third species can be constructed similarly, with $\deg C = \ell/2$ and $\deg F = (\ell - 2)/2$, respectively.

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The coefficients in each Lamé polynomial are computed from the appropriate recurrence relation by setting the coefficient of the highest power of \( x \) to unity and working downward. Unless \( B \) is specially chosen, the coefficients of negative powers of \( x \) may be non-zero. However, by examination, they will be zero if the coefficient of \( x^{-1} \) equals zero.

**Definition 3.2.** The Type I Lamé spectral polynomial \( L_\ell^I(B; g_2, g_3) \) is the polynomial monic in \( B \) which is proportional to the coefficient \( d_{-1} \) if \( \ell \) is odd, and to \( c_{-1} \) if \( \ell \) is even. (The former assumes \( \ell \geq 3 \); by convention \( L_1^I := 1 \).) The Type II Lamé spectral polynomial \( L_\ell^II(B; e_\gamma, g_2, g_3) \) is similarly obtained from the coefficient \( e_{-1} \) if \( \ell \) is odd and \( f_{-1} \) if \( \ell \) is even. Each spectral polynomial may be regarded as \( \prod_{s}[B - B_s(g_2, g_3)] \), respectively, \( \prod_{s}[B - B_s(e_\gamma, g_2, g_3)] \), where the roots \( \{B_s\} \) are the values of \( B \) for which a Lamé equation solution of the indicated type exists, counted with multiplicity.

By examination, \( N_\ell^I := \deg L_\ell^I \) is \((\ell - 1)/2\) if \( \ell \) is odd, and \( \ell/2 + 1 \) if \( \ell \) is even; and \( N_\ell^II := \deg L_\ell^II \) is \((\ell + 1)/2\) if \( \ell \) is odd and \( \ell/2 \) if \( \ell \) is even. Hence, as expected

\[
L_\ell(B; g_2, g_3) := L_\ell^I(B; g_2, g_3) \prod_{\gamma=1}^3 L_\ell^II(B; e_\gamma, g_2, g_3), 
\]

(3.5)

the full Lamé spectral polynomial, has degree \( N_\ell^I + 3N_\ell^II = 2\ell + 1 \) in \( B \).

It should be noted that \( \prod_{\gamma=1}^3 L_\ell^II(B; e_\gamma, g_2, g_3) \), the full Type II Lamé spectral polynomial, is a function only of \( B; g_2, g_3 \), since any symmetric polynomial in \( e_1, e_2, e_3 \) can be written in terms of \( g_2, g_3 \). For example \( e_1 e_2 e_3 = g_3/4 \). This is why \( e_\gamma \) is absent on the left-hand side of (3.5). When using the recurrences (3.1)–(3.4), one should also note that \( 4e_\gamma^3 - g_2 e_\gamma - g_3 = 0 \), so \( e_\gamma^3 = (1/4)(g_2 e_\gamma + g_3) \). Any polynomial in \( e_\gamma, g_2, g_3 \) can be reduced to one which is of degree at most 2 in \( e_\gamma \), much as any polynomial in \( x, y \) can be reduced to one of degree at most 1 in \( y \).

The Lamé polynomials of Types I and II are listed in table 2 and the corresponding spectral polynomials in table 3. They replace the table of Guerritore (1909), with its many unfortunate errors. The spectral polynomials with \( \ell \leq 7 \) were recently computed by a different technique (van der Waall 2002, table A.3). The table of van der Waall displays the full Type II spectral polynomials, rather than the more fundamental \( e_\gamma \)-dependent polynomials \( L_\ell^II(B; e_\gamma, g_2, g_3) \). The spectral polynomials can also be computed by the technique of Gesztesy & Weikard (1995a), which employs the Weierstrassian counterpart of the Ansatz used by Hermite in his solution of the Jacobi-form Lamé equation (Whittaker & Watson 1927, §23.71).

The roots of the spectral polynomials are the energies \( B \) for which the Lamé polynomials are solutions of the Lamé equation. It is clear that when \( \ell \geq 9 \), the Type II energies cannot be expressed in terms of radicals, since the degree of the spectral polynomial will be 5 or above. When \( \ell = 8 \) or \( \ell \geq 10 \), the Type I energies cannot be so expressed. These statements apply also to the coefficients of the Lamé polynomials, which depend on \( B \). Hence when \( \ell \geq 10 \), the symbolic computation of the Lamé polynomials is impossible, and when \( \ell = 8 \) or 9, it is possible only in part. However, when \( g_2, g_3 \) take on special values, what would otherwise be impossible may become possible. For instance, when \( g_3 = 0 \) (the lemniscatic case, including \( m = 1/2 \)), the quintic spectral polynomial \( L_8^I(B; g_2, g_3) \) reduces to \( B^5 - 1044g_2B^3 + 112320g_2^2B \), the roots of which can obviously be expressed in terms of radicals.

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Table 2. Lamé polynomials of Types I and II.

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>Lamé polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I solution: ( C(x; B, g_2, g_3) ) or ( D(x; B, g_2, g_3) )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( x - (1/6) B )</td>
</tr>
<tr>
<td>2</td>
<td>( y )</td>
</tr>
<tr>
<td>3</td>
<td>( x^2 - (1/14) B x^2 + ((1/280) B^2 - (3/20) g_2) y )</td>
</tr>
<tr>
<td>4</td>
<td>( [x - (1/18) B] y )</td>
</tr>
<tr>
<td>5</td>
<td>( x^3 - (1/22) B x^2 + ((1/792) B^2 - (5/24) g_2) x + (- (1/33264) B^3 + (13/1584) B g_2 - (1/7) g_3) )</td>
</tr>
<tr>
<td>6</td>
<td>( [x^2 - (1/26) B x + (1/1144) B^2 - (5/44) g_2] y )</td>
</tr>
<tr>
<td>7</td>
<td>( x^3 - (1/30) B x^2 + ((1/1560) B^2 - (7/26) g_2) x^2 + (- (1/102960) B^3 + (9/1144) B g_2 - (2/11) g_3) x + (1/741320) B^4 - (7/51480) B^2 g_2 + (7/1320) B g_3 + (7/624) g_2^3 )</td>
</tr>
<tr>
<td>Type II solution: ( E(x; B, e_\gamma, g_2, g_3) \sqrt{x - e_\gamma} ) or ( F(x; B, e_\gamma, g_2, g_3) y / \sqrt{x - e_\gamma} )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \sqrt{x - e_\gamma} )</td>
</tr>
<tr>
<td>2</td>
<td>( y / \sqrt{x - e_\gamma} )</td>
</tr>
<tr>
<td>3</td>
<td>( [x + (- (1/10) B + (1/2) e_\gamma)] \sqrt{x - e_\gamma} )</td>
</tr>
<tr>
<td>4</td>
<td>( [x + (- (1/14) B - (1/2) e_\gamma)] y / \sqrt{x - e_\gamma} )</td>
</tr>
<tr>
<td>5</td>
<td>( [x^2 + (- (1/18) B + (1/2) e_\gamma)] x + ((1/4504) B^2 - (1/36) B e_\gamma + (3/8) e_\gamma^2 - (5/28) g_2)] \sqrt{x - e_\gamma} )</td>
</tr>
<tr>
<td>6</td>
<td>( [x^2 + (- (1/22) B + (1/2) e_\gamma)] x + ((1/792) B^2 + (1/44) B e_\gamma - (1/8) e_\gamma^2 - (1/12) g_2)] y / \sqrt{x - e_\gamma} )</td>
</tr>
<tr>
<td>7</td>
<td>( [x^3 + (- (1/26) B + (1/2) e_\gamma)] x^2 + ((1/1144) B^2 - (1/52) B e_\gamma + (3/8) e_\gamma^2 - (21/88) g_2) x )</td>
</tr>
<tr>
<td>8</td>
<td>( \left[ x^3 + (- (1/30) B - (1/2) e_\gamma) \right] x^2 + ((1/1560) B^2 + (1/60) B e_\gamma - (1/8) e_\gamma^2 - (15/104) g_2) x )</td>
</tr>
<tr>
<td>8</td>
<td>( \left[ x^3 + (- (1/102960) B - (1/240) B e_\gamma + (1/240) B e_\gamma^2 + (127/34320) B g_2 + (47/832) e_\gamma g_2 - (51/704) g_3) \right] y / \sqrt{x - e_\gamma} )</td>
</tr>
</tbody>
</table>

In the context of the Jacobi form, the \( 2 \ell + 1 \) values \( \{ E_x(m) \}^{2 \ell}_{x=0} \) for which a Lamé polynomial solution exists can be thought of as the \( 2 \ell + 1 \) branches of a spectral curve that lies over the triply punctured sphere \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \), the space of values of the modular parameter \( m \). Turbiner (1989) showed that if \( \ell \geq 2 \), this curve has only four connected components, not \( 2 \ell + 1 \). The reason is clear now. These are the Type I component and the three Type II components, one associated with each point \( e_\gamma \). Since each of the four is defined by a polynomial in \( E \) and \( m \), each can be extended to an algebraic curve over \( \mathbb{P}^1 \). At the values \( m = 0, 1, \infty \), the four curves may touch one another. (Refer to Li et al. (2000, fig. 3), for the behaviour of the real portions of the \( \ell = 1, 2 \) curves as \( m \to 0, 1, \) and Alhassid et al. (1983) for \( \ell = 3 \).) These three values of \( m \) correspond to two of \( e_1, e_2, e_3 \) coinciding and the elliptic curve \( y^2 = 4 \prod_{j=1}^{3} (x - e_j) \) becoming rational rather than elliptic. Level crossings of this sort are perhaps less interesting than ‘intra-curve’ ones.

In the present context, \( E \) is replaced by the transformed energy \( B \), and \( m \) by the pair \( g_2, g_3 \) or the Klein invariant \( J \), with \( J = \infty \) corresponding to \( m = 0, 1, \infty \). It is easy to determine which finite values of \( J \) yield coincident values of \( B \). One simply computes the discriminants of the Type I and full Type II spectral polynomials, \( \sum_{\ell=1}^{J} L_\ell^{I}(B; g_2, g_3) \) and \( \prod_{j=1}^{3} L_{\ell}^{II}(B; e_\gamma, g_2, g_3) \). Each discriminant is zero if and only if there is a double root. By using \( J = g_3^2 / (g_3^2 - 27 g_3^2) \), one can eliminate
Table 3. Lamé spectral polynomials of Types I and II. (Most of the ones with $\ell \geq 5$ disagree with those published by Guerritore (1909).)

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>Lamé spectral polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$L_1^\ell(B; g_2, g_3)$</td>
</tr>
<tr>
<td>2</td>
<td>$B^2 - 3 g_2$</td>
</tr>
<tr>
<td>3</td>
<td>$B$</td>
</tr>
<tr>
<td>4</td>
<td>$B^3 - 52 g_2 B + 560 g_3$</td>
</tr>
<tr>
<td>5</td>
<td>$B^2 - 27 g_2$</td>
</tr>
<tr>
<td>6</td>
<td>$B^3 - 294 g_2 B^2 + 7776 g_3 B + 3465 g_3^2$</td>
</tr>
<tr>
<td>7</td>
<td>$B^3 - 196 g_2 B + 2288 g_3$</td>
</tr>
<tr>
<td>8</td>
<td>$B^2 - 1044 g_2 B^3 + 48816 g_3 B^2 + 112320 g_3^2 B - 4665600 g_3 g_2$</td>
</tr>
</tbody>
</table>

$g_2, g_3$ and obtain a polynomial equation for $J$. For each of Types I and II, there are coincident values of $B$ if and only if $J$ is a root of what we shall call a Cohn polynomial.

In Table 4, the Cohn polynomials are listed. Since the coefficients are rather large integers that may have number-theoretic significance, each is given in a fully factored form. An interesting feature of these polynomials is that none has a zero on the real half-line $[1, \infty)$. Since $J \in [1, \infty)$ corresponds to $m \in (0, 1)$, the existence of such a zero would imply that for some $m \in (0, 1)$, two of the $2\ell + 1$ band edges become degenerate. That this cannot occur follows from a Sturmian argument (Whittaker & Watson 1927, §23.41). It also follows from the analysis of Gesztesy & Weikard (1995a, §3).

**Proposition 3.2.** For any integer $\ell \geq 1$, the degeneracies of the algebraic spectrum of the Lamé operator, which comprise the $2\ell + 1$ roots (up to multiplicity) of the spectral polynomial $L_\ell(B; g_2, g_3)$, are fully captured by the Cohn polynomials of Types I and II. As the parameters $g_2, g_3$ are varied, a pair of roots will coincide, reducing the number of distinct roots from $2\ell + 1$ to $2\ell$, if and only if the Klein invariant $J$ is a root of one of the two Cohn polynomials; and there are no multiple coincidences.

This proposition will be proved in §4. The following conjecture is based on a close examination of the spectral and Cohn polynomials for all $\ell \leq 25$.
Conjecture 3.1.

(i) As a polynomial in $J$ with integer coefficients, no Cohn polynomial has a non-trivial factor, except for the Type I Cohn polynomials with $\ell \equiv 2(\text{mod } 3)$, each of which is divisible by $J$. (These factors of $J$ are visible in Table 4.)

(ii) If $N_I^\ell$ and $N_{II}^\ell$ denote the degrees of the spectral polynomials of Types I, II, which are given above, then the Cohn polynomials of Types I, II have degrees $[([N_I^\ell]^2-N_I^\ell+4)/6]$ and $N_{II}^\ell(N_{II}^\ell-1)/2$, respectively. (In the first expression, [...] is the integer part, or ‘floor’ function.)

The conjectured degree formulae constitute a conjecture as to the number of points in elliptic moduli space (elliptic curve parameter space), labelled by $J$, at which the $2\ell+1$ distinct energies in the algebraic spectrum are reduced to $2\ell$. For example, $N_I^3, N_{II}^3 = 1, 2$, so when $\ell = 3$ the Cohn polynomials of Types I, II have degrees 0, 1, respectively. In other words, if $\ell = 3$ there is no Type I polynomial, and the Type II one is linear in the invariant $J$. According to Table 4, it equals $4J+1$. A non-degeneracy condition equivalent to the linear condition $4J+1 \neq 0$ was previously worked out by Treibich (1994, §6.6), namely $\prod_{g=1}^{3}(5g_2 - 12e_{g_2}^2) \neq 0$.

The remarks regarding extra $J$ factors amount to a conjecture that in the equianharmonic case $m = (1/2) \pm (\sqrt{3}/2)i$ (i.e. $J = 0$ or $g_2 = 0$), there are only $2\ell$ distinct energies if and only if $\ell \equiv 2(\text{mod } 3)$. For those values of $\ell$, the double energy eigenvalue is evidently located at $B=0$. It should be mentioned that when $J = 0$ and $\ell \equiv 0(\text{mod } 3)$, there is also an eigenvalue at $B=0$, but it is a simple one.
A periodicity of length 3 in \( \ell \) is present in the equianharmonic case of a third-order equation resembling (2.12), now called the Halphen equation (Halphen 1888, pp. 571–4). By the preceding, a similar periodicity appears to be present in the equianharmonic case of the Lamé equation. This was not previously realized.

(b) Twisted Lamé polynomials

The twisted Lamé polynomials are exponentially modified Lamé polynomials. They will play a major role in theorem 4.1 and in the hyperelliptic reductions following from the Hermite–Krichever Ansatz, but they are of independent interest.

**Definition 3.3.** A solution of the Lamé equation (2.9) on the elliptic curve \( E_{q_2,q_1} \) is said to be a twisted Lamé polynomial, of Type I, or of Type II associated with the point \( e_\gamma \), if it has the respective form

\[
\left\{ \frac{C(x) + D(x) y}{E(x)\sqrt{x - e_\gamma} + F(x) y/\sqrt{x - e_\gamma}} \right\} \times \exp \left[ \kappa \int \frac{dx}{y} \right],
\]

with \( \kappa \in \mathbb{C} \) non-zero. Here \( C, D, E, F \) are polynomials.

On the level of differential equation solutions, there is little to distinguish between twisted Lamé polynomials and ordinary Lamé polynomials, which are simply twisted polynomials with \( \kappa = 0 \). A function \( \Psi(x, y) = \Psi(x, y)\exp[\kappa \int dx/y] \) will be a solution of the Lamé equation (2.9) if and only if \( \Psi \) satisfies

\[
\left\{ \left( y \frac{d}{dx} \right)^2 + 2\kappa \left( y \frac{d}{dx} \right) - [\ell(\ell + 1)x + B - \kappa^2] \right\}\Psi = 0. \tag{3.6}
\]

This is a differential equation on \( E_{q_2,q_1} \) that generalizes but strongly resembles (2.9). It has a single singular point, \( (x, y) = (\infty, \infty) \), the characteristic exponents of which are \{\( -\ell, \ell + 1 \)\}. Like \( B, \kappa \) is an accessory parameter that does not affect these exponents. The values \( (B, \kappa) \) for which a twisted or conventional Lamé polynomial solution of (2.9) exists can be viewed as the points in a two-dimensional parameter space at which (3.6) has single- or double-valued solutions.

If \( C(x) = \sum_j c_j x^j \), \( D(x) = \sum_j d_j x^j \), \( E(x) = \sum_j e_j x^j \) and \( F(x) = \sum_j f_j x^j \), substituting the expression for each type of twisted polynomial solution into (2.9) and equating the coefficients of powers of \( x \) yields the coupled pairs of recurrences

\[
(2j - \ell)(2j + \ell + 1)c_j + (\kappa^2 - B)c_{j+1} - (j + 2)\left( j + \frac{3}{2} \right) g_2 c_{j+2} = 0,
\]

\[
-(j + 2)(j + 3)g_3 c_{j+3} + 2\kappa(4j + 2) d_{j-1} - 2\kappa \left( j + \frac{3}{2} \right) g_2 d_{j+1} = 0,
\]

\[
-2\kappa(j + 2)g_3 d_{j+2} = 0, \tag{3.7}
\]

\[
(2j - \ell + 3)(2j + \ell + 4) d_j + (\kappa^2 - B) d_{j+1} - (j + 2) \left( j + \frac{5}{2} \right) g_2 d_{j+2} = 0,
\]

\[
-(j + 2)(j + 3)g_3 d_{j+3} + 2\kappa(j + 2) c_{j+2} = 0; \tag{3.8}
\]
is the proper handling of the case 
computed by polynomial elimination, e.g. by computing resultants. A minor problem
respectively, $C$ and $F$, and working downward
are computed by setting the coefficient of the highest power of $x$ to unity in $D$ and $E$,
odd, respectively, even, then
degrees are 12 and 16). The following proposition will be proved in

\[ (2j - \ell + 1)(2j + \ell + 2)e_j + [(4j + 5)e_\gamma + k^2 - B]e_{j+1} \]
\[ + \left[-\left(j + \frac{5}{2}\right)g_2 + 4e_\gamma^2\right](j + 2)e_{j+2} - (j + 2)(j + 3)g_3e_{j+3} \]
\[ + 2\kappa(4j + 4)f_j + 2\kappa(4j + 6)e_\gamma f_{j+1} + 2\kappa(4j + 8)\left(e_\gamma^2 - \frac{1}{4}g_2\right)f_{j+2} = 0, \quad (3.9) \]
\[ (2j - \ell + 2)(2j + \ell + 3)f_j + [-4j + 7)e_\gamma + k^2 - B]f_{j+1} \]
\[ + \left[-\left(j + \frac{3}{2}\right)g_2 - 4e_\gamma^2\right](j + 2)f_{j+2} - (j + 2)(j + 3)g_3f_{j+3} \]
\[ + 2\kappa\left(j + \frac{3}{2}\right)e_{j+1} - 2\kappa(j + 2)e_\gamma e_{j+2} = 0. \quad (3.10) \]

It is easy to determine the maximum value of the exponent $j$ in each of $C$, $D$, $E$, $F$.

**Proposition 3.3.** Non-zero twisted Lamé polynomials of Type I (if $\ell \geq 3$) and of
Type II (if $\ell \geq 2$) can in principle be constructed from these recurrence relations. If $\ell$ is odd, respectively, even, then $\deg C, D; E, F$ are $(\ell - 1)/2, (\ell - 3)/2;$
$(\ell - 1)/2, (\ell - 3)/2$, respectively, $\ell/2, (\ell - 4)/2; \ell/2 - 1, \ell/2 - 1$. The coefficients are computed by setting the coefficient of the highest power of $x$ to unity in $D$ and $E$,
respectively, $C$ and $F$, and working downward.

Unless $B, \kappa$ are specially chosen, the coefficients of negative powers of $x$ may be
non-zero. However, by examination, they will be zero if the coefficients of $x^{-1}$ in $C$ and
$D$ (for Type I) or $E$ and $F$ (for Type II), equal zero. $c_{-1} = 0, d_{-1} = 0$ and $e_{-1} = 0,$
$f_{-1} = 0$, are coupled polynomial equations in $B, \kappa$, and their solutions may be
computed by polynomial elimination, e.g. by computing resultants. A minor problem
is the proper handling of the case $\kappa = 0$, in which (3.7)–(3.10) reduce to (3.1)–(3.4). If
$\ell$ is odd, respectively, even, then $c_{-1}$ and $f_{-1}$, respectively, $d_{-1}$ and $e_{-1}$, turn out to be
divisible by $\kappa$. By dividing the appropriate equations by $\kappa$ before solving each pair of
coupled equations, the spurious $\kappa = 0$ solutions can be eliminated.

**Definition 3.4.** The Type I twisted Lamé spectral polynomial $Lt^I_\ell(B; g_2, g_3)$ is
the polynomial monic in $B$ which is proportional to the resultant of $c_{-1}, d_{-1}$ with
respect to $\kappa$, with $\kappa$ factors removed as indicated. (This assumes $\ell \geq 3$; by
convention $Lt^I_1 = Lt^I_3 := 1$.) The Type II twisted Lamé spectral polynomial
$Lt^II_\ell(B; \gamma, g_2, g_3)$ is similarly obtained from $e_{-1}, f_{-1}$, (This assumes $\ell \geq 2$; by
convention $Lt^II_1 := 1$.) Each twisted spectral polynomial may be regarded as
$\prod_s[B - B_s(g_2, g_3)]$, respectively, $\prod_s[B - B_s(\gamma, g_2, g_3)]$, where the roots \{Bs\} are
the values of $B$ for which a Lamé equation solution of the specified type exists,
counted with multiplicity.

The twisted Lamé spectral polynomials for $\ell \leq 8$ are listed in table 5. The polynomials $Lt^II_3$ and $Lt^II_8$ are omitted on account of lack of space (their respective
degrees are 12 and 16). The following proposition will be proved in §4.

**Proposition 3.4.**

(i) For any integer $\ell \geq 3$, respectively, $\ell \geq 2$, there is a non-trivial twisted
spectral polynomial of Type I, respectively, II.

*Phil. Trans. R. Soc. A* (2008)
Table 5. Twisted Lamé spectral polynomials of Types I and II.

<table>
<thead>
<tr>
<th>ℓ</th>
<th>Twisted Lamé spectral polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$L^1_{11}(B;g_2,g_3)$ [ 1 ]</td>
</tr>
<tr>
<td>2</td>
<td>$L^2_{11}(B;g_2,g_3)$ [ 1 ]</td>
</tr>
<tr>
<td>3</td>
<td>$B^2 - (75/4)g_2 + (1715/2)g_3$ [ 1 ]</td>
</tr>
<tr>
<td>4</td>
<td>$B^3 - (343/4)g_2B - (1895/4)g_3B^2 + (54699/16)g_2B^3 + (58939/65)g_2g_3B + (41100625/4)g_2^2B$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$- (5063435715/16)g_2^3$ [ 1 ]</td>
</tr>
<tr>
<td>5</td>
<td>$B^4 - (987/2)g_2B^3 - (19845/4)g_3B^4 + (5469936/16)g_2B^5 + (5893965/8)g_2g_3B$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$- (5063435715/16)g_2^2B^2$ [ 1 ]</td>
</tr>
<tr>
<td>6</td>
<td>$B^5 - (2751/2)g_2B^4 - (181521/2)g_2B^5 + (3407481/16)g_2^2B^6 + (164862621/8)g_2g_3B^3$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (677951505/16)g_2^3B^2 - (3362086035/8)g_2^2g_3B + (15890285475/4)g_2^3B$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (166423587325/16)g_2^4$ [ 1 ]</td>
</tr>
<tr>
<td>7</td>
<td>$B^{12} - 418693B^{10} - (1048223/2)g_2B^9 + (17433633/8)g_2B^8 + (3510785355/8)g_2g_3B$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (4590946825/16)g_2^2B^7 + (5947328487/32)g_2^2B^6 - (314684877773/32)g_2^3B^5$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (1458559449258296875/128)g_2^4B^4 + (1647921227262688125/256)g_2^5$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (16766233150463677881/128)g_2^3g_3^2 + (285799721595172159375/256)g_2^6$ [ 1 ]</td>
</tr>
<tr>
<td>8</td>
<td>$B^{13} - 1018893B^{12} - (4944861/2)g_2B^{11} + (48623733/8)g_2B^{10} + (33098210361/8)g_2g_3B$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (210211163451/8)g_2^2B^9 - (1634908193451/4)g_2^3B^8 + (4666788317745/32)g_2^4B^7 + (19891264758855699190625/128)g_2^5B^6$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (124730602375822866515625/256)g_2^6 + (15203913100824300328125/128)g_3g_2^5B^5 + (83437068242769811171875/256)g_3^2B^4$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (10040853832454565875909625/8)g_3^3g_2^2B^3 + (10031234162997777251171875/16)g_3^2g_2^3B$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (140478132732445019532152/2)g_2g_3 + (305679140836374550781250g_2^3g_3$ [ 1 ]</td>
</tr>
<tr>
<td></td>
<td>$+ (626738284763018974609375/2)g_2^5$ [ 1 ]</td>
</tr>
</tbody>
</table>

$L^1_k(B;c_\nu,g_2,g_3)$
(ii) For any integer \( \ell \geq 1 \), \( \text{deg } \Lambda^1 \ell k \) is \((\ell^2 - 1)/4\) if \( \ell \) is odd and \( \ell^2/4 - 1 \) if \( \ell \) is even; and \( \text{deg } \Pi^1 \ell k \) is \((\ell^2 - 1)/4\) if \( \ell \) is odd and \( \ell^2/4 \) if \( \ell \) is even.

Hence, the full twisted Lamé spectral polynomial \( \Lambda^1 \ell k \), the definition is \( \Lambda^1 \ell k (B; g_2, g_3) \), which is of degree \( \text{deg } \Lambda^1 \ell k \) is \((\ell^2 - 1)/4\) if \( \ell \) is odd and \( \ell^2/4 - 1 \) if \( \ell \) is even.

(c) Theta-twisted Lamé polynomials

Lamé equation solutions of a third sort can be constructed for certain values of the spectral parameter \( B \). These are linear combinations, over polynomials in the coordinate \( x \), of (i) the multivalued meromorphic function \( \Phi(x, y; x_0, y_0) \) parametrized by the point \((x_0, y_0) \in E_{g_2, g_3} \setminus \{ (\infty, \infty) \} \), and (ii) its derivative

\[
\Phi^{(1)}(x, y; x_0, y_0) := \left( \frac{y}{d} \right) \Phi(x, y; x_0, y_0) = \frac{1}{2} \left( \frac{y + y_0}{x - x_0} \right) \Phi(x, y; x_0, y_0).
\]

One way of seeing that \( \Phi, \Phi^{(1)} \) are a natural basis is to note that when \((x_0, y_0) = (e_\gamma, 0)\), they reduce to \( \sqrt{x - e_\gamma}, (1/2)y/\sqrt{x - e_\gamma} \). Hence, the class of functions constructed from them will include the Lamé polynomials of Type II.

**Definition 3.5.** A solution of the Lamé equation (2.9) on the elliptic curve \( E_{g_2, g_3} \) is said to be a theta-twisted Lamé polynomial, if it is of the form \( A(x) \Phi(x, y; x_0, y_0) + 2B(x) \Phi^{(1)}(x, y; x_0, y_0) \), with \((x_0, y_0) \neq (e_\gamma, 0)\) for \( \gamma = 1, 2, 3 \).

Here, \( A, B \) are polynomials, and the innocuous ‘2’ factor compensates for the ‘1/2′ factor of (3.11).

If \( A(x) = \sum_j a_j x^j \) and \( B(x) = \sum_j b_j x^j \), substituting this expression into (2.9) and equating the coefficients of powers of \( x \) yields the coupled pair of recurrences

\[
(2j - \ell + 1)(2j + \ell + 2) a_j + [(4j + 3)x_0 - B] a_{j+1} \\
+ \left( - \left( j + \frac{5}{2} \right) g_2 + 4x_0^2 \right) (j + 2)a_{j+2} - (j + 2)(j + 3)g_3 b_{j+3} \\
- 2y_0(4j + 6)b_{j+1} - 4x_0y_0(j + 2)b_{j+2} = 0,
\]

\[
(2j - \ell + 2)(2j + \ell + 3) b_j + [(4j + 3)x_0 - B] b_{j+1} \\
+ \left( -(j + \frac{3}{2})g_2 - 4x_0^2 \right) (j + 2)b_{j+2} - (j + 2)(j + 3)g_3 a_{j+3} \\
y_0(j + 2)a_{j+2} = 0.
\]

**Proposition 3.5.** If \( \ell \geq 4 \), non-zero theta-twisted polynomials can in principle be computed from these recurrences. If \( \ell \) is odd, respectively, even, then \( \text{deg } A, B \) are \((\ell - 1)/2, (\ell - 5)/2\) respectively, \( \ell/2 - 2, \ell/2 - 1 \). The coefficients are computed by setting the coefficient of the highest powers of \( x \) in \( A \), respectively, \( B \), to unity, and working downward.

Unless \( B \) and the point \((x_0, y_0)\) are specially chosen, the coefficients of negative powers of \( x \) may be non-zero. However, by examination, they will be zero if the coefficients of \( x^{-1} \) in \( A \) and \( B \) are both zero. \( a_{-1} = 0, b_{-1} = 0 \) are equations in
To (3.3) and (3.4). If \[ polynomial \ monic \ in \ \text{computing \ their \ resultants \ against \ the \ third \ equation, \ and \ then \ eliminate} \]

solutions with \[ convention \ dividing \ the \ appropriate \ equation \ by \ \sqrt{a} \]

Alternatively, a Grobner basis calculation may be performed (Brezhnev 2004).

Irrespective of which procedure is followed, there is a minor problem: the handling of the improper case \( (x_0, y_0) = (e_1, 0) \), in which (3.12) and (3.13) reduce to (3.3) and (3.4). If \( \ell \) is odd, respectively, even, then the left-hand side of the equation \( b_{-1} = 0 \), respectively, \( a_{-1} = 0 \), turns out to be divisible by \( y_0 \). By dividing the appropriate equation by \( y_0 \) before eliminating \( x_0, y_0 \), the spurious solutions with \( y_0 = 0 \) can be eliminated.

**Definition 3.6.** The theta-twisted Lamé spectral polynomial \( L_{\theta_\ell}(B; g_2, g_3) \) is the polynomial monic in \( B \) which is obtained by eliminating \( x_0, y_0 \) from the equations \( a_{-1} = 0, b_{-1} = 0, \) with \( y_0 \) factors removed as indicated. (This assumes \( \ell \geq 4; \) by convention \( L_{\theta_1} = L_{\theta_2} = L_{\theta_3} := 1. \) Each theta-twisted spectral polynomial may be regarded as \( \prod_{s} [B - B_s(g_2, g_3)] \), where the roots \( \{ B_s \} \) are the values of \( B \) for which a theta-twisted Lamé polynomial exists, counted with multiplicity.

The theta-twisted Lamé spectral polynomials for \( \ell \leq 8 \) are listed in table 6. The following proposition will be proved in §4.

**Proposition 3.6.**

(i) For any integer \( \ell \geq 4 \), there is a non-trivial theta-twisted spectral polynomial \( L_{\theta_\ell} \).

(ii) For any integer \( \ell \geq 2 \), \( N_{\theta_\ell} := \deg L_{\theta_\ell} \) is \( (\ell + 1)(\ell - 3)/4 \), if \( \ell \) is odd, and \( \ell(\ell - 2)/4 \) if \( \ell \) is even.

**Table 6.** Theta-twisted Lamé spectral polynomials.

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( L_{\theta_\ell}(B; g_2, g_3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>( B^2 - (196/3) g_2 )</td>
</tr>
<tr>
<td>5</td>
<td>( B^3 - (1053/4) g_2 B - (25515/4) g_3 )</td>
</tr>
<tr>
<td>6</td>
<td>( B^4 - (4599/4) g_2 B^3 - (12085/2) g_3 B^2 + 160083 g_2^2 B + 20376279/2 g_2 B g_3 ) + ( (-576357606 g_2^3 + (96850215/4) g_2 g_3) )</td>
</tr>
<tr>
<td>7</td>
<td>( B^5 - (19565/6) g_2 B^4 - (832843/2) g_3 B^3 + (26049731/48) g_2^3 B^2 + (4205970769/24) g_2 g_3 B^3 ) + ( ((37048456991/48) g_2^3 - (20496641251/16) g_2^2 g_3 + (8684628953/6) g_2 g_3^2) B^2 ) + ( (-552623218875/4) g_2^4 + (4390244356771/12) g_2 g_3^2) )</td>
</tr>
<tr>
<td>8</td>
<td>( B^{12} - (18063/2) g_2 B^{10} - (4067739/2) g_3 B^9 + (73174185/16) g_2^8 ) + ( (22697632971/8) g_2 g_3 B^7 + (28431/16) (10247115 g_2^2 - 167402573 g_2 g_3) B^6 ) - ( (1385229823965/2) g_2^2 g_3 B^5 + (492075/4) (-164228833 g_2^4 + 3606494307 g_2 g_3^2) B^4 ) + ( 63969750000 (2175 g_2^2 g_3 - 101062 g_2^3) B^3 + 98415000 (62738863 g_2^5 - 1656031845 g_2^2 g_3^2) B^2 ) + ( 9216640000 (-256036 g_2^4 g_3 + 7098507 g_2 g_3^2) B ) + ( 19683000000 (-35153041 g_2^6 + 669725199 g_2^2 g_3^2 + 7578832716 g_3^4) )</td>
</tr>
</tbody>
</table>

For any integer \( (i) \) and \( (ii) \), \( N_{\theta_\ell} := \deg L_{\theta_\ell} \) is \( (\ell + 1)(\ell - 3)/4 \), if \( \ell \) is odd, and \( \ell(\ell - 2)/4 \) if \( \ell \) is even.
4. The Hermite–Krichever Ansatz

The Hermite–Krichever Ansatz is a tool for solving any Schrödinger-like differential equation, not necessarily of second order, with coefficient functions that are elliptic. Such an equation should ideally have one or more independent solutions that, according to the Ansatz, are expressible as finite series in the derivatives of an elliptic Baker–Akhiiezer function, including an exponential factor (cf. (1.5)).

In the context of (2.9), the elliptic-curve algebraic form of the Lamé equation, this means that one hopes to be able to construct a solution on the curve $E_{g_2,g_3}$, except at a finite number of values of the spectral parameter $B \in \mathbb{C}$, as a finite series in the functions $\Phi^{(j)} := (y \, d/dx)^j \Phi$, $j \geq 0$, multiplied by a factor $\exp[\kappa \, \int dx/y]$. Here, $\Phi(\cdot, \cdot ; x_0, y_0)$ is the fundamental multivalued meromorphic function on $E_{g_2,g_3}$ introduced in \$2. Actually, a different but equivalent sort of series is easier to manipulate symbolically. By examination $\Phi^{(2)} = (2x + x_0) \Phi$, from which it follows by induction on $j$ that any finite series in $\Phi^{(j)}$, $j \geq 0$, is a combination (over polynomials in $x$) of the basis functions $\Phi, \Phi^{(1)}$. This motivates the following definition.

**Definition 4.1.** A solution of the Lamé equation (2.9) on the elliptic curve $E_{g_2,g_3}$ is said to be an Hermite–Krichever solution if it is of the form

$$\begin{aligned}
[A(x) \Phi(x, y, x_0, y_0) + 2B(x) \Phi^{(1)}(x, y, x_0, y_0)] \exp[\kappa \, \int \frac{dx}{y}] &= [A(x) + B(x) \left( \frac{y + y_0}{x - x_0} \right)] \Phi(x, y, x_0, y_0) \exp[\kappa \, \int \frac{dx}{y}],
\end{aligned}$$

(4.1)

for some $(x_0, y_0) \in E_{g_2,g_3} \setminus \{(\infty, \infty)\}$ and $\kappa \in \mathbb{C}$. Here $A, B$ are polynomials.

As defined, Hermite–Krichever solutions subsume most of the solutions explored in \$3. If $\kappa = 0$, they reduce to theta-twisted Lamé polynomials. If $(x_0, y_0) = (e_\gamma, 0)$ for $\gamma = 1, 2, 3$, in which case $\Phi, \Phi^{(1)}$ degenerate to $\sqrt{x - e_\gamma}, (1/2) y/\sqrt{x - e_\gamma}$, they reduce to twisted Lamé polynomials of Type II. If both specializations are applied, they reduce to ordinary Lamé polynomials of Type II. The Lamé polynomials of Type I, both ordinary and twisted, are not of the Hermite–Krichever form, but they can be viewed as arising from a passage to the $(x_0, y_0) \to (\infty, \infty)$ limit.

If $A(x) = \sum_j a_j x^j$ and $B(x) = \sum_j b_j x^j$, substituting (4.1) into (2.9) and equating the coefficients of powers of $x$ yields the coupled pair of recurrences

$$\begin{aligned}
(2j - \ell + 1)(2j + \ell + 2) a_j &+ [(4j + 5)x_0 + \kappa^2 - B] a_{j+1} \\
+ \left[ -\left( j + \frac{5}{2} \right) g_2 + 4x_0^2 - 2\kappa y_0 \right] (j + 2) a_{j+2} - (j + 2)(j + 3) g_3 a_{j+3} \\
+ 8\kappa (j + 1) b_j &+ 4 \left( k x_0 - y_0 \right) (2j + 3) b_{j+1} + 2 \left[ k (4x_0^2 - g_2) - 2x_0 y_0 \right] (j + 2) b_{j+2} = 0, \\
\end{aligned}$$

(4.2)

$$\begin{aligned}
(2j - \ell + 2)(2j + \ell + 3) b_j &+ [-(4j + 7)x_0 + \kappa^2 - B] b_{j+1} \\
+ \left[ -\left( j + \frac{3}{2} \right) g_2 - 4x_0^2 + 2\kappa y_0 \right] (j + 2) b_{j+2} - (j + 2)(j + 3) g_3 b_{j+3} \\
+ \kappa (2j + 3) a_{j+1} - (2\kappa x_0 - y_0)(j + 2) a_{j+2} = 0.
\end{aligned}$$

(4.3)
If $\kappa = 0$, (4.2) and (4.3) reduce to (3.12) and (3.13), and if $(x_0, y_0) = (e_\gamma, 0)$, they reduce to (3.9) and (3.10). If both specializations are applied, they reduce to (3.3) and (3.4).

**Proposition 4.1.** For all $\ell \geq 2$, Hermite–Krichever solutions can in principle be computed from these recurrences. If $\ell$ is odd, respectively, even, then $\deg A, B$ are $(\ell - 1)/2, (\ell - 3)/2$, respectively, $\ell/2 - 1, \ell/2 - 1$. The coefficients are computed by setting the coefficient of the highest power of $x$ in $A, B$, to unity, and working downward.

Unless $B, \kappa$ and the point $(x_0, y_0)$ are specially chosen, the coefficients of negative powers of $x$ may be non-zero. However, by examination, they will be zero if the coefficients of $x^{-1}$ in $A$ and $B$ are both zero. $a_{-1} = 0$, $b_{-1} = 0$ are equations in $B; \kappa; x_0, y_0$. They are ‘compatibility conditions’ similar to those that appear in other applications of the Hermite–Krichever Ansatz. Together with the identity $y_0^2 = 4x_0^3 - g_2x_0 - g_3$, they make up a set of three equations for these four unknowns.

Informally, one may eliminate any two of the four unknowns $B; \kappa; x_0, y_0$, thereby deriving an algebraic relation between the remaining two (involving $g_2, g_3$ of course). A rigorous investigation must be more careful. For example, if the ideal generated by the three equations contained a polynomial involving only $B$ (and $g_2, g_3$), then a solution of the Hermite–Krichever form would exist for very few values of $B$ (Brezhnev 2004). In practice, this problem does not arise: except at a finite number of values of $B$ (at most), the Ansatz can be employed (Geszttesy & Weikard 1998). In fact, in previous work, an algebraic curve in $(B, \kappa)$ has been derived for each $\ell \leq 5$. The $(B, \kappa)$-curve is the one with the most physical significance, since $B$ is a transformed energy and $\kappa$ is related to the crystal momentum.

Solving for $(x_0, y_0)$ as functions of $(B, \kappa)$ reveals that $x_0$ is a rational function of $B$, and that if $\kappa$ is not identically zero, $y_0$ is a rational function of $B$, times $\kappa$. These facts can be interpreted in terms of the following seemingly different curve.

**Definition 4.2.** The $\ell$th Lamé spectral curve $G_\ell := G_\ell(g_2, g_3)$ is the hyper-elliptic curve over $\mathbb{P}^1 \ni B$ comprising all $(B, \nu)$ satisfying $\nu^2 = L_\ell(B; g_2, g_3)$, where $L_\ell$ is the full Lamé spectral polynomial, of degree $2\ell + 1$ in $B$. ($G_\ell$ was informally introduced as $\tilde{G}_\ell$ in §1, where the original energy parameter $E$ was used.) $G_\ell$ will have genus $\ell$ unless two roots of $\tilde{L}_\ell(\cdot; g_2, g_3)$ coincide, i.e. unless the Klein invariant $J$ is a root of one of the two Cohn polynomials of table 4, in which case the genus equals $\ell - 1$.

For each $\ell$, there must exist a parametrization of Lamé equation solutions by a point $(B, \nu)$ on the punctured curve $G_\ell \backslash \{(\infty, \infty)\}$, by the general theory of Hill’s equation on $\mathbb{R}$ (McKean & van Moerbeke 1979). For any finite-band Schrödinger equation on an elliptic curve, including the integer-$\ell$ Lamé equation, the Baker–Akhiezer function (2.13) provides such a parametrization of solutions. In the general theory, the parametrizing hyperelliptic curve $\Gamma$ for any finite-band Hill’s equation arises from *differential–difference bispectrality*; as a uniformization of the relation between the energy parameter $B$ and the crystal momentum $k$ (Treibich 2001). This curve $\Gamma \ni (B, \nu)$ is defined by an irrationality of the form $\nu^2 = L(B)$, and $B$ and $k$ are meromorphic functions on it; the former single valued, and the latter additively multivalued. The energy is computed from the
degree-2 map \( B : \Gamma \to \mathbb{P}^1 \) given by \( (B, \nu) \mapsto B \), and the crystal momentum from the formula
\[
k = -i \oint \left[ \frac{1}{2} \left( \frac{y + y_0(B, \nu)}{x - x_0(B, \nu)} \right) + \kappa(B, \nu) \right] \frac{dx}{y},
\]
in which the line integrals on \( E_{g_2,g_3} \) are taken over the appropriate fundamental loop. Here, \( (B, \nu) \mapsto (x_0, y_0) \) is a certain projection \( \pi : \Gamma \to E_{g_2,g_3} \) and \( \kappa : \Gamma \to \mathbb{P}^1 \) is a certain auxiliary meromorphic function. These two morphisms of complex manifolds are ‘odd’ under the involution \( (B, \nu) \mapsto (B, -\nu) \), i.e. \( x_0(B, -\nu) = x_0(B, \nu) \) and \( y_0(B, -\nu) = -y_0(B, \nu) \), and \( \kappa(B, -\nu) = -\kappa(B, \nu) \). Hence, \( x_0 \) must be a rational function of \( B \), and each of \( y_0 \) and \( \kappa \) must be a rational function of \( B \), times \( \nu \).

In the general theory of finite-band equations, the Baker–Akhiezer uniformization is viewed as more fundamental than the Hermite–Krichever Ansatz. However, in the case of the integer-\( \ell \) Lamé equation, one can immediately identify the curve in \( (B, \kappa) \), derived from the Ansatz as explained previously, with the \( \ell \)th Lamé spectral curve \( \Gamma_\ell \equiv (B, \nu) \). It is isomorphic to it by a birational equivalence of a simple kind: the ratio \( \kappa/\nu \) is a rational function of \( B \).

This interpretation makes possible a geometrical understanding of each of the types of Lamé spectral curve worked out in §3. Owing to oddness, each of the finite Weierstrass points \( \{(B_s, 0)\}_{s=0}^{2k} \) on \( \Gamma_\ell \), which correspond to band edges, must be mapped by the projection \( \pi_\ell : \Gamma_\ell \to E_{g_2,g_3} \) to one of the finite Weierstrass points \( \{(e_\gamma, 0)\}_{\gamma=1}^{N} \) or to \((\infty, \infty)\). Bearing in mind that Type I Lamé polynomials, both ordinary and twisted, are not of the Hermite–Krichever form, on account of \( (x_0, y_0) \) formally equaling \((\infty, \infty)\) and \( \kappa \) equalling \( \infty \), one has the following proposition.

**Proposition 4.2.**

(i) The roots of the Type I Lamé spectral polynomial \( L^I_{\ell}(B) \) are the \( B \)-values of the finite Weierstrass points \( \{(B_s, 0)\}_{s=0}^{2k} \) that are projected by \( \pi_\ell : \Gamma_\ell \to E_{g_2,g_3} \) to the infinite Weierstrass point \((\infty, \infty)\). Moreover, the roots of the Type I twisted Lamé spectral polynomial \( L^I_{\ell}(B) \) include the \( B \)-values of the finite non-Weierstrass points that are projected to \((\infty, \infty)\) and do not include the \( B \)-value of any finite point that is not projected to \((\infty, \infty)\).

(ii) For \( \gamma = 1, 2, 3 \), the roots of the Type II Lamé spectral polynomial \( L^{II}_{\ell}(B; e_\gamma) \) are the \( B \)-values of the finite Weierstrass points \( \{(B_s, 0)\}_{s=0}^{2k} \) that are projected by \( \pi_\ell : \Gamma_\ell \to E_{g_2,g_3} \) to the finite Weierstrass point \((e_\gamma, 0)\). Moreover, the roots of the Type II twisted Lamé spectral polynomial \( L^{II}_{\ell}(B; e_\gamma) \) include the \( B \)-values of the finite non-Weierstrass points that are projected to \((e_\gamma, 0)\) and do not include the \( B \)-value of any finite point that is not projected to \((e_\gamma, 0)\).

(iii) The roots of the theta-twisted Lamé spectral polynomial \( L^{0\ell}_{\theta}(B) \) include the \( B \)-values of the finite non-Weierstrass points that are zeroes of \( \kappa_\ell : \Gamma_\ell \to \mathbb{P}^1 \) and do not include the \( B \)-value of any finite point that is not a zero.

**Remark 4.1.** The phrasing of the proposition leaves open the possibilities that (i) a root of \( L^{I}_{\ell}(B) \) may be a root of \( L^{I}_{\ell}(B) \), (ii) a root of \( L^{II}_{\ell}(B, e_\gamma) \) may be a root of \( L^{II}_{\ell}(B, e_\gamma) \) and (iii) a root of \( L^{0\ell}_{\theta}(B) \) may be the \( B \)-value of a finite Weierstrass point, i.e. a band edge. Generically, these three types of coincidence do not occur, but instances are not difficult to find. One is the case \( \ell \equiv 0 \pmod{3} \) and \( g_2 = 0 \) (i.e. \( J = 0 \)), in which by examination \( L^{I}_{\ell} \) and \( L^{I}_{\ell} \) have the common root \( B = 0 \).
To proceed beyond the proposition, a significant result from finite-band integration theory is needed. For the integer-$\ell$ Lamé equation, the covering $\pi_\ell : \Gamma_\ell \to E_{g_2, g_3}$ and the auxiliary map $\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1$ are both of degree $\ell(\ell + 1)/2$, irrespective of the choice of elliptic curve $E_{g_2, g_3}$. From this fact, supplemented by proposition 4.2, the two propositions 3.4 and 3.6, which were left unproved in §3, immediately follow.

The fibre over any point $(x_0, y_0) \in E_{g_2, g_3}$ must comprise $\ell(\ell + 1)/2$ points of $\Gamma_\ell$, the counting being up to multiplicity. Consider the fibre over $(\infty, \infty)$, which by examination includes with unit multiplicity the point $(B, \nu) = (\infty, \infty)$. It also includes each finite Weierstrass point of $\Gamma_\ell$ that corresponds to a Type I Lamé polynomial. As was shown in §3, the number of these up to multiplicity, $N^I_\ell := \deg L^I_\ell$, equals $(\ell - 1)/2$ if $\ell$ is odd and $\ell/2 + 1$ if $\ell$ is even. Hence, the number of additional points above $(\infty, \infty)$ is $\ell(\ell + 1)/2 - 1 - (\ell - 1)/2 = (\ell^2 - 1)/2$ if $\ell$ is odd and $\ell(\ell + 1)/2 - 1 - (\ell/2 + 1) = \ell^2/2 - 2$ if $\ell$ is even. Since the projection $\pi_\ell$ is odd under the involution $(B, \nu) \mapsto (B, -\nu)$, these occur in pairs. Hence, $Nt^I_\ell := \deg Lt^I_\ell$ must equal $(\ell^2 - 1)/4$ if $\ell$ is odd and $\ell^2/4 - 1$ if $\ell$ is even; which is the formula for $Nt^I_\ell$ as given in proposition 3.4. A similar computation applied to the fibre above any finite Weierstrass point $(e_{\gamma}, 0)$ yields the formula for $Nt^I_\ell$ given in that proposition.

The formula for $N\theta_\ell := \deg L\theta_\ell$ stated in proposition 3.6 can also be derived with the aid of proposition 4.2. Since the map $\kappa : \Gamma_\ell \to \mathbb{P}^1$ has degree $\ell(\ell + 1)/2$, the fibre above 0 comprises that number of points, up to multiplicity. It includes each finite Weierstrass point of $\Gamma_\ell$ that corresponds to a Type II Lamé polynomial (but not the Weierstrass points corresponding to Type I Lamé polynomials, since those are not of the Hermite–Krichever form and formally have $\kappa = \infty$). The number of these up to multiplicity is three times $N^I_\ell := \deg L^I_\ell$, which equals $3(\ell + 1)/2$ if $\ell$ is odd and $3\ell/2$ if $\ell$ is even. Hence, the number of additional points above 0 is $\ell(\ell + 1)/2 - 3(\ell + 1)/2 = (\ell + 1)(\ell - 3)/2$ if $\ell$ is odd and $\ell(\ell + 1)/2 - 3\ell/2 = \ell(\ell - 2)/2$ if $\ell$ is even. They come in pairs, and division by two yields the formula for $N\theta_\ell$ given in proposition 3.6.

A geometrized version of proposition 3.2 can also be proved, with the aid of an additional result that goes beyond the Hermite–Krichever Ansatz. The Lamé spectral curve $\Gamma_\ell(g_2, g_3)$ is non-singular with genus $\ell$ for generic values of the Klein invariant $J = J(g_2, g_3)$, and when it degenerates to a singular curve $\Gamma_\ell^* := \Gamma_\ell(g_2^*, g_3^*)$, the singular curve has genus $\ell - 1$, with singularities that are limits as $(g_2, g_3) \to (g_2^*, g_3^*)$ of Weierstrass points of $\Gamma_\ell(g_2, g_3)$. It follows that the $\ell$th Types I and II Cohn polynomials, which characterize the pairs $(g_2, g_3)$ for which the $\ell$th Lamé operator on $E_{g_2, g_3}$ has degenerate algebraic spectrum of the specified type, i.e. for which $\Gamma_\ell(g_2, g_3)$ has a pair of degenerate finite Weierstrass points of the specified type, in fact do more: they characterize the $(g_2, g_3)$ for which $\Gamma_\ell(g_2, g_3)$ is singular. Owing to the reduction of the genus by at most unity, there can be, as proposition 3.2 states, no multiple coincidences of the algebraic spectrum.

The problem of explicitly constructing a Hermite–Krichever solution of the integer-$\ell$ Lamé equation, of the form (4.1), will now be considered. What are needed are the quantities $x_0, y_0, \kappa$, or equivalently $x_0, y_0/\nu, \kappa/\nu$. Each of the latter is a rational function of the spectral parameter $B$.

One way of deriving these functions is to eliminate variables from the system of three polynomial equations in $B; \kappa; x_0, y_0$, as previously explained. Coupled with the spectral equation $\nu^2 = L_\ell(B)$, this yields explicit expressions for the
three desired functions. By the standards of polynomial elimination algorithms, this procedure is not time consuming. It is much more efficient than the manipulations of compatibility conditions that other authors have employed. Beginning with Halphen (1888), it has been the universal practice to apply the Hermite–Krichever Ansatz to the Weierstrassian form of the Lamé equation, i.e. to (2.6), rather than to the elliptic form (2.9). It is now clear that this Weierstrassian approach is far from optimal. For example, the computation of the covering map \((B,v) \mapsto (x_0, y_0)\) in the case \(\ell = 5\), by Eilbeck & Enol’skii (1994), required 7 h of computer time. The just-sketched elliptic curve approach requires only a fraction of a second.

It turns out that for constructing Hermite–Krichever solutions, this revised elimination scheme is also not optimal. Remarkably, at this point no elimination needs to be performed at all, since the covering \(\pi_\ell : \Gamma_\ell \to E_{g_2,g_3}\) and auxiliary function \(\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1\) can be computed directly from the spectral polynomials of §3.

**Theorem 4.1.** For all integer \(\ell \geq 1\), the covering map \(\pi_\ell : \Gamma_\ell \to E_{g_2,g_3}\) appearing in the Ansatz maps \((B,v)\) to \((x_0, y_0)\) according to

\[
x_0(B; g_2, g_3) = e_\gamma + \frac{4}{[\ell(\ell + 1)]^2} \frac{L^{II}_\ell(B; e_\gamma, g_2, g_3)[Lt^{II}_\ell(B; e_\gamma, g_2, g_3)]^2}{L^{I}_\ell(B; g_2, g_3)[Lt^{I}_\ell(B; g_2, g_3)]^2},
\]

(4.5)

\[
y_0(B, v; g_2, g_3) = \frac{16}{[\ell(\ell + 1)]^3} \left\{ \frac{3}{\gamma = 1} \frac{Lt^{II}_\ell(B; e_\gamma, g_2, g_3)}{[Lt^{I}_\ell(B; g_2, g_3)]^2[Lt^{I}_\ell(B; g_2, g_3)]^2} \right\} v,
\]

(4.6)

with \(\gamma\) in (4.5) being any of 1, 2, 3. The auxiliary function \(\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1\) is given by

\[
\kappa(B, v; g_2, g_3) = - \frac{(\ell - 1)(\ell + 2)}{\ell(\ell + 1)} \left[ \frac{L\theta_\ell(B; g_2, g_3)}{L^{I}_\ell(B; g_2, g_3)Lt^{I}_\ell(B; g_2, g_3)} \right] v.
\]

(4.7)

**Proof.** With the exception of the three \(\ell\)-dependent prefactors, such as \(4/[\ell(\ell + 1)]^2\), the formulae (4.5)–(4.7) follow uniquely from proposition 4.2, regarded as a list of properties that \(\pi_\ell\) and \(\kappa_\ell\) must satisfy.

\(\pi_\ell\) must map each point \((B_s, 0)\), where \(B_s\) is a root of \(L^{I}_\ell(B)\), singly to \((\infty, \infty)\), and each point \((B_s, 0)\), where \(B_s\) is a root of \(L^{II}_\ell(B; e_\gamma)\), singly to \((e_\gamma, 0)\). It must also map each point \((B_t, \pm v_t)\), where \(B_t\) is a root of \(Lt^{I}_\ell(B)\), singly to \((\infty, \infty)\), and each point \((B_t, \pm v_t)\), where \(B_t\) is a root of \(Lt^{II}_\ell(B; e_\gamma)\), singly to \((e_\gamma, 0)\). In all these statements, the counting is up to multiplicity.

In addition, \(B \mapsto \kappa/v\) must map each point \((B', \pm v')\), where \(B'\) is a root of \(L\theta_\ell(B)\), singly to zero, and must map each point \((B_s, 0)\), where \(B_s\) is a root of \(L^{I}_\ell(B)\), and each point \((B_t, \pm v_t)\), where \(B_t\) is a root of \(Lt^{I}_\ell(B)\), singly to \(\infty\). In these statements as well, the counting is up to multiplicity.

The \(\ell\)-dependent prefactors in (4.5)–(4.7) can be deduced from the leading-order asymptotic behaviour of \(x_0\) and \(\kappa/v\) as \(B \to \infty\).

The remarkably simple formulae of the theorem permit Hermite–Krichever solutions of the form (4.1) to be constructed for very large values of \(\ell\), since the Lamé spectral polynomials \(L, Lt, L\theta\) (ordinary, twisted and theta-twisted) are relatively easy to work out, as §3 made clear. Tables 3, 5 and 6 may be consulted.

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It should be stressed that in the formula (4.5) for $x_0$, the same right-hand side results, irrespective of which of the three values of $\gamma$ is chosen. All terms explicitly involving $e_\gamma$ will cancel. Of course, all powers of $e_\gamma$ higher than the second must first be rewritten in terms of $g_2$, $g_3$ by using the identity $e_\gamma^3 = (1/4)(g_2 e_\gamma + g_3)$. In the same way, it is understood that the numerator of the right-hand side of (4.6), the terms of which are symmetric in $e_1$, $e_2$, $e_3$, should be rewritten in terms of $g_2$, $g_3$. (This can always be done: for example, the symmetric polynomial $e_1^2 e_2^2 + e_2^2 e_3^2 + e_3^2 e_1^2$ equals $g_2^2/16$).

The application of theorem 4.1 to the cases $\ell = 1$, 2, 3 may be illuminating. — If $\ell = 1$, then $(x_0, y_0) = (B, 2\nu)$ and $\kappa = 0$. The map $\pi_1 : \Gamma_1 \to E_{g_2, g_3}$ is a mere change of normalization, since $\Gamma_1$ is isomorphic to $E_{g_2, g_3}$; cf. (2.15).

— If $\ell = 2$, then

$$x_0 = e_\gamma + \frac{1}{9} \frac{(B + 3e_\gamma)(B - 6e_\gamma)^2}{B^2 - 3g_2} = \frac{B^3 + 27g_3}{9(B^2 - 3g_2)}, \quad (4.8)$$

$$y_0 = \frac{2}{27} \frac{\prod_{\gamma=1}^{3} (B - 6e_\gamma)}{(B^2 - 3g_2)^2} \nu = \frac{2(B^3 - 9g_2 B - 54g_3)}{27(B^2 - 3g_2)^2} \nu, \quad (4.9)$$

and $\kappa = -\{2/[3(B^2 - 3g_2)]\} \nu$.

— If $\ell = 3$, then

$$x_0 = e_\gamma + \frac{1}{36} \frac{(B^2 - 6e_\gamma B + 45e_\gamma^2 - 15g_2)(B^2 - 15e_\gamma B - 225e_\gamma^2 + \frac{75}{4} g_2)^2}{B(B^2 - \frac{75}{4} g_2)^2}$$

$$= \frac{(16B^6 + 360g_2 B^4 + 27000g_3 B^3 - 3375g_2^2 B^2 - 303750g_2g_3 B - 84375g_3^2 + 2278125g_3^3)}{36B(4B^2 - 75g_2)^2}, \quad (4.10)$$

$$y_0 = \frac{1}{108} \frac{\prod_{\gamma=1}^{3} (B^2 - 15e_\gamma B - 225e_\gamma^2 + \frac{75}{4} g_2)}{B^2(B^2 - \frac{75}{4} g_2)^3} \nu$$

$$= \frac{(16B^6 - 1800g_2 B^4 - 54000g_3 B^3 - 16875g_2^2 B^2 + 421875g_3^2 - 11390625g_3^3)}{27B^2(4B^2 - 75g_2)^3} \nu, \quad (4.11)$$

and $\kappa = -\{10/[3B(4B^2 - 75g_2)]\} \nu$.

The formulae for $\ell = 2, 3$ were essentially known to Hermite. Setting $\ell = 4, 5$ in theorem 4.1 yields the less familiar and more complicated formulae that Enol’skii & Kostov (1994) and Eilbeck & Enol’skii (1994) derived by eliminating variables from compatibility conditions. Theorem 4.1 readily yields the covering map $\pi_\ell$ and auxiliary function $\kappa_\ell$ for far larger $\ell$.  

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5. Hyperelliptic reductions

The cover \( \pi_\varphi: \Gamma_\ell(g_2, g_3) \to E_{g_2, g_3} \) introduced as part of the Hermite–Krichever Ansatz, i.e. the map \((B, \nu) \mapsto (x_0, y_0)\), is of independent interest, since explicit examples of coverings of elliptic curves by higher genus algebraic curves are few, and the problem of determining which curves can cover \( E_{g_2, g_3} \), for either specified or arbitrary values of the invariants \( g_2, g_3 \), remains unsolved. \( \Gamma_\ell(g_2, g_3) \) generically has genus \( g = \ell + 1 \), as noted, and the cover will always be of degree \( N = \ell(\ell + 1)/2 \). The formula for \( x_0 = x_0(B; g_2, g_3) \) given in theorem 4.1 is consistent with this, since \( N \) equals \( \max(N^I_\ell + 2N^I_{tq}, N^II_\ell + 2N^II_{tq}) \), the maximum of the degrees in \( B \) of the numerator and denominator of \( x_0 \). The degrees \( N^I_\ell, N^II_\ell \), as well as the twisted degrees \( Nt^I_\ell, Nt^II_\ell \), were computed in \S 3 (for the latter, see proposition 3.4).

Since \( \Gamma_\ell(g_2, g_3) \) is hyperelliptic (defined by the irrationality \( \nu^2 = L_\ell(B; g_2, g_3) \)) and \( E_{g_2, g_3} \) is elliptic (defined by the irrationality \( g_0^2 = 4x_0^3 - g_2x_0 - g_3 \)), the map \( \pi_\varphi \) enables certain hyperelliptic integrals to be reduced to elliptic ones. Just as \( E_{g_2, g_3} \) is equipped with the canonical holomorphic 1-form \( dx_0/y_0 \), so can \( \Gamma_\ell(g_2, g_3) \) be equipped with the holomorphic 1-form \( dB/\nu \). Any integral of a function in the function field of a hyperelliptic curve (here, any rational function \( R(B, \nu) \)) against its canonical 1-form is called a hyperelliptic integral. Hyperelliptic integrals are classified as follows (Belokolos et al. 1986). The linear space of meromorphic 1-forms of the form \( R(B, \nu) \, dB/\nu \), i.e. of Abelian differentials, is generated by 1-forms of the first, second and third kinds. These are (i) holomorphic 1-forms, with no poles; (ii) 1-forms with one multiple pole and (iii) 1-forms with a pair of simple poles, the residues of which are opposite in sign. The indefinite integrals of (i)–(iii) are called hyperelliptic integrals of the first, second and third kinds. They generalize the three kinds of elliptic integral (Abramowitz & Stegun 1965, ch. 17).

Hyperelliptic integrals of the first kind are the easiest to study, since the linear space of holomorphic 1-forms is finite-dimensional and is spanned by \( B^{r-1} \, dB/\nu \), \( r = 1, \ldots, g \), where \( g \) is the genus. Hence, there are only \( g \) independent integrals of the first kind. A consequence of the map \( \pi_\varphi: \Gamma_\ell(g_2, g_3) \to E_{g_2, g_3} \) is that on any hyperelliptic curve of the form \( \Gamma_\ell(g_2, g_3) \), there are really only \( g - 1 \) independent integrals of the first kind, modulo elliptic integrals (considered trivial by comparison). Changing variables in \( \int dx_0/y_0 \), the elliptic integral of the first kind, yields

\[
\int \left[ \frac{(y_0/\nu)^{-1}}{dB} \frac{dB}{\nu} \right] = \int \frac{dx_0}{y_0}. \tag{5.1}
\]

The quantity in square brackets is rational in \( B \) and, in fact, is guaranteed to be a polynomial in \( B \) of degree less than or equal to \( g - 1 \), since the left-hand integrand is a pulled-back version of the right-hand one and must be a holomorphic 1-form. Equation (5.1) is a linear constraint relation on the \( g \) basic hyperelliptic integrals of the first kind. It reduces the number of independent integrals from \( g \) to \( g - 1 \).

The cases \( \ell = 2, 3 \) of (5.1) may be instructive. The maps \((B, \nu) \mapsto (x_0, y_0)\) were given in (4.8), (4.10), and the degree-\((2\ell + 1)\) spectral polynomials \( L_\ell(\cdot; g_2, g_3) \) follow from table 3. If \( \ell = 2 \), one obtains the hyperelliptic-to-elliptic reduction

\[
\int \frac{[\frac{3}{2} B] \, dB}{\sqrt{(B^2 - 3g_2)(B^3 - \frac{9}{4} g_2B + \frac{27}{4} g_3)}} = \int \frac{dx_0}{\sqrt{4x^3_0 - g_2x_0 - g_3}}, \tag{5.2}
\]

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Table 7. Polynomials specifying the holomorphic 1-forms pulled back from $E_{g_2;g_3}$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\hat{P}_\ell(B;g_2,g_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1$</td>
</tr>
<tr>
<td>2</td>
<td>$B$</td>
</tr>
<tr>
<td>3</td>
<td>$B^2 -(15/4)g_2$</td>
</tr>
<tr>
<td>4</td>
<td>$B^3 -(91/4)g_2B+(175/2)g_3$</td>
</tr>
<tr>
<td>5</td>
<td>$B^4 -(321/4)g_2B^2+(2835/4)g_3B+(891/2)g_2^2$</td>
</tr>
<tr>
<td>6</td>
<td>$B^5 -(861/4)g_2B^3+(12879/4)g_3B^2+(24255/4)g_2^2B-(280665/4)g_2g_3$</td>
</tr>
<tr>
<td>7</td>
<td>$B^6 -(973/2)g_2B^4+10813g_2B^3+(681373/16)g_2^2B^2-(2145143/2)g_2g_3B$</td>
</tr>
<tr>
<td></td>
<td>$+(54071875/16)g_3^2-(5417685/16)g_2^3$</td>
</tr>
<tr>
<td>8</td>
<td>$B^7 -(1553/2)g_2B^5+29916g_3B^4+(3335445/16)g_2^2B^3$</td>
</tr>
<tr>
<td></td>
<td>$-(34152435/4)g_2g_3B^2+((-122490225/16)g_2^2+(937038375/16)g_2^3)B$</td>
</tr>
<tr>
<td></td>
<td>$+179425125g_2^2g_3$</td>
</tr>
</tbody>
</table>

where the change of variables is performed by (4.8). If $\ell = 3$, one obtains

$$
\int \frac{3(B^2 - \frac{15}{4}g_2)}{\sqrt{L_3(B; g_2, g_3)}} dB = \int \frac{dx_0}{\sqrt{4x_0^3 - g_2x_0 - g_3}},
$$

(5.3)

where the full spectral polynomial $L_3(B; g_2, g_3)$ is

$$
B \left( B^6 - \frac{63}{2}g_2B^4 + \frac{297}{2}g_3B^3 + \frac{4185}{16}g_2^2B^2 - \frac{18225}{8}g_2g_3B - \frac{3375}{16}g_2^3 + \frac{91125}{16}g_3 \right),
$$

and the change of variables is performed by (4.10). These reductions were known to Hermite (Königsberger 1878; Belokolos et al. 1986). More recently, the reductions induced by the $\ell = 4, 5$ coverings have been worked out (Eilbeck & Enol’skii 1994; Enol’skii & Kostov 1994). However, the reductions with $\ell > 5$ proved to be too difficult to compute. Theorem 4.1 makes possible the computation of many such higher reductions.

The following proposition specifies the normalization of the pulled-back 1-form. It follows from the known leading order asymptotic behaviour of $x_0, y_0/\nu$ as $B \to \infty$.

**Proposition 5.1.** For all integer $\ell \geq 1$, the polynomial function $P_\ell(B; g_2, g_3) := [(y_0/\nu)^{-1} dx_0/dB](B; g_2, g_3)$ in the hyperelliptic-to-elliptic reduction formula

$$
\int \frac{P_\ell(B; g_2, g_3)}{\sqrt{L_\ell(B; g_2, g_3)}} dB = \int \frac{dx_0}{\sqrt{4x_0^3 - g_2x_0 - g_3}},
$$

where the change of variables $x_0 = x_0(B; g_2, g_3)$ is given by theorem 4.1, equals $\ell(\ell + 1)/4$ times a polynomial $P_\ell(B; g_2, g_3)$ which is monic and of degree $\ell - 1$ in $B$.

The polynomials $\hat{P}_\ell$ are listed in table 7. $\hat{P}_4, \hat{P}_5$ agree with those found by Enol’skii et al. if allowance is made for a difference in normalization conventions.

A complete analysis of Lamé-derived elliptic covers will need to consider exceptional cases of several kinds. The covering curve $\Gamma_\ell(g_2, g_3)$ generically has
genus $g=\ell$, but if the Klein invariant $J = g_3^3/(g_2^3 - 27g_3^2)$ is a root of one of the two Cohn polynomials of table 4, the genus will be reduced to $\ell-1$. According to conjecture 3.1, this will happen, for instance, if $\ell \equiv 2(\text{mod } 3)$ and $g_2=0$ (i.e. $J=0$), so that the base curve $E_{g_2,g_3}$ is equiharmonic. When $g$ is reduced to $\ell-1$ in this way, the linear space of holomorphic 1-forms will be spanned by $B^{r-1}(B-B_0)B/B$, $r=1,\ldots,\ell-1$, where $B_0$ is the degenerate root of the spectral polynomial; but (5.1) will still provide a linear constraint on the associated hyperelliptic integrals.

Another sort of degeneracy takes place when the modular discriminant $\Delta := g_3^3 - 27g_2^2$ equals zero, i.e. when $J=\infty$. In this case, $E_{g_2,g_3}$ will degenerate to a rational curve due to two or more of $e_1$, $e_2$, $e_3$ being coincident. Lamé-derived reduction formulae such as (5.2) and (5.3) will continue to apply. (They are valid though trivial even in the case $e_1=e_2=e_3$, in which case $g_2=g_3=0$.) Hence, these formulae include as special cases certain hyperelliptic-to-rational reductions.

Subtle degeneracies of the covering map $\pi_\ell$ can occur, even in the generic case when $\Gamma_\ell$ has genus $\ell$ and $E_{g_2,g_3}$ has genus 1. The branching structure of $\pi_\ell$ is determined by the polynomial $P_\ell$ of table 7, which is proportional to $d x_0/d B$. If $P_\ell$ has distinct roots $\{B^{(i)}\}_{i=1}^{\ell-1}$, then $\pi_\ell$ will normally have $2\ell-2$ simple critical points on $\Gamma_\ell$, of the form $\{(B^{(i)}, \pm P^{(i)})\}_{i=1}^{\ell-1}$. However, if any $B^{(i)}$ is located at a band edge, i.e. at a branch point of the hyperelliptic $(B,\nu)$-curve, then $(B^{(i)}, 0)$ will be a double critical point. This appears to happen when $\ell \equiv 0(\text{mod } 3)$ and the base curve $E_{g_2,g_3}$ is equiharmonic; the double critical point being located at $(B,\nu) = (0, 0)$. Even if no root of $P_\ell$ is located at a band edge, it is possible for it to have a double root, in which case each of a pair of points $(B^{(i)}, \pm P^{(i)})$ will be a double critical point. By examination, this happens when $\ell = 4$ and $J = -2^{25}/3^5.53$.

A few hyperelliptic-to-elliptic reductions, similar to the quadratic ($N=2$) reduction of Legendre and Jacobi, can be found in the handbooks of elliptic integrals (Byrd & Friedman 1954, §§575 and 576). The Lamé-derived reductions, indexed by $\ell$, should certainly be included in any future handbook. It is natural to wonder whether they can be generalized in some straightforward way. The problem of finding the genus-2 covers of an elliptic curve was intensively studied in the nineteenth century, by Weierstrass and Poincaré among many others, and one may reason by analogy with results on $\ell=2$. One expects that for all $\ell \geq 2$ and for arbitrarily large $N$, a generic $E_{g_2,g_3}$ can be covered by some genus-$\ell$ curve via a covering map of degree $N$. Each Lamé-derived covering $\pi_\ell : \Gamma_\ell(g_2,g_3) \to E_{g_2,g_3}$ has degree $N = \ell(\ell+1)/2$ and may be only a low-lying member of an infinite family. Generalizing the Lamé-derived coverings may be possible even if one confines oneself to $N = \ell(\ell+1)/2$. One can of course pre-compose with an automorphism of $\Gamma_\ell(g_2,g_3)$ and post-compose with an automorphism of $E_{g_2,g_3}$ (a modular transformation). However, when $\ell = 2$, a rather different covering map with the same degree is known to exist (Belokolos et al. 1986). $\pi_2$ has two simple critical points on $\Gamma_2$, but the other degree-3 covering map has a single double critical point on its analogue of $\Gamma_2$. Both can be generalized to include a free parameter (Burnside 1892; Belokolos & Enol’skii 2000). It seems possible that when $\ell > 2$, similar alternatives to the Lamé-derived coverings may exist, with degree $\ell(\ell+1)/2$ but different branching structures.
6. Dispersion relations

It is now possible to introduce dispersion relations and determine the way in which the Hermite–Krichever Ansatz reduces higher-\(\ell\) to \(\ell=1\) dispersion relations. The starting point is the fundamental multivalued function \(\Phi\) introduced in §2. As noted, if the parametrization point \((x_0, y_0)\) on the punctured elliptic curve \(E_{g_2,g_3} \setminus \{(\infty, \infty)\}\) is over \(x_0 = B \in \mathbb{C}\), then \(\Phi(\cdot, \cdot; x_0, y_0)\) will be a solution of the \(\ell=1\) case of the Lamé equation (2.9). \(E_{g_2,g_3}\) is defined by \(y^2 = 4x^3 - g_2x - g_3\), so the hypothesis here is that \((x_0, y_0)\) should equal \((B, \pm \sqrt{4B^3 - g_2B - g_3})\).

In the Jacobi form (with independent variable \(\alpha\)), respectively the Weierstrassian form (with independent variable \(u\)), the crystal momentum \(k\) characterizes the behaviour of a solution of the Lamé equation under \(\alpha \mapsto \alpha + 2K\), respectively \(u \mapsto u + 2\omega\). Both shifts correspond to motion around \(E_{g_2,g_3}\), along a fundamental loop that passes between \((x, y) = (e_1, 0)\) and \((\infty, \infty)\), and cannot be shrunk to a point. (If \(e_1, e_2, e_3\) are defined by (2.1), this will be because \(y\) is positive on one-half of the loop, and negative on the other.) By definition, the solution will be multiplied by \(\xi = \exp[iK(2\omega)] = \exp[ik(2\omega)]\). It follows from the definition (2.11) of \(\Phi\) that when \(\ell=1\),

\[
\xi = \exp[iK(2\omega)] = \exp\left[\frac{1}{2} \oint \frac{(y + y_0)}{(x - x_0)} \frac{dx}{y}\right].
\]

(6.1)

In other words, when \(\ell=1\), the crystal momentum is given by a complete elliptic integral. In the context of finite-band integration theory, this is a special case of (4.4).

It was pointed out in §4 that the spectral curve \(\Gamma_1\) that parametrizes \(\ell=1\) solutions can be identified with \(E_{g_2,g_3}\) itself, via the identification \((B, \nu) = (x_0, y_0/2)\). This suggests a subtle but important reinterpretation of \(k\). In §1, it was introduced as a function of the energy parameter, here \(B\), which is determined only up to integer multiples of \(\pi/K = \pi/\omega\), and which is also undetermined as to sign. If the presence of \(y_0 = 2\nu\) in (6.1) is taken into account, it is clear that the \(\ell=1\) crystal momentum, called \(k_1\) henceforth, should be regarded as a function not on \(\mathbb{P}^1 \setminus \{\infty\} \ni B\), but rather on \(\Gamma_1 \setminus \{(\infty, \infty)\} \ni (B, \nu)\). In this interpretation, the indeterminacy of sign disappears. The additive indeterminacy, on account of which \(k_1\) is an elliptic function of the second kind, remains but can be viewed as an artefact: it is due to \(k_1 \propto \log \xi\), where \(\xi\) is the Floquet multiplier. The behaviour of \(k_1\) near the puncture \((B, \nu) = (\infty, \infty)\) is easily determined. It follows from (6.1) that as \((B, \nu) \to (\infty, \infty)\), i.e. \((x_0, y_0) \to (\infty, \infty)\), each branch of \(k_1\) is asymptotic to \(i\nu/B\) to leading order. Since \(B = x_0, \nu = 2y_0\) have double and triple poles there, respectively, it follows that each branch of \(k_1\) has a simple pole at the puncture.

The multiplier \(\xi\) is a true single-valued function on the punctured spectral curve \(\Gamma_1 \setminus \{(\infty, \infty)\}\), and moreover is entire. One can write \(\xi : \Gamma_1 \setminus \{(\infty, \infty)\} \to \mathbb{P}^1 \setminus \{0, \infty\}\), since the multiplier is never zero. Like \(k_1\), this function is not algebraic; it necessarily has an essential singularity at the puncture. The \((B, \nu, \xi)\)-curve over \(\Gamma_1 \setminus \{(\infty, \infty)\} \ni (B, \nu)\), which is a single cover, and the \((B, \xi)\)-curve over \(\mathbb{C} \ni B\), which is a double cover, are both transcendental curves.

The crystal momentum for each integer \(\ell \geq 1\) may similarly be viewed as an additively multivalued function on the punctured spectral curve \(\Gamma_{\ell} \setminus \{(\infty, \infty)\}\). It will be written as \(k_{\ell}(B, \nu; g_2, g_3)\), with the understanding that for this to be well

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defined, $B, \nu$ must be related by the spectral curve relation $\nu^2 = L_\ell(B; g_2, g_3)$. The quantity $k_\ell(B, \nu; g_2, g_3)$ will not be undetermined as to sign. Suppose now that the projections $\pi_\ell : \Gamma_\ell \to E_{g_2, g_3}$ of the Hermite–Krichever Ansatz are regarded as maps $\pi_\ell : \Gamma_\ell \to \Gamma_1$. In other words, $\pi_\ell$ maps $(B, \nu) \in \Gamma_\ell$ to the point $(B', \nu') := (x_0, y_0/2) \in \Gamma_1$. The reductions $(B, \nu) \mapsto (B', \nu')$ for $\ell = 2, 3$, for example, follow from (4.8)–(4.11).

**Proposition 6.1.** If the integration of the Lamé equation on the elliptic curve $E_{g_2, g_3}$, for integer $\ell \geq 1$, can be accomplished in the framework of the Hermite–Krichever Ansatz by maps $\pi_\ell : \Gamma_\ell \to \Gamma_1$ and $\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1$, where $\pi_\ell$ and $\kappa_\ell$ map the point $(B, \nu)$ to $(B_\ell(B; g_2, g_3), \nu_\ell(B, \nu; g_2, g_3))$ and $\kappa_\ell(B; g_2, g_3)\nu$, respectively, then the dispersion relation for the Hermite–Krichever solutions will be

$$k_\ell(B, \nu; g_2, g_3) = k_1(B_\ell(B, \nu; g_2, g_3), \nu_\ell(B, \nu; g_2, g_3)) - i \kappa_\ell(B, \nu; g_2, g_3)\nu. \quad (6.2)$$

This proposition follows immediately from the form of the Hermite–Krichever solutions (4.1). The first term in (6.2) arises from the $\Phi$, $\Phi'$ factors, and the second from the exponential. The factors $A, B$ in (4.1) do not contribute to the crystal momentum. Equation (6.2) could also be derived from the general theory of finite-band integration, specifically from the formula (4.4). However, a derivation from the Hermite–Krichever Ansatz seems more natural in the present context.

The effort expended in replacing two-valuedness by single-valuedness is justified by the following observation. As a function of $B$ alone, rather than of the pair $(B, \nu)$, each of the two terms of (6.2) would be undetermined as to sign. This ambiguity could cause confusion and errors. The present formulation, though a bit pedantic, facilitates the determination of the correct relative sign.

The form of the $(B, \nu) \mapsto (B', \nu')$ map assumed in the proposition is of course the form supplied by theorem 4.1. Substituting the formulae of the theorem into (6.2) yields an explicit expression for $k_\ell$. It follows readily from this expression that for all integer $\ell \geq 1$, each branch of $k_\ell$ on the hyperelliptic $(B, \nu)$-curve $\Gamma_\ell(g_2, g_3)$ satisfies $k_\ell \sim \nu/B_\ell$, $(B, \nu) \to (\infty, \infty)$. Since $B, \nu$ have double and order-$(2\ell + 1)$ poles at the puncture $(\infty, \infty)$, respectively, this implies that irrespective of $\ell$, each branch of the crystal momentum has a simple pole at the puncture.

### 7. Band structure of the Jacobi form

The results of §§3–6 were framed in terms of the elliptic-curve algebraic form of the Lamé equation. Most work on Lamé dispersion relations has used the Jacobi form instead, and has led accordingly to expressions involving Jacobi theta functions. To derive dispersion relations that can be compared with previous work, the formulation of §6 must be converted to the language of the Jacobi form. The relationships among the several forms were sketched in §2. In the Weierstrassian and Jacobi forms, the Lamé equation is an equation on $\mathbb{C}$ with doubly periodic coefficients, rather than an equation on the curve $E_{g_2, g_3}$. In the conversion to the Jacobi form, the invariants $g_2, g_3$ are expressed in terms of the modular parameter $m$ by (2.2), with $A = 1$ by convention. The coordinate $x$ on $E_{g_2, g_3}$ is interpreted as the function $\varphi(u; g_2, g_3)$, i.e. as $m \text{sn}^2(\alpha|m) - (1/3)(m + 1)$, where...

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\( u \in \mathbb{C} \) and \( \alpha := u - iK'(m) \in \mathbb{C} \) are the respective independent variables of the Weierstrassian and Jacobi forms. The holomorphic differential \( dx/y \) corresponds to \( du \) or \( d\alpha \), and the derivative \( yd/dx \) to \( d/du \) or \( d/d\alpha \). The coordinate \( \varphi = (yd/dx)x \) on \( E_{g_2,g_3} \) is interpreted as \( \varphi'(u; g_2, g_3) \), i.e. as the doubly periodic function \( 2m \text{sn}(\alpha|m) \text{cn}(\alpha|m) \text{dn}(\alpha|m) \) on the complex \( \alpha \)-plane. The functions \( \sqrt{x-c_\gamma}, \gamma=1, 2, 3 \), correspond to \(-i \text{dn}(\alpha|m), -im^{1/2} \text{cn}(\alpha|m) \) and \( m^{1/2} \text{sn}(\alpha|m) \). The equation

\[
B = -E + \frac{1}{3} \ell(\ell + 1)(m + 1),
\]

relates the accessory parameters \( B, E \) of the different forms.

With these formulae, it is easy to convert the Lamé polynomials of table 2 to polynomials in \( \text{sn}(\alpha|m), \text{cn}(\alpha|m), \text{dn}(\alpha|m) \), and the spectral polynomials of table 3 to polynomials in \( E \), for comparison with the list given by Arscott (1964, §9.3.2).

**Definition 7.1.** The Jacobi-form spectral polynomial \( \tilde{L}_\ell(E|m) \) is the negative of the spectral polynomial \( L_\ell(B; g_2, g_3) \), when \( B; g_2, g_3 \) are expressed in terms of \( E \) and \( m \). It is a monic degree-(2\( \ell+1 \)) polynomial in \( E \) with coefficients polynomial in \( m \), and can be regarded as \( \prod_{s=0}^{2\ell}[E-E_s(m)] \), where the roots \( \{E_s\} \) are the values of the energy \( E \) for which there exists a Lamé polynomial solution of the Lamé equation, counted with multiplicity. (The negation is due to the relative minus sign in the \( B \leftrightarrow E \) correspondence (7.1)).

**Definition 7.2.** The \( \ell \)th Jacobi-form spectral curve \( \tilde{\Gamma}_\ell := \tilde{\Gamma}_\ell(m) \) is the hyperelliptic curve over \( \mathbb{P}^1 \supset E \) comprising all \((E, \tilde{\nu})\) satisfying \( \tilde{\nu}^2 = \tilde{L}_\ell(E|m) \). In the usual case when \( m \) is real, \( \tilde{\nu} \) will be real if \( E \) is in a band, and non-real if \( E \) is in a lacuna. In both cases, it is determined only up to negation. By convention, the correspondence between the curve \( \tilde{\Gamma}_\ell \supset (E, \tilde{\nu}) \) and the previously introduced curve \( \Gamma_\ell \supset (B, \nu) \), which was defined by \( \nu^2 = L_\ell(B; g_2, g_3) \), is given by \( \nu = +i\tilde{\nu} \).

The following cases are examples. When \( \ell = 1, 2, 3 \), the spectral polynomial factors over the integers into polynomials at most quadratic in \( E \). In full,

\[
\tilde{L}_1(E|m) = (E-1)(E-m)(E-m-1),
\]

\[
\tilde{L}_2(E|m) = [E^2 - 4(m + 1)E + 12m](E-m-1)(E-4m-1)(E-m-4),
\]

\[
\tilde{L}_3(E|m) = (E - 4m - 4)[E^2 - 2(2m + 5)E + 3(8m + 3)] \times [E^2 - 2(5m + 2)E + 3(3m^2 + 8m)] \times [E^2 - 10(m + 1)E + 3(3m^2 + 26m + 3)].
\]

In (7.3) and (7.4), the first factor arises from \( L_\ell^I(B; g_2, g_3) \) and the remaining three from the factors \( L_\ell^{II}(B; e_\gamma, g_2, g_3) \), \( \gamma = 1, 2, 3 \). In (7.2), there is no Type I factor. The polynomials (7.2)–(7.4) agree with those obtained by Arscott.

The derivation of the Jacobi-form spectral polynomial \( L_\ell(E|m) \) from \( L_\ell(B; g_2, g_3) \) sheds light on a regularity noted by Ince (1940a, §7), which arises in the lemniscatic case \( m = 1/2 \). Ince observed that if \( \ell \leq 6 \), at least, then \( L_\ell(E|1/2) \) has an integer root, namely \( E = \ell(\ell + 1)/2 \). In fact, this is the case for all integer \( \ell \).
By (7.1), the presence of this root is equivalent to the full spectral polynomial $L_{\ell}(B; g_2, 0)$ having $B=0$ as a root. But if $m=1/2$, it follows from (2.1) that $e_2=0$. A glance at the pattern of coefficients in table 3 reveals that if $g_3=0$ and a singular point $e_\gamma$ also equals zero, then either the Type I spectral polynomial $L_{\ell}^I(B; g_2, g_3)$ (if $\ell \equiv 0, 3 \pmod{4}$) or one of the three Type II spectral polynomials $L_{\ell}^{II}(B; e_\gamma, g_2, g_3)$ (if $\ell \equiv 1, 2 \pmod{4}$) will necessarily have $B=0$ as a root.

Dispersion relations in their Jacobi form can now be investigated. Recall that if $\ell=1$, the Jacobi-form Lamé equation (1.1) has $\Phi(\cdot; \alpha_0|m)$ as a solution, where the theta quotient $\Phi$ (the Jacobi-form version of Halphen’s l’élément simple) is defined in (1.3), and the multivalued parameter $\alpha_0$ is defined by $\text{dn}^2(\alpha_0|m)=E-m$. This solution has crystal momentum $k=k_1$ equal to $-iZ(\alpha_0|m) + \pi/2K(m)$, which is undetermined as to sign, and is also determined only up to integer multiples of $\pi/K(m)$. The sign indeterminacy is due to $\text{dn}^2(\cdot|m)$ being even. This causes $\alpha_0$ to be undetermined as to sign, and $k_1$ as well, since the function $Z(\cdot|m)$ is odd.

The parametrization point $\alpha_0$, or the equivalent point $u_0:=\alpha_0 + iK'(m)$ of the Weierstrassian form, corresponds to the parametrization point $(x_0, y_0)$ of the fundamental function $\Phi$ on the elliptic curve $E_{g_2,g_3}$. The correspondence is the usual one: $x_0 = \wp(u_0; g_2, g_3), y_0 = \wp'(u_0; g_2, g_3)$. The first of these two equations says that $x_0 = mn^2(\alpha_0|m) - \frac{1}{4}(m+1)$, and the latter that $y_0 = 2m\text{sn}(\alpha_0|m)\text{cn}(\alpha_0|m)\text{dn}(\alpha_0|m)$. The formula which computes $\alpha_0$ from $E$, namely $\text{dn}^2(\alpha_0|m)=E-m$, is readily seen to be a translation to the Jacobi form of the familiar condition $x_0 = B$, which simply says that the parametrization point $(x_0, y_0)$ must be ‘over’ $B \in \mathbb{C}$.

The correspondence between the Jacobi and elliptic-curve forms motivates the following reinterpretation of the crystal momentum of the fundamental solution $\Phi$, which is modelled on the reinterpretation of the last section. $k_1$ should be viewed as a function not of the energy $E \in \mathbb{C}$, but rather of a point $(E, \tilde{v})$ on the punctured Jacobi-form spectral curve $\tilde{T}_1(m) \setminus \{(\infty, \infty)\}$. There are two such points for each energy $E$, except when $E$ is a band edge. This is the source of the sign ambiguity in the parameter $\alpha_0$. Since $y_0 = 2\nu = 2i\tilde{v}$, the equation

$$m\text{sn}(\alpha_0|m)\text{cn}(\alpha_0|m)\text{dn}(\alpha_0|m) = i\tilde{v},$$

(7.5)

determines a unique sign for $\alpha_0$, provided that $\tilde{v}$ is specified in addition to $E$. $k_1$ will be written as $k_1(E, \tilde{v}|m)$, with the understanding that for this to be well defined, the pair $E, \tilde{v}$ must be related by the spectral curve relation $\tilde{v}^2 = \tilde{L}_\ell(E|m)$. The additively undetermined quantity $k_1(E, \tilde{v}|m)$ will not be undetermined as to sign. It is easily checked that on each branch, $k_1 \sim \tilde{v}/E$ as $(E, \tilde{v}) \to (\infty, \infty)$.

**Definition 7.3.** A solution of the Jacobi-form Lamé equation (1.1) is said to be a Hermite–Krichever solution if it is of the form

$$[\tilde{A}(\text{sn}^2(\alpha|m))\tilde{\Phi}(\alpha; \alpha_0|m) + 2\tilde{B}(\text{sn}^2(\alpha|m))\tilde{\Phi}'(\alpha; \alpha_0|m)]\exp(\kappa \alpha),$$

(7.6)

for some $\alpha_0 \in \mathbb{C}$ and $\kappa \in \mathbb{C}$. Here $\tilde{A}, \tilde{B}$ are polynomials, and $\tilde{\Phi}' := (d/d\alpha)\tilde{\Phi}$.

The expression (7.6) is a replacement for the original Jacobi-form expression (1.5), to which it is equivalent. Regardless of which is used, it is easy to compute the crystal momentum of an Hermite–Krichever solution. The momentum computed from (7.6) will be $[-iZ(\alpha_0|m) + \pi/2K(m)] - i\kappa$, up to additive
multivaluedness. The first term arises from the $\tilde{\Phi}, \tilde{\Phi'}$ factors, and the second from the exponential. The factors $A, B$ do not contribute, since $\sin^2(\alpha|m)$ is periodic in $\alpha$ with period $2K(m)$.

The Jacobi form of the Hermite–Krichever Ansatz asserts that for all integer $\ell$ and $m \in \mathbb{C} \setminus \{0, 1\}$, there is an Hermite–Krichever solution for all but a finite number of values of the energy $E$. On the elliptic curve $E_{g_2,g_3}$, these solutions were constructed from two maps: a projection $\pi_\ell : \Gamma_\ell \to \Gamma_1$ and an auxiliary function $\kappa_\ell : \Gamma_1 \to \mathbb{P}^1$. However, $\pi_\ell$ should really be regarded as a map from $\Gamma_\ell$ to $\Gamma_1$, on account of the correspondence between $E_{g_2,g_3}$ and $\Gamma_1$ provided by $(x_0, y_0) = (B, 2v)$. The following is the Jacobi-form version of proposition 6.1.

**Proposition 7.1.** Suppose that the integration of the Lamé equation on the elliptic curve $E_{g_2,g_3}$, for integer $\ell \geq 1$, can be accomplished in the framework of the Hermite–Krichever Ansatz by the maps $\pi_\ell : \Gamma_\ell \to \Gamma_1$ and $\kappa_\ell : \Gamma_1 \to \mathbb{P}^1$, where $\pi_\ell$ and $\kappa_\ell$ map the point $(B, v)$ to $(B_\ell(B; g_2, g_3), v_\ell(B; v; g_2, g_3))$ and $\tilde{k}_\ell(B; g_2, g_3)v$, respectively. Then the dispersion relation for the solutions of the Jacobi form of the Lamé equation will be $k = k_\ell(E, \tilde{v}|m)$, up to additive multivaluedness

$$k_\ell(E, \tilde{v}|m) = k_1(\mathcal{E}_\ell(E|m), \tilde{v}_\ell(E, \tilde{v}|m)|m) + \tilde{k}_\ell(E|m)\tilde{v},$$

in which

$$\mathcal{E}_\ell(E|m) := -B_\ell\left(-E + \frac{1}{3}\ell(\ell + 1)(m + 1); g_2(m), g_3(m)\right) + \frac{2}{3}(m + 1),$$

$$\tilde{v}_\ell(E, \tilde{v}|m) := -iv_\ell\left(-E + \frac{1}{3}\ell(\ell + 1)(m + 1), i\tilde{v}; g_2(m), g_3(m)\right),$$

$$\tilde{k}_\ell(E|m) := \tilde{k}_\ell\left(-E + \frac{1}{3}\ell(\ell + 1)(m + 1); g_2(m), g_3(m)\right).$$

The formula (7.7) follows by inspection. The projection $\pi_\ell$ reduces the integration of the Lamé equation to the integration of an $\ell=1$ equation, the ‘$B$’ parameter of which equals $B_\ell(B; g_2, g_3)$. By (7.1), the ‘$E$’ parameter of the $\ell=1$ equation will be the right-hand side of (7.8). The two terms of (7.7) are simply the two terms of $[-iz(\alpha_0|m) + \pi/2K(m)] - iK$. The equality $v = i\tilde{v}$ has been used.

It is straightforward to apply proposition 7.1 to the cases $\ell=2, 3$, since the coverings $\pi_2, \pi_3$ and auxiliary functions $\kappa_2, \kappa_3$ were worked out in §4. A brief discussion of the $\ell=2$ case should suffice. After some algebra, one finds

$$\mathcal{E}_2(E|m) = \frac{E^3 - 12(m + 1)^2E - 4(m + 1)(4m^2 - 19m + 4)}{9[E^2 - 4(m + 1)E + 12m]},$$

$$\tilde{v}_2(E, \tilde{v}|m) = -\frac{(E + 2m - 4)(E - 4m + 2)(E - 4m - 4)}{27[E^2 - 4(m + 1)E + 12m]^2}\tilde{v},$$

$$\tilde{k}_2(E|m) = -\frac{2}{3[E^2 - 4(m + 1)E + 12m]}.$$
from which \( k_2(E, \tilde{v}|m) \) may be computed by (7.7). Like \( k_1 \), \( k_2 \) is determined only up to integer multiples of \( \pi/K := \pi/K(m) \). Each branch of \( k_2 \) has the property that \( k_2(E, \tilde{v}|m) \sim \pm E^{1/2}, \ E \to +\infty \), with ‘\( \pm \)’ determined by the sign of \( \tilde{v} = \tilde{v}(E) \). This is a special case of a general fact: for all integer \([R_1, k]([E, \tilde{v}|m] w \ G E_1 = 2, E/C N, \] with ‘\( G \)’ determined by the sign of \( \tilde{v} \) \( \tilde{v}(E) \). This is a special case of a general fact: for all integer \([R_1, k]([E, \tilde{v}|m] w \ G E_1 = 2, E/C N, \] since each branch of \( k \) is asymptotic to \((\pi/K)^{k_1} \sim E^{1/2}, E \to +\infty \), as \((E, \tilde{v}) \to (\infty, \infty) \).

The real portions of the dispersion relations \( k = k_1(E, \tilde{v}|1/2), k_2(E, \tilde{v}|1/2) \) and \( k_3(E, \tilde{v}|1/2) \) are graphed in figure 1. For ease of viewing, each crystal momentum is regarded as lying in the interval \([0, \pi/2K]\); which is equivalent to choosing the sign of \( \tilde{v} = \tilde{v}(E) \) in an \( E \)-dependent way. As (7.2)–(7.4) imply, the two \( \ell = 1 \) bands are \([1/2, 1], [3/2, \infty) \), the three \( \ell = 2 \) bands are \([3 - \sqrt{3}, \sqrt{3} 3/2], [3, 3/2], [3 + \sqrt{3}, \infty) \), and the four \( \ell = 3 \) bands are \([9/2 - \sqrt{6}, 6 - \sqrt{15}], [15/2 - \sqrt{6}, 6], [9/2 + \sqrt{6}, 6 + \sqrt{15}], [15/2 + \sqrt{6}, \infty) \).

The \( \ell = 1 \) graph agrees with that of Li et al. (2000, fig. 6), and for confirmation, with that of Sutherland (1973, fig. 1). Unfortunately, the \( \ell = 2 \) graph disagrees with that of Li et al. in the placement or direction of curvature of each of the two upper bands. The algorithm they used for reducing \( \ell = 2 \) to \( \ell = 1 \), which was based on Hermite’s solution of the Jacobi-form Lamé equation (Whittaker & Watson 1927, §23.71), evidently yielded incorrect results for these bands. It appears that for the middle band, at least, the discrepancy can be traced to an incorrect choice of relative sign for the two terms of \( k = k_2 \). The reinterpretation of the crystal momentum as a function on the spectral curve, rather than a function of the energy, eliminates such sign ambiguities.

### 8. Summary and final remarks

A new approach to the closed-form solution of the Lamé equation has been introduced. Our key result, theorem 4.1, provides a formula for the covering map of the Hermite–Krichever Ansatz in terms of certain polynomials which are of independent interest, namely twisted spectral polynomials. The theorem permits an efficient computation of Lamé dispersion relations, of a mixed symbolic–numerical kind. Cohn polynomials, which are a new concept, have also been
introduced. The roots of such a polynomial are the points in elliptic moduli space at which a Lamé spectral polynomial has a double root, so that the Lamé spectral curve becomes singular. Twisted and theta-twisted Cohn polynomials could be defined, as well.

The approach of this paper can be extended from the Lamé equation to the Heun equation, which as a differential equation on the elliptic curve $E_{g_2,g_3}$ has up to four regular singular points, positioned at the finite Weierstrass points $\{(e_γ,0)\}_{γ=1}^3$ as well as at $(∞,∞)$. Its Weierstrassian form is called the Treibich–Verdier equation (Smirnov 2002), and its Jacobi form, at least when only two of the Weierstrass points are singular points, the associated Lamé equation (Magnus & Winkler 1979, §7.3).

The ‘four triangular numbers’ condition for the Heun equation to have the finite-band property, due to Treibich & Verdier (1992) and Gesztesy & Weikard (1995b), is now well known. In the finite-band case, the number of points in the algebraic spectrum has been computed up to multiplicity (Gesztesy & Weikard 1995b). The corresponding band-edge solutions are Heun polynomials. Applying the Hermite–Krichever Ansatz to the finite-band Heun equation leads to a greater variety of coverings of $E_{g_2,g_3}$ than arise in the solution of the integer-$ℓ$ Lamé equation; for example, coverings by a genus-2 hyperelliptic curve that have degrees 3, 4, 5 (Belokolos & Enol’skii 2000). These coverings play a role in the construction of elliptic soliton solutions of certain nonlinear evolution equations that occur in fibre optics (Christiansen et al. 2000). A treatment of the Heun equation along the lines of this paper will appear elsewhere.

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