Dynamical systems and the transition to turbulence in linearly stable shear flows

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Plane Couette flow and pressure-driven pipe flow are two examples of flows where turbulence sets in while the laminar profile is still linearly stable. Experiments and numerical studies have shown that the transition has features compatible with the formation of a strange saddle rather than an attractor. In particular, the transition depends sensitively on initial conditions and the turbulent state is not persistent but has an exponential distribution of lifetimes. Embedded within the turbulent dynamics are coherent structures, which transiently show up in the temporal evolution of the turbulent flow. Here we summarize the evidence for this transition scenario in these two flows, with an emphasis on lifetime studies in the case of plane Couette flow and on the coherent structures in pipe flow.

Keywords: pipe flow; strange saddles; strange attractors

1. Introduction

The transition to turbulence is straightforward to discuss in cases where the laminar profile becomes unstable when the driving of the flow is increased (Chandrasekhar 1961; Drazin & Reid 1981; Koschmieder 1993). Usually, the laminar profile then gives way to new flow patterns, typically with spatial or temporal modulations, which undergo further bifurcations at even stronger driving. The transitions occur at well-defined values of the control parameter, and these values as well as the characteristics of the new states are amenable to experimental scrutiny. The dynamics of liquids heated from below (Rayleigh–Bénard flow) and of flows between rotating cylinders (Taylor–Couette flow) are outstanding examples of the success of this programme.

The flows between two parallel plates moving relative to each other or pressure-driven flow down a pipe of circular cross section behave differently. Their linear and parabolic laminar profiles are linearly stable for all Reynolds numbers (Romanov 1973; Schmid & Henningson 1999; Meseguer & Trefethen 2003). Finite-amplitude perturbations can trigger a transition to turbulence, but the requirements for this to occur are not easily described. One indication of this is the wide ranges of ‘critical’ Reynolds numbers afloat in the literature (Kerswell...)

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Furthermore, the non-laminar flow pattern is not a set of simple vortices or cells, as in the Rayleigh–Bénard or the Taylor–Couette flow, but immediately spatially and temporally fluctuating. An additional unexpected and complicating feature is that the turbulent state very often does not persist indefinitely, but decays without any precursors or indications of the imminent decay (as first noted by Brosa (1989)).

Plane Couette flow and pipe flow stand out owing to their simple geometry and practical relevance. The related pressure-driven flow between parallel plates has a linear instability (Drazin & Reid 1981; Schmid & Henningson 1999). However, the transition to turbulence is observed at much lower Reynolds numbers and shows many of the characteristics of plane Couette flow and pipe flow: there does not seem to be a connection between the linear instability and the transition. A similar situation arises in the case of the flow between rotating cylinders, which approaches the case of plane Couette flow in the limit of large radii and small gap; then the linear instability to Taylor vortices moves to very high Reynolds numbers and the transition to turbulence is observed well below this value (Faisst & Eckhardt 2000). In addition to these so-called internal flows, bounded by walls, there are external flows, like boundary layers or flows around bluff bodies, that show similar features (Holmes et al. 1996; Schmid & Henningson 1999). Therefore, we expect that an understanding of the dynamics in the two internal flow cases can contribute to an understanding of the other cases as well.

Starting with Schmiegel & Eckhardt (1997), we have applied tools from dynamical system theory to understand and explain the behaviour of these flows. The basic method is a very simple one and can be used in both experiments and numerical simulations: pick a flow field as the initial condition to the Navier–Stokes equation and follow its evolution. But rather than focusing on one initial condition and a long-time evolution in order to converge the statistical properties, we take many initial conditions and integrate them over time intervals of some intermediate length. Intermediate means long enough to see more than just the initial transients, but not necessarily long enough to obtain the statistical averages of the turbulent flow. Owing to the chaotic nature of the turbulent flow, the point of reference is the Lyapunov exponent and its inverse, the Lyapunov time. ‘Intermediate’ then refers to a time interval that is several Lyapunov times long.

A second method focuses on the invariant structures in a state space: there are perhaps stationary states, travelling waves, periodic solutions or other coherent structures. Unless they are stable, they correspond to hyperbolic objects with stable and unstable manifolds. The network of connections set up by these manifolds and their crossings then contains Smale horseshoes as the basic elements of chaos and complexity (Guckenheimer & Holmes 1983; Ott 1993).

These approaches to the transition problem are inspired by the ones used in transient chaos (Kantz & Grassberger 1985; Tél 1991) and chaotic scattering (Eckhardt 1988; Smilansky 1992): upon scattering of one molecule off another, or, closer to the hydrodynamic context, one set of vortex pairs by another (Manakov & Shchur 1983; Aref et al. 1988; Eckhardt & Aref 1988), the entities can form a transient compound state in which all constituents are close together. During the close-up phase, the dynamics has features of a chaotic motion; in particular, there is a positive Lyapunov exponent. Then, suddenly, and without much advance notice, the compound state falls apart and the constituents escape. The collapse is a
statistical process and is often described by an exponential distribution of lifetimes (Kadanoff & Tang 1984; Tél 1991). The chaos forms around indefinitely living trapped periodic motions, and the hyperbolic tangles that form in their neighbourhood (Guckenheimer & Holmes 1983; Ott 1993).

To a great extent, the same programme can be and has been pursued in the case of shear flows. Plane Couette flow is the historically first example, where lifetimes have been studied (Bottin & Chaté 1998; Bottin et al. 1998b) and coherent structures have been identified (Nagata 1990, 1996; Clever & Busse 1992, 1997; Eckhardt et al. 2002). Similar investigations of pipe flow came later: lifetimes are studied in the papers by Faisst & Eckhardt (2004), Hof et al. (2005, 2006), Mullin & Peixinho (2006a,b) and Peixinho & Mullin (2006), and coherent states in the papers by Faisst & Eckhardt (2003), Hof et al. (2004), Wedin & Kerswell (2004) and Schneider et al. (2007b). The status of pipe flow has been reviewed by Kerswell (2005) and Eckhardt et al. (2007) and the related studies of other flows may be traced from Mullin & Kerswell (2004).

As in the case of transient chaos, a complete investigation then requires a study of the lifetimes and of coherent states. In view of previous publications and reviews, we here focus on previously unpublished results and summarize the related results. Therefore, in §2, we present an extensive discussion of the lifetimes of perturbations in plane Couette flow, as contained in the thesis of Schmiegel (1999), and only a brief summary of the studies of coherent states. Section 3 deals with pipe flow, where the focus is on the properties of the coherent states, complemented by a brief summary of the ongoing discussion about the lifetimes. We conclude with a brief summary and an outlook in §4.

2. Plane Couette flow

After a general introduction, we turn to the discussion of lifetime experiments with suddenly and adiabatically introduced initial conditions and conclude this section with a summary of the work on coherent states.

(a) General remarks

In plane Couette flow, the fluid is bounded by two infinitely extended parallel plates moving in opposite directions. The velocities of the plates are ± U₀, so that the mean velocity vanishes. We measure velocities in units of U₀, distances in units of d/2, i.e. half the gap widths, and time in units of τ = d/2U₀. Then the Reynolds number is Re = U₀d/2ν, with ν the kinematic viscosity. For a comparison with pipe flow and other shear flows, we note that the Reynolds number should be based on the full velocity difference and the full distance over which it occurs. For plane Couette flow, such a Reynolds number would be (2 U₀)(d)/ν and hence four times as large as the one commonly used. Coordinates are chosen such that eₓ points in the streamwise, eᵧ in the spanwise and eᶻ in the wall-normal direction. The laminar velocity profile is then given by U₀ = z·eₓ.

A finite disturbance u = (u, v, w) superimposed on the laminar profile evolves according to

\[ \frac{∂u}{∂t} = -(u·∇)u - z\frac{∂u}{∂x} - we_x - \text{grad} p + Re^{-1}\Delta u. \]  (2.1)
The flow field \( \mathbf{u} \) is incompressible and vanishes at the plates,
\[
\text{div } \mathbf{u} = 0 \quad \text{and} \quad \mathbf{u}|_{z=\pm 1} = 0.
\]
For the numerical representation of the flow field, we expand in Fourier modes in the spanwise and streamwise directions and normalized Legendre polynomials \( \phi_p(z) \) in the wall-normal direction,
\[
\mathbf{u}(x, t) = \sum_{n_x, n_y, p} \mathbf{u}_{k,p}(t) \exp(i(2\pi n_x x/L_x + 2\pi n_y y/L_y))\phi_p(z),
\]
where \( L_x \) and \( L_y \) are the streamwise and spanwise widths of the integration domain, respectively. Following Clever & Busse (1992, 1997) and Nagata (1996), we take \( L_x = 2\pi d \) and \( L_y = \pi d \), where \( d \) is the gap width. The wavevectors in these directions are constrained by \( |n_x| + |n_y| \leq N \), where in most of our simulations \( N=6 \). The maximal order \( N_z \) of Legendre polynomials used was \( N_z = 9 \). The code was verified by reproducing the eigenvalues of the linearized problem and the coherent state of Clever & Busse (1997).

We worked with two kinds of initial conditions: a sudden perturbation, introduced by instantaneously adding it to the laminar profile, and a second one realized by adiabatic switching of a perturbation. The sudden perturbation has a transient period in which it adjusts to the flow. The adiabatic one is free of this initial transient and gives information about the approach to the turbulent state.

(b) Evolution of suddenly introduced vortex rings

The suddenly introduced perturbations consist of poloidal vortex rings that are superimposed on the linear profile to form the initial flow field. The vortex rings are given by
\[
\mathbf{u}_0 = A \text{curl curl} \exp(-\sigma^2(x^2 + y^2 + z^2))\mathbf{e}_j,
\]
where \( A \) is the amplitude and \( \mathbf{e}_j \) is a unit vector pointing in the direction of the axis of the vortex. With a dimensionless inverse width of \( \sigma = 8 \), they are strongly localized in directions perpendicular to \( \mathbf{e}_j \). For \( j = z \), i.e. a vortex with axis pointing in the wall-normal direction, the disturbances are similar to those generated in the experiments of Daviaud et al. (1992) and Tillmark & Alfredsson (1992).

The initial evolution of the vortex rings is governed by the shear and stretch of the laminar profile and is similar to that of singular vortex lines that carry all the vorticity. However, vorticity spreads out very quickly and the dynamics becomes less obvious. In our numerical simulations, we monitor two scalar variables: the energy in the perturbation and the mean shear. The energy density in the perturbation \( E \) divided by the energy density of the laminar profile is given by
\[
E = \frac{1}{V} \int \mathbf{u} \cdot \mathbf{u}^* \, dV,
\]
where \( V \) is the volume and \( \mathbf{u}^* \) is the complex conjugate of \( \mathbf{u} \). The shear rate is calculated from the downstream component \( u \) of the velocity field and the area \( A \) of one plate,
\[
M = \frac{1}{AS_0} \int \frac{\partial u}{\partial z} \bigg|_{z=1} \, dA,
\]
normalized by the laminar shear rate \( S_0 = 2U_0/d \).
All perturbations decay when the Reynolds numbers are low and the initial amplitudes of the perturbations are small. This regime is by and large governed by the linearized equation. For larger amplitude and Reynolds numbers, the perturbations show nonlinearly sustained oscillations for some time before finally decaying. Energy is continually shifted between the components, but eventually the fluctuations decay and the system approaches the laminar state. Once it gets sufficiently close, it cannot escape owing to the linear stability of the laminar profile. Another way to see this is by studying the process by which the perturbation can draw energy from the basic profile (Boberg & Brosa 1988; Grossmann 2000): it becomes ineffective once the fluctuations in the normal component are too small. Hence, the criterion that we use to conclude that a state will decay is that the energy in the wall-normal component falls below $10^{-4}$.

For even higher values of the Reynolds number and a sufficiently large initial amplitude, a typical perturbation does not decay within the observation time. This type of dynamics is closest to what one might consider a turbulent state. However, since there does not seem to be a precursor to the decay, it is not clear that this type of trajectories indeed lives forever. In particular, it can well happen that some of the initial conditions in this regime decay anytime after the maximal integration time, which here was typically limited to $t_{\text{max}} = 3000$ dimensionless units. While earlier studies suggested that there indeed was a Reynolds number beyond which no trajectories would ever decay (Bottin & Chaté 1998), more recent studies give evidence that a decay will always be possible (see Hof et al. (2006) and below).

In figure 1, we show the lifetimes for vortex ring perturbations with axis pointing in the wall-normal directions, calculated on a grid of $40 \times 40$ points in the amplitude versus Reynolds number plane. The short-lived laminar regions and the long-lived turbulent ones clearly stand out, with a rugged transition region in-between. Magnifications of the intermediate region show huge variations in

Figure 1. Lifetimes in plane Couette flow. (a) Lifetimes of perturbations in plane Couette flow as a function of the Reynolds number and amplitude of perturbations for a vortex ring with axis pointing in the normal direction. The lifetimes are calculated on a $40 \times 40$ grid. (b) Median (filled symbols) and maximum (open symbols) lifetimes at a fixed Reynolds number for different orientations of the vortices and 40 values of the amplitude.
lifetimes from one lattice point to the other. This sensitivity to perturbations is a stable feature, also observed for other aspect ratios (Schmiegel & Eckhardt 1997) and in simpler models (Eckhardt & Mersmann 1999; Moehlis et al. 2004a,b).

In order to extract the trend in lifetimes, one can collect results for different amplitudes but for a fixed Reynolds number. Two simple measures are the median and the maximum of the observed lifetimes at a given Reynolds number. Figure 1 shows the results for all three different orientations of the vortices. In the linear regime, the dynamics should be independent of the amplitude of the perturbation so that the median and the maximum of the lifetimes coincide. Within the numerical fluctuations, this is the case for the Reynolds numbers below approximately 215, 225 and 255 for vortex rings with axis pointing in the downstream ($e_x$), spanwise ($e_y$) and wall-normal direction ($e_z$), respectively. For a higher Reynolds number, the median and the maximum of the lifetimes differ. The first lifetimes exceeding the maximum integration time are found for the Reynolds numbers of 290, 295 and 285, respectively (the same order of the perturbation orientation as above). The median reaches the maximum integration time at Reynolds numbers of 320 for streamwise and wall-normal orientations and 305 for spanwise orientation.

The median lifetimes and, even more so, the maximum of the observed lifetimes depend on the realizations and the choice of an initial ensemble (further studies of this point are part of ongoing investigations). The quantity that is most robust against variations in the choice of initial conditions is the long-time behaviour of the probability $P(t)$ to remain turbulent for at least a time $t$: for long time, when the influences from the initial conditions and transients are gone, it should fall off exponentially (Kadanoff & Tang 1984; Tél 1991). Such an exponential distribution has been seen in experiments (Bottin & Chaté 1998; Bottin et al. 1998b) and direct numerical simulations (Eckhardt et al. 2002; Eckhardt & Faisst 2004).

As a possible operational definition for the threshold Reynolds number for the transition to turbulence, the occurrence of the first states with lifetimes exceeding the observation time is less suitable since it depends sensitively on initial conditions and the occurrence of rare events. A statistical measure like the median is perhaps more stable and reproducible. Ideally, the median should be determined from the asymptotic exponential tails, but since there are almost no immediately decaying trajectories in the data introduced in figure 1, we determined the values from the full $P(t)$. As the median exceeds the maximum integration time if half the initial conditions do, one can define a threshold Reynolds number as the one where more than half of all initial conditions are not observed to decay. This gives Reynolds numbers of 320 and 305 for the different initial conditions mentioned above. The consequences of lowering the observation time are illustrated in figure 2, where the fraction of initial conditions with lifetimes exceeding 2500 is shown. More than half of the initial conditions live longer than this time for Reynolds numbers above approximately 315; for spanwise orientation of the vortex, the Reynolds number is a bit lower. Experiments by Bottin & Chaté (1998) with a cut-off of 20 000 time units show a critical Reynolds number of approximately 328. Thus, an eightfold increase in lifetimes changes the Reynolds numbers by only approximately 4%.

In the spirit of Darbyshire & Mullin (1995), we also studied the effects of averaging over a narrow window in initial conditions and Reynolds number. The amplitude was taken from the interval $A \in [0.18 \pm 0.12]$ and the Reynolds...
number was within an interval of ±5 from the indicated value (the widths of the Reynolds number interval are the experimental resolution of Bottin et al. (1998)). A state was classified as long-lived if the time was above the level $t_c$ as indicated. Figure 2 shows the probability to hit a long-lived initial condition. On the assumption that the probabilities are drawn from a binominal distribution, a variance of $p(1-p)$ can be assigned to each value; this gives the error bars shown in the figure. The probability of obtaining a long-living state grows rapidly between $Re=290$ and 325. This interval shifts to higher $Re$ for larger lifetimes. The probability does not reach 1 within the range of Reynolds numbers studied, indicating that transient trajectories are still possible.

The calculations show that even with the probabilistic approach to a definition of a turbulence threshold, not all dependencies can be removed. However, the variation of the threshold Reynolds number with cut-off time is weak, so that for Reynolds numbers above approximately 320 more than half the runs should be turbulent in a typical experiment. Doubling the observation time can shift this value upwards by approximately 10 and changing the perturbation to a ring with spanwise axis can lower it by approximately 15. The picture that emerges is compatible with the amplitude–Reynolds number dependence documented experimentally, say, in fig. 11 of Bottin & Chaté (1998).

An important feature of the probabilities in figure 2 is their curvature and S-shape: as was emphasized by Hof et al. (2006), a divergence of the lifetimes at a finite $Re$ would imply that these curves reach a maximal value at a Reynolds number that is independent of the observation time. However, all curves bend towards increasing $Re$, suggesting that there is always a small but non-negligible
fraction of initial conditions which decays. The analysis of the characteristic times then shows that they seem to increase exponentially with Reynolds number. This observation is at the focus of an ongoing discussion (see Hof et al. 2006; Peixinho & Mullin 2006; Willis & Kerswell 2007).

\((c)\) Triggering the transition by adiabatic perturbations

The second class of perturbations is driven by a slowly switched body force that allows one to adiabatically introduce the perturbation on top of the laminar profile. It allows one to explore a wider range of intermediate energies and to study the approach to the turbulent flow. In many experiments, the perturbation is applied in the form of a finite time pulse in a small region near the wall, thus introducing a non-symmetric perturbation with a strong wall-normal component. We model it by adding to the equations of motion (2.1) a time-dependent volume force,

\[
\frac{\partial \mathbf{u}}{\partial t} = \mathcal{N}(\mathbf{u}) + f(t)\mathbf{F}(x),
\]

where \(\mathcal{N}\) refers to the r.h.s. of the Navier–Stokes equation (2.1). With the maximum amplitude \(A_0\) and the duration of the impulse \(t_{\text{dist}}\), we take the amplitude function on the interval \(0 \leq t \leq t_{\text{dist}}\) to be

\[
f(t) = A_0 \left( \sin \left( \pi \left( 2 \frac{t}{t_{\text{dist}}} - \frac{1}{2} \right) \right) + 1 \right).
\]

In order to break the reflexion symmetries of the system, a volume force with inflow and outflow that is asymmetric with respect to the mirror symmetry in the spanwise direction, \(u \rightarrow -u\) and \(v \rightarrow -v\), is used. The analytical form of the \(z\)-component is

\[
F_z = \sqrt{\pi} \exp(-\sigma(y^2 + x^2) - 20(z + 1)) \int_0^{\sqrt{\sigma x}} d\tilde{y} \, e^{-\tilde{y}^2},
\]

and the other components are also non-zero after projecting onto the divergence-free subspace. The inverse width is \(\sigma = 32\).

Time traces for a weak perturbation at \(Re=400\) are shown in figure 3. The perturbation ends at \(t_{\text{dist}} = 10\), before the first data point in the figure. The traces show that there is a gradual increase in energy until a value of approximately 0.05 is reached. Then, the energy quickly builds up to the value of the fully developed turbulent state, approximately 0.3. The collection of time traces for perturbations with slightly different amplitudes shows that the slope of the initial increase can differ widely and is mapped to a wide range of times after which the perturbations swing up to the turbulent state. However, the dynamical approach to the turbulent state, including the overshoot in energy, is very similar, once an energy level of approximately 0.05 is crossed. This variability with initial conditions can affect the short time part of the lifetime distribution, illustrating the need to extract the characteristic times from the tails.

\((d)\) Permanently living states

The general theory of transient chaos and chaotic scattering shows that the long-lived turbulent states form around a dense set of permanently lived structures: they include stationary states, periodic states, homoclinic and heteroclinic connections between them and aperiodic states hopping in a seemingly chaotic fashion between the simpler states. Are there such states in plane Couette flow?
Owing to the reflection symmetry across the middle, plane Couette flow sustains stationary flows. They were first discovered by Clever & Busse (1992, 1997) and Nagata (1996; see also Waleffe (2003) for more recent and more accurate numerical characterizations of the states). Interestingly, the flow consists of vortices that are weakly modulated in the downstream direction and the associated streaks (figure 4a). Breaking the up–down symmetry weights the top and bottom half differently, and the structures turn into travelling waves, moving with a constant speed without changing their form (figure 4b; Nagata 1997; Eckhardt et al. 2002). Secondary bifurcations can bring in additional time dependencies. A variety of such modes have been found in plane Couette flow (Schmiegel 1999). Some can be grouped according to their topology, with different numbers of vortices per cell or different arrangements, including staggered ones with several vortices in the normal directions, and others look alike and differ only in minute details in flow pattern (and hence are numerically difficult to discern). Additional sets of coherent states, in particular periodic ones (Kawahara & Kida 2001), can be expected in the neighbourhood of homoclinic and heteroclinic connections. In principle, all these states can contribute to the turbulent dynamics.

In order to obtain some insight into the dynamical relevance of the stationary states, we project them onto a two-dimensional space spanned by the two variables: energy and shear as introduced in §2b. The stationary states then are points in this space. We picked a few initial conditions in the neighbourhood of these periodic solutions and mapped out their further evolution in this space. As the actual trajectories are quite irregular, we replaced them with sketched arrows in figure 5. At $Re=200$, most of the stationary states are never visited and there

Figure 3. Approaching the turbulent state with adiabatically switched perturbations. (a) Time evolution of the energy in the perturbation for 100 runs with a finite time perturbation of a randomly selected amplitude $A = 0.025 \pm 0.005$ at a fixed Reynolds number $Re=400$. Two traces are connected by straight line segments. Of the other runs, only the instantaneous values are indicated by dots. The dots clearly separate into two groups with an energy less than approximately 0.1 and an energy approximately 0.3. The lack of points in the intermediate region is due to the rapid increase in energy in that range. (b) Time evolution for an ensemble with larger perturbation amplitudes $A = 0.035 \pm 0.005$. The line at approximately $E=0.3$ indicates the time average of a turbulent signal at $Re=400$. Also indicated are the mean energy and the variance at each Reynolds number. They are connected by straight line segments to highlight the oscillatory relaxation to the turbulent state.
are only a few connections. At $Re=240$, where the annealing experiments in Schmiegel & Eckhardt (2000) did show some turbulence, more states participate in the dynamics and there are more loops and some begin to link up. What is particularly noticeable is the concentration of the arrows and links near $E=0.2$,
which is also the region where the values from a typical turbulent trajectory come to lie, clearly suggesting a link between the turbulent dynamics and the coherent states and their connections. Another approach to linking coherent states and turbulence has been developed and followed for pipe flow (see Schneider et al. 2007b; §3). Ultimately, one would like to collect all the states, weigh them with the inverse of their stability and calculate all statistical averages from periodic orbit theory ( Cvitanovic & Eckhardt 1991; Ott & Eckhardt 1994). This, however, requires more information about the states and their organization than is currently available.

3. Pipe flow

(a) General remarks

The laminar profile for flow down an infinitely long pipe is parabolic,

\[ u_z(r) = u_c (1 - (r/R)^2) e_z, \quad (3.1) \]

where \( u_c \) is the centreline velocity and \( R \) is the radius. A Reynolds number may be based on the maximum velocity \( u_c \) at the centre and the radius, or the mean velocity \( \bar{u} \) and the diameter,

\[ Re = \frac{u_c R}{\nu} = \frac{\bar{u}(2R)}{\nu}. \quad (3.2) \]

While the Reynolds numbers are the same, the characteristic time scales vary by a factor of 4 since \( t_c = 2R/\bar{u} = 4R/u_c = 4t_c \). Following experimental conventions, we take the second definition with the mean velocity.

For the numerical representation of the Navier–Stokes equation, we use a Fourier–Legendre collocation method in cylindrical coordinates with Lagrange multipliers to account for no-slip boundary conditions at the wall and the constraints that the flow field is solenoidal, analytical and regular in the centre for \( r = 0 \). The programme was verified by reproducing literature values for the linearized problem (Schmid & Henningson 1994), for the nonlinear dynamics of optimal modes (Zikanov 1996; Meseguer 2003) and for the statistical properties of the fully developed turbulent flow up to Reynolds numbers of 5000 (Eggels et al. 1994). We used a resolution of \( |n/11| + |m/11| < 1 \), where \( n \) and \( m \) are the azimuthal and downstream wavenumbers, respectively, and up to 56 Legendre polynomials radially.

The laminar profile is stable for all Reynolds numbers, but experiments start showing turbulence for Reynolds numbers of approximately 1700. Probing the phase space with initial conditions as in the case of plane Couette flow reveals similar characteristics: for low Reynolds number, all initial conditions decay; and for increasing Reynolds number, more and more initial conditions show turbulence and the turbulent lifetimes increase until eventually it becomes virtually impossible to find an initial condition that decays. Averaging over initial conditions at a fixed Reynolds number, the distribution of lifetimes can be measured, and once again it follows an exponential law that can be characterized by the time over which the probability drops to 1/e. Following earlier observations in plane Couette flow, it was suspected that this time would diverge at a critical Reynolds number somewhere between 1750 and 2250.

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However, experiments in a very long pipe suggest that there is no divergence, but that the characteristic time increases rapidly. There is no direct theoretical reason for any specific behaviour (Eckhardt & Faisst 2004), but the analysis by Hof et al. (2006) shows that the data are compatible with an increase that is exponential in Reynolds number. This suggests that (i) there is no transition to an attractor, (ii) the turbulent dynamics and the laminar one are always dynamically connected without barrier, and (iii) there should be new possibilities for turbulence control.

(b) Travelling waves

As in the case of plane Couette flow, there exist in pipe flow coherent structures around which the chaotic dynamics develops. They were found by adiabatically following the states that bifurcate from downstream vortices superimposed on the parabolic flow (Faisst & Eckhardt 2003; Wedin & Kerswell 2004). The initial vortices are translation invariant downstream and are sustained by a suitable body force which will eventually be reduced to zero. The details of the body force are not important, as long as the initial state undergoes a bifurcation that breaks the translational symmetry, since it can be shown that a translationally invariant flow must decay. This bifurcation typically is supercritical when the Reynolds number is low and the forcing large, but turns subcritical when the Reynolds number increases and the forcing decreases. The symmetry broken state can then be followed to the case of zero forcing. Different types of coherent states can be obtained for different topologies of the forcing.

All coherent states identified thus far are dominated by downstream vortices that drive streaks, i.e. modulations in the downstream velocity. This is shown in figure 6 for a state with four vortices, arranged in two pairs. The low-speed streaks in the middle show wiggles similar to the ones in figure 4, but the high-speed streaks close to the wall are rather more stationary. Similar behaviour is seen for states with three, four and five vortex pairs (Faisst & Eckhardt 2003). The number of vortices does not uniquely identify the states, and there can exist neighbouring ones of similar topology (Wedin & Kerswell 2004).

The radial flow changes the mean flow profile as shown in figure 7. There is a steepening near the walls and a flattening in the centre. The values differ for the upper and the lower branches that emerge at the point of bifurcation. For the state with six vortex pairs, the differences between the two states are larger (figure 8). It is also noticeable that in the central region of the pipe, the influence of the vortices is reduced and the mean profile again develops an almost quadratic shape, reminiscent of an unperturbed pipe flow in a narrower tube. This effect becomes even more pronounced for more vortices, as they tend to line up closer to the wall.

The coherent states in pipe flow have one free parameter, the downstream wavelength. At the point of bifurcation, only a single length is possible, but for higher Reynolds numbers a whole interval of wavelengths becomes accessible. Over this interval, the properties of the states change. Figure 9 shows the variations of the phase speed and the friction factor with wavelength for the four-vortex state at different Reynolds numbers. The phase speed of the travelling waves is higher than the mean velocity, but slower than the maximal
velocity at the centre of a parabolic profile with the same mean speed. The variations in friction factor are small and tend to lower values as the Reynolds number increases.

The prominence of the coherent structures in pipe flow has triggered experimental efforts to detect them (Hof et al. 2004). They can appear in signals transiently, when the flow state approaches the flow along the stable manifold and

(c) Correlation functions for the coherent structures

The prominence of the coherent structures in pipe flow has triggered experimental efforts to detect them (Hof et al. 2004). They can appear in signals transiently, when the flow state approaches the flow along the stable manifold and
leaves it along the unstable one. Ideally, one would like to obtain statistics on the frequency with which a neighbourhood of the travelling waves is visited in the state space. To clearly identify such a state, a sufficient number of d.f. have to be used, but then it becomes exceedingly difficult to obtain statistically reliable probabilities. Therefore, we devised a correlation function, which projects the full state space onto a low-dimensional subset and allows us to extract the number

Figure 8. Mean profiles for the six-vortex state at Reynolds number (Re) (a) 1250 (bifurcation), (b) 1500 and (c) 2000. The dashed line shows the parabolic profile with the same mean velocity and the dotted line shows the parabolic profile with the same friction coefficient, which agrees with the slope near the wall. The upper branch is shown to the left and the lower to the right.

Figure 9. Wave number dependence of a four-vortex travelling wave at Re=2000. (a) Downstream phase velocity $v$ and (b) friction factor $f$. The states with higher friction have lower phase velocity. The four-vortex travelling wave exists over a wave number interval $(0.83,1.96)$, i.e. up to 45% lower and 30% higher than the critical wave number $k_z=1.5$ at bifurcation. The numerical resolution is $|n/10| + |m/10| \leq 1$, i.e. 66 Fourier modes, and 56 Legendre polynomials, corresponding to 5595 active d.f.
of vortices in the state as an approximation to the visitation of a travelling wave with the same number of states (Schneider et al. 2007b). This measure also takes into account the myriad of similar travelling waves that exist near a hyperbolic one (Guckenheimer & Holmes 1983).

With \( u \) the downstream velocity component and \( \phi \) the azimuthal angle, the correlator in a cross section is

\[
C(\phi; r) = \langle u(\phi' + \phi, r)u(\phi', r) \rangle_{\phi}. \tag{3.3}
\]

The radius \( r \) where it is calculated should cut across the centre of the vortices and will be taken to be \( r=0.81 \). Averaging over the angle \( \phi' \) eliminates a continuous d.f. in azimuthal direction where the vortices can be located. A vortex pair will then give a positive value at zero, a zero crossing in the middle between the vortices and a negative maximum at the position of the second vortex in the pair. In order to highlight the variations with \( \phi \), we show in figure 10 the derivative of the correlation function: it vanishes at the origin and at the position of the extrema and it is maximal near the zero crossings.

As an example of how close a time evolution can come to a coherent state, we show in figure 11 a time signal and a cross section of a state that is tantalizingly close to a coherent travelling wave described by Wedin & Kerswell (2004); this shows that the coherent states indeed appear transiently in the time dynamics.

The advantages of this correlation function are its ease of calculation and the clear signal it provides for the vortices. It enables a statistical decomposition of the flow into situations with different numbers of vortices, a study of the times it stays there and the transitions between them. This should then allow a statistical analysis of the coherent structures and their effects on the turbulent properties.

**4. Outlook**

The dynamical system approach to the transition to turbulence has helped to characterize the transient nature of the turbulent state in the transition region and explains why a diversity of critical Reynolds numbers is quoted in the literature. It links the transition to turbulence to the appearance of three-dimensional states,
which in their simplest form are stationary states or travelling wave solutions. These states typically are saddles and do not persist forever, but they can appear transiently during the evolution of the flow. Dynamical system theory suggests that a complete classification of all periodic states and their statistical weights could be used to estimate the statistical properties, but we are far from a sufficient understanding of the organization of these states. Perhaps more accessible is the challenge to explain why there is a gap between the Reynolds numbers when coherent structures first appear (125 in plane Couette flow and 780 (Pringle & Kerswell 2007) in pipe flow) and the ones where turbulence first appears in experiments (approx. 280 and 1700, respectively).

Pushing the use of initial-value studies and dynamical system ideas further, one can ask about the nature of the transition state between the laminar and the turbulent dynamics. An indication of a reasonably clear distinction comes from the trigger experiments in plane Couette flow shown in figure 3, where trajectories stayed at intermediate energies for a while, before swinging up to the turbulent one. This can be formalized: increasing the amplitude of the initial perturbation from zero, the lifetimes will increase, and there will be a critical initial condition that neither decays to zero nor swings up to the turbulent case. The studies on a low-dimensional model by Skufca et al. (2006) indicate that this state indeed exists, and in the meantime it has been found in pipe flow (Schneider & Eckhardt 2006; Schneider et al. 2007a). Specifically, there is a surface that separates states that decay immediately and those that become turbulent and within that surface states evolve towards a specific relative attractor. Not unexpectedly, the relative attractor is dominated by a pair of vortices.

In addition to plane Couette flow and pipe flow as discussed here, dynamical systems ideas can also be applied to other transitional flows (Itano & Toh 2001; Toh & Itano 2003) and boundary layers (Holmes et al. 1996). All these efforts demonstrate the value of dynamical systems ideas in the transitional regime and suggest many lines of continuation. They could even be valuable in the fully developed turbulence regime, as shown in the studies by van Veen et al. (2006).

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Figure 11. Transient appearance of a coherent state. (a) Indicator function for the appearance of states with three, four, five and six pairs of vortices. Note the wide region for times between 50 and 100 where the three-vortex signal becomes very strong at the expense of the others. (b) Cross section of the flow in the prominent three-vortex region, averaged in time for approximately 20 $D/u$ between 65 and 85. This state is very similar to the one shown in figs. 8 and 9 of Wedin & Kerswell (2004). The Reynolds number is 2500 and the length of the pipe is 5$D$.  

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