Maxwell and the classical wave particle dualism

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Maxwell’s equations are one of the greatest theoretical achievements in physics of all times. They have survived three successive theoretical revolutions, associated with the advent of relativity, quantum mechanics and modern quantum field theory. In particular, they provide the theoretical framework for the understanding of the classical wave particle dualism.

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1. Introduction

Having used Maxwell’s equations almost every day for nearly three decades, I was particularly pleased to have been in Aberdeen for the commemoration of Maxwell’s achievements. In this presentation, I will discuss a somewhat ignored issue, the classical wave particle dualism that in the case of radiation phenomena can be entirely understood in the frame of Maxwell’s theory of radiation.

The wave particle dualism is usually considered in the context of quantum physics, and we have the tendency to forget that it also takes place in classical physics. In particular, these complementary aspects of nature, when applied to electromagnetic radiation, can be both completely described by Maxwell’s equations, or from equations that can be derived from them.

I will start this presentation with a short historical account of wave and particle theories of light and of the central role played by Maxwell in this theoretical debate. I will then give a modern perspective on the classical theory of radiation, by describing the various levels of understanding of the radiation phenomena. First of all, light can be seen as a wave process, as described by the wave equation. This equation is a direct consequence of Maxwell’s equations. Second, light can be seen as particles of light (nowadays called photons), which behave in a medium according to the well-known geometric optics or ray equations. Third, light can be seen as a fluid and its evolution is described by a wave kinetic equation. Under certain conditions, this kinetic equation states the conservation of the number of photons. We will establish a link between these three levels of description (wave, fluid and single particle) and indicate their range of application. The interesting and most relevant aspect of these three different approaches to radiation is that they are all based on Maxwell’s equations.

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We will also describe less known aspects of classical radiation, related to the particle-like behaviour of light. We discuss several different aspects of the photon (seen here as a classical particle), such as its effective mass and its effective charge or electric dipole, the possible occurrence of photon acceleration, diffusion and trapping, or the occurrence of inelastic photon–photon collisions, associated with photon mixing and splitting in nonlinear optical media. These nonlinear processes obey well-defined energy and momentum conservation relations that can be derived from Maxwell’s classical theory and coincide with the corresponding quantum relations.

2. Classical complementarity

The history of the concepts of the nature of light and radiation (Whittaker & History 1989), from the ancient times to the present, is dominated by two main and closely related problems. One is the nature of empty space, or ether or void, which corresponds to the universal background scenario where radiation phenomena take place. The other is the nature of radiation itself: is light a vibration of this basic and primordial medium or is it made of independent particles moving in empty space? Maxwell’s (1891) equations will give a definite answer to both problems, at least as definitive as classical theory can be considered.

Although not denying the existence of the ether, Newton assumed that light is something else and, following the atomistic tradition and along the lines of Gassendi, described light as made of particles, with different energies and sizes. Therefore, light rays could be described as streams of corpuscles. This view led the foundations of geometric optics still used in the present day to describe light propagation in non-uniform media, such as optical circuits with lenses, mirrors and prisms, as well as radio wave propagation in the ionosphere, or in dense fusion plasmas. The most impressive success of the corpuscular theory of light was the explanation of the (simple or multiple) rainbows that can be seen on a rainy day. But its most important discovery is associated with Newton’s prism experiment, which corresponds to the production of artificial rainbows in a laboratory and led to the invention of spectroscopy. His atomistic or corpuscular view of light led Newton to an intuitive explanation of his experiment, and also of the phenomenon of total reflection that would be recovered later by Maxwell’s theory of radiation, as discussed below.

At about the same time, Huygens formulated his competing wave theory of light that followed the tradition of Descartes and anticipated the fundamental work of Fresnel, Young and others, and culminated with the modern theory of electromagnetism. Unable to explain interference and other phase-dependent effects, the corpuscular theory of light was rejected. But then, in 1905 with Einstein, the particle description of light came back, with the introduction of the modern concept of photons.

Here we will concentrate on the ability of Maxwell’s equations to describe both the wave and particle aspects of light; in a sense, to incorporate complementarity. This property has been known for many years, but is very often overlooked because our usual way of thinking is dominated by the opposition between classical and quantum mechanics. We therefore erroneously tend to identify the particle behaviour with classical physics and the wave behaviour with quantum
mechanics. The fact is that the classical theory of radiation provided by Maxwell’s equations is able to describe the wave–particle dualism. Only in the modern field theory are light and matter described on equal footing. We should, however, keep in mind that a particle has different meanings in classical and quantum theories. In classical terms, a particle, such as a photon, can be associated with a small wave packet. In quantum field theory, a particle becomes an elementary excitation of a given quantum field.

3. Classical theory of radiation

Maxwell’s equations can be written in many equivalent forms (Jackson 1975; Landau & Lifshitz 1975), but we can use, as a starting point, the following simple and compact form:

\[ \partial_k F^{kj} = J^k, \quad \partial_k \tilde{F}^{kj} = 0, \]  

where \( F^{kj} \) represents the electromagnetic field tensor and \( \tilde{F}^{kj} = (1/2)\epsilon^{kijm} F_{jm} \) its dual; \( J^k \) is the four-current; and \( \partial_k \equiv \partial/\partial x^k \) is the differential operator associated with the four-position \( x^k \). The components of the electromagnetic field tensor are (apart from irrelevant multiplying factors that depend on the choice of the unit system) equal to those of the electric and magnetic fields, \( E \) and \( B \), respectively. From these equations, we can derive a wave equation for the electric and magnetic fields.

In the modern version of the classical theory of radiation, we can establish a clear connection between the wave and the ray equations of geometric optics, or equivalently, between the wave description and the corpuscular description of light. This is illustrated in the scheme of figure 1, where four different descriptions of the classical electromagnetic radiation, and their connections, are represented. First, the wave equation for the electric or magnetic field gives the usual wave description. Second, the wave kinetic equation, which is exactly equivalent to the wave equation, describes the space–time evolution of the field correlations. The Fourier transformation of field correlations can be used to represent the electromagnetic field in a classical phase space, where the Heisenberg inequalities apply. These classical uncertainty relations are a direct consequence of the Fourier representation. The wave kinetic equation is the exact analogue of the Wigner–Moyal equation used for quantum systems.

As a third step, we can take the geometric optics approximation, valid when the background medium varies slowly in space and time in comparison with the typical values for the wave period and wavelength. In this approximation, the wave kinetic equation reduces to a Vlasov or one-particle Liouville equation, which states the conservation of the number of photons. The photons can therefore be seen as the classical quasi particles of the electromagnetic radiation. Similarly, geometric optics can be seen as the quasi classical approximation for electromagnetic field. But this statement is misleading because both descriptions of light (the exact and the approximate) are purely classical.

The Vlasov equation for the photon occupation number can be used to describe radiation as a fluid of quasi particles. Photon kinetics is the global result of individual quasi particle dynamics. The equations that describe such a dynamics are nothing but the ray equations of geometric optics. They can be written in a way that takes into account both the spatial and temporal changes in...
These changes are responsible for the usual refraction, as well as for the equivalent temporal effect, which we have called ‘time refraction’ (Mendonça 2001a). The transition from kinetics to dynamics is equivalent to the transition between a fluid description to a particle corpuscular description of light. This theoretical structure, represented in figure 1, based on Maxwell’s equations (3.1) shows that, on purely classical grounds, light can be seen as a wave, as a fluid or as particles. The exact theory covers both the wave and the fluid views, and geometric optics covers the transition from fluid to particles. In order to illustrate these four levels of representation, we will consider the case of a plasma. This example is particularly useful because plasma is a dispersive medium with a simple and well-known dielectric function.

Let us start with the exact equation for the electric field $E$ of a transverse wave in a plasma. This equation can be written as

$$\left[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\omega_p^2(r, t)}{c^2} \right] E = 0.$$  \hspace{1cm} (3.2)

For simplicity, we have considered here a non-magnetized or isotropic plasma, and used explicit space and time variables. In this equation, $c$ is the velocity of light in vacuum and $\omega_p$ is the plasma frequency, as defined (in SI units) by $\omega_p^2(r, t) = \frac{e^2 n(r, t)}{\epsilon_0 m}$, where $e$ and $m$ are the electron charge and mass, respectively, and $n$ is the electron mean density.

In the simple case of a homogeneous plasma, $n = n_0 = \text{const.}$, $\omega_p = \omega_{p0} = \text{const.}$, and the solution of such an equation is trivial. The wave field will be given by $E(r, t) = E_0 \exp(ik \cdot r - i\omega t)$, where the frequency $\omega$ and the wavenumber $k$ satisfy the well-known dispersion relation $\omega^2 = \omega_{p0}^2 + k^2 c^2$.

However, the plasma frequency is not constant in general, due to the existence of plasma fluctuations, or due to the nonlinear contributions of the transverse wave itself. In this more general situation, it is sometimes more useful to replace the wave equation by a wave kinetic equation, describing the field autocorrelation

$$C(r, s) = E(r + s/2, t + \tau/2) \cdot E^*(r - s/2, t - \tau/2).$$  \hspace{1cm} (3.3)
It is also useful to introduce its double Fourier transformation

\[ F(\omega, k; \mathbf{r}, t) = \int \! ds \int \! d\tau C(\mathbf{r}, s) \exp(-i \mathbf{k} \cdot \mathbf{s} + i \omega \tau). \]  

(3.4)

This quantity is the Wigner function for the electromagnetic field, and corresponds to a straightforward generalization of the original Wigner function, proposed for quantum systems (Hillery et al. 1984). We can then derive, from the wave equation (3.1), an evolution equation of the form

\[ i \left( \partial_t + \frac{c^2 k \cdot \nabla}{\omega} \right) F = \int \frac{d\Omega}{(2\pi)^3} \frac{d\mathbf{q}}{2\pi} \omega_0^2(\Omega, \mathbf{q}) [F_- - F_+] \exp(i \mathbf{q} \cdot \mathbf{r} - i \Omega t), \]  

(3.5)

where \( \omega_0^2(\Omega, \mathbf{q}) \) are the double Fourier components of the space and time varying plasma frequency, and \( F_\pm \equiv F(\omega \pm \Omega/2, k \pm \mathbf{q}/2) \).

This equation is exactly equivalent to the wave equation (3.1), and can be used to represent the radiation field in a phase space \((\omega, t)\), where the Heisenberg uncertainty relations \( \Delta \omega \Delta t \) can become apparent, as shown in figure 2. At first sight, the wave kinetic equation (3.5) looks much more complicated than the original equation (3.1), but quite surprisingly it can easily be applied to specific problems (Mendonça & Serbeto 2006).

Let us now introduce a simplifying assumption, and assume that the typical plasma perturbations are very slow, in the sense that \( \Omega \ll \omega \) and \( q \ll k \). This is the geometric optics approximation, where photons slowly (or adiabatically) adapt to the local dispersive properties of the medium. In this case, we can define a local dispersion relation \( \omega_k^2 = k^2 c^2 + \omega_0^2(\mathbf{r}, t) \). This means that the photon frequency is not an independent variable, but a given function of space, time and wavenumber \( \omega_k \equiv \omega(\mathbf{k}, \mathbf{r}, t) \). In this approximation, the complicated wave kinetic equation (3.5) reduces to

\[ (\partial_t + v_k \cdot \nabla + f_k \cdot \nabla_k) N(\mathbf{k}, \mathbf{r}, t) = 0, \]  

(3.6)

Figure 2. \((a,b)\) Phase space representation of the exact Wigner function of a laser beam propagating in a plasma, at two different times: normalized laser field amplitude \( a_0 = 1.3 \), laser pulse duration \( \Delta t = 32 \text{ fs} \), \( \lambda_0 = 19 \mu m \), plasma density \( n_0 = 6 \times 10^{18} \text{ cm}^{-3} \). Results obtained with the relativistic particle-in-cell code Osiris. The variable \( Z \) is a normalized distance.
where $\nabla_k \equiv \partial / \partial k$, and

$$N(k, r, t) = \frac{c_0}{4\hbar \omega_k^3} F(k, r, t) \quad (3.7)$$

is the photon occupation number. Here we also have used the photon group velocity $v_k$ and the force acting on the photons $f_k$ as determined by

$$v_k = \frac{c^2 k}{\omega_k}, \quad f_k = -\frac{1}{2\omega_k} \nabla \omega_k^2. \quad (3.8)$$

Note that equation (3.6) is formally identical to a one-particle Liouville equation, also called a Vlasov equation (Balescu 1975). It simply states that the photon occupation number is a constant of motion. Therefore, the number of photons is conserved in geometric optics (a dissipative or a source term could, however, be included). At this level of description, radiation can be seen as a gas of photons.

A final step corresponds to the dynamics of individual photons moving with velocity $v_k$ under the action of a force $f_k$. Such dynamics is determined by the photon ray equations, which can be represented in canonical form as follows:

$$\frac{dr}{dt} = \nabla_k \omega_k = v_k, \quad \frac{dk}{dt} = -\nabla \omega_k = f_k. \quad (3.9)$$

These equations coincide with the characteristic curves of the kinetic equation (3.6) and describe the single-particle trajectories embedded in the kinetic process. Here $r$ and $k$ represent the (mean) position and (mean) wavevector of a chosen photon trajectory, respectively, where the photon can be seen as the classical quasi particle for the electromagnetic field. In these equations, $\omega_k$ plays the role of the Hamiltonian. In physical terms, the force appearing in equation (3.9) describes refraction in a static and non-homogeneous medium. However, for a time-dependent medium, it can also describe time refraction (the temporal equivalent to refraction) which leads to a frequency shift. The conjugated influence of space and time variations of the refractive index can then lead to a cumulative effect of photon frequency shifts, known as photon acceleration (Mendonça 2001a) which has only been understood in recent years.

We can then conclude that the classical theory of radiation, provided by Maxwell’s equations, is able to describe light as a wave, as a fluid and as an individual particle. These particles represent small wave packets that are the classical analogue of the quantum concept of photon. Thus, in a sense, Maxwell’s theory reconciles the wave description of Huygens with the corpuscular description of Newton (revised by Einstein through the modern concept of photons). It is, therefore, not surprising that many corpuscular or particle-like properties of the electromagnetic radiation are described by Maxwell’s theory, as briefly illustrated in the following section.

4. Classical photon behaviour

The behaviour of light as a fluid, as discussed above, is not surprising, having in mind the well-known radiation pressure. This pressure is due to photon reflection or absorption at a given surface. Even the extreme case of the electromagnetic vacuum shows a residual radiation pressure, known as the Casimir effect.
This effect results from the difference in pressure of the virtual photons at both sides of a plane boundary. Of course, this effect can only be described by a quantum theory of radiation, and therefore has to be taken outside the classical Maxwell’s theory. In contrast, its dynamical counterpart, the dynamical Casimir effect, corresponding to the emission of radiation from an oscillating boundary, can be understood on purely classical grounds. It can indeed be shown that an empty oscillating cavity is classically unstable (Silva & Mendonça 1996).

We also know that light can be transported in pipelines like an ordinary fluid, such as in optical fibres and waveguides. Actually in such confined media, we can identify another corpuscular property of light, the photon effective mass. This photon mass is proportional to the cut-off frequency of a given waveguide propagation mode, \( \omega_0 \), or more specifically \( m_{\text{eff}} = \frac{\hbar \omega_0}{c^2} \). Such a mass is derived from Maxwell’s equations with appropriate boundary conditions. In a plasma, a photon mass can also be defined, even in the absence of boundary conditions, as determined by the plasma frequency, \( m_{\text{eff}} = \frac{\hbar \omega_p}{c^2} \). Even more surprisingly, a photon effective charge can also be defined in a plasma (Mendonça et al. 1998b). This charge is a direct consequence of radiation pressure because a light beam tends to expel electrons from out of the regions where it propagates. For high-intensity laser pulses, nearly all the electrons can be expelled from the laser pulse region as shown by relativistic numerical simulations. The ions, due to their large inertia, are not affected. This means that the photon effective mass is negative owing to the electron repulsion. This concept can be explored further. In terms of charge, we can see a laser beam in a plasma as equivalent to an electric current. Therefore, two nearby laser pulses will attract each other, in the same way as two parallel currents do. If the currents are slightly bent with respect to each other, this attractive force will make the two laser pulses to spiral around each other. The resulting braided light behaviour was demonstrated by numerical simulations (Ren 2000) but not yet observed in experiments.

In contrast with a plasma, where electrons and ions can move freely in space, in a non-ionized optical medium, such as an optical fibre, charge separation induced by photons can only occur at the atomic or molecular level. In this case, we can nevertheless derive an equivalent electric dipole for a photon bunch as considered in Mendonça et al. (1998a).

Another particle-like property of light that can be described by Maxwell’s equations was revealed by a recent experiment (Chauvat 2005). A short laser pulse at total reflection was shown to suffer a time delay, depending on the angle of incidence. The time delay increases for angles approaching the critical angle. This is very much in accordance with Newton’s view of total reflection. The particles of light would jump out of the surface of total reflection for a few moments, before coming back to the propagating medium like fish jumping out of the sea. At exactly the critical angle, such a jump would be so large that the corpuscles would never come back, and the delay time would tend to infinity, in accordance with the experiment. This photon time delay is similar to the so-called Wigner time, a time delay predicted for neutron scattering, and can be accurately described by a wave theory (Wigner 1955). In this case, Maxwell’s theory of radiation confirms Newton’s intuition and the wave field in the evanescent region mimics the jumping corpuscles of light.
Another interesting aspect of classical Maxwell’s theory of light is related to nonlinear optics. A parametric process, such as the generation of sub-harmonics of a laser beam in a crystal, can be described as an inelastic process where a photon with frequency (energy) \( \omega \) breaks into two equal fragments with half of the initial frequency (energy) \( \omega /2 \). The inverse process of harmonic generation is the coalescence of two photons of a given laser beam into a single one. But, more surprisingly, this classical description of nonlinear wave mixing reveals a quantum-like behaviour, because the classical nonlinear wave equations imply the existence of two conservation relations, the energy and the momentum conservation relations. For a generic three-wave process, these relations are (Boyd 1991)

\[
\omega_1 = \omega_2 + \omega_3, \quad k_1 = k_2 + k_3.
\]  

These equations coincide with those predicted by quantum optics and can be generalized to four- or higher-order photon processes. One of the most spectacular manifestation of ‘photons breaking into fragments’ is the observation of photon cascades, capable of generating showers of secondary photons of different colours, from the infrared to the ultraviolet (Crespo & Mendonça 2000). All these nonlinear optical processes can also be seen as inelastic photon–photon processes in a medium, where the classical theory of radiation fully anticipates a quantum description.

Finally, photons can be accelerated and trapped in many different conditions. In this sense, they behave as any other particle. This occurs, for instance, in a plasma, where relativistic electron plasma waves (produced by an intense laser pulse or by a relativistic electron bunch) can trap and accelerate photons belonging to another test pulse or to the driver pulse itself (Murphy 2006). This can also occur in a dielectric crystal, where infrared or visible photons can be accelerated by terahertz polaritons (Mendonça 2006). Or even, in more exotic situations, photons travelling in vacuum can be accelerated (or energized, or frequency upshifted) by moving gravitational fields (Mendonça 2001b; Mendonça & Drury 2001), or going outside the frame of classical theory, by nonlinear quantum electrodynamics effects (Mendonça et al. 2006). Other quantum-like aspects of photons, including quantization of geometric optics, have also been discussed by Leontovich (1944), Leontovich & Fock (1946), Gloge (1968) and Fedele & Man’ko (1999).

5. Conclusions

Maxwell’s equations are one of the greatest scientific achievements of all times. They allow us to describe light as waves and as particles. We have shown the equivalence of the wave equation with the wave kinetic equation, which explains the transition from wave to particle behaviour. The corpuscular aspects of radiation become clear when we introduce the geometric optics approximation and reduce the exact wave kinetic equation to the conservation equation for the number of photons. This equation, which is formally identical to a Vlasov or one-particle Liouville equation, describes light as a fluid. Each individual particle of such a fluid obeys the well-known ray-tracing equations and can be identified with a photon. In this classical description, a photon is nothing but a small wave packet evolving in a slowly varying medium.
After having described the two distinct aspects of light in the frame of Maxwell’s classical theory of radiation, we have given several examples of phenomena where light shows a predominantly corpuscular character. This includes the existence of an effective photon mass and an effective photon charge (in a plasma) or electric dipole (in a fibre), as well as the possible occurrence of photon acceleration and trapping in different optical media, and even in vacuum.

Maxwell’s equations were able to survive three consecutive scientific revolutions: relativity, quantum mechanics and modern quantum field theory. They survived relativity because they are Lorentz invariant, and therefore have anticipated special relativity. They also survived quantum theory because they describe the wave behaviour of light in the same way as Schrödinger’s or Dirac’s equations describe the wave nature of matter. And they survived field theory because the electromagnetic field operators still obey Maxwell’s equations.

In addition to its intrinsic scientific value, Maxwell’s theory of radiation has also inspired and guided the modern theories of general relativity, quantum electrodynamics and quantum chromodynamics. Therefore, they are of incommensurable value and will lead us to many new discoveries in the future.

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References


