Gauge formulation of general relativity using conformal and spin symmetries

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The gauge symmetry inherent in Maxwell’s electromagnetics has a profound impact on modern physics. Following the successful quantization of electromagnetics and other higher order gauge field theories, the gauge principle has been applied in various forms to quantize gravity. A notable development in this direction is loop quantum gravity based on the spin-gauge treatment. This paper considers a further incorporation of the conformal gauge symmetry in canonical general relativity. This is a new conformal decomposition in that it is applied to simplify recently formulated parameter-free construction of spin-gauge variables for gravity. The resulting framework preserves many main features of the existing canonical framework for loop quantum gravity regarding the spin network representation and Thiemann’s regularization. However, the Barbero–Immirzi parameter is converted into the conformal factor as a canonical variable. It behaves like a scalar field but is somehow non-dynamical since the Hamiltonian constraint does not depend on its momentum. The essential steps of the mathematical derivation of this parameter-free framework for the spin-gauge variables of gravity are spelled out. The implications for the loop quantum gravity programme are briefly discussed.

Keywords: general relativity; quantum gravity; conformal geometry

1. Extended conformal gauge symmetry in canonical gravity

One of the most striking features of Maxwell’s electromagnetics is that it has a gauge symmetry and is the earliest theory in physics to have such a property. This provides a basis for the successful quantization of electrodynamics in the mid-twentieth century. Higher order gauge symmetries further led to quantum chromodynamics for strong nuclear interactions as well as unified electroweak theory in the Standard Model of particle physics. These developments have stimulated efforts to reformulate general relativity (GR) into a gauge field theory. The choice of the (spatial) spin gauge has led to the development of loop quantum gravity. See Rovelli (2004) for an excellent review.

In a recent series of work (Wang 2005a,b, 2006a,b), the canonical theory of GR has been constructed in terms of the conformal equivalence classes of spatial metrics. (See Wang (2006c) for a recent overview.) The work is motivated by the
following two ideas. First, the conformal three-geometry, rather than the three-geometry, may well carry the true dynamics of GR (York 1971, 1972; Wang 2005c). Second, the free parameter responsible for the Barbero–Immirzi ambiguity of the present loop quantum gravity is of conformal nature, and may be removed by an extension of the phase space for GR using the conformal symmetry.\(^1\) This led to the canonical framework in Wang (2005a, b), in which the conformal three-metric, the mean extrinsic curvature and their respective momenta act as canonical variables. A new first-class constraint, the conformal constraint, was introduced to offset the conformal redundancy. However, its quantum implementation appears to be impeded by the complexity of the formalism in which the conformal factor as a key subexpression is a highly non-polynomial function of other variables. This will cause the required regularization for quantization to be hard to resolve.

It was soon realized that the conformal factor need not be treated this way and that it can receive a canonical variable status if a suitable canonical transformation is performed (Wang 2006b). A quick way of demonstrating this is as follows: starting with the Arnowitt–Deser–Misner (ADM) variables \(g_{ab}\) and \(p^{ab}\), we introduce the conformally related quantities \(\tilde{g}_{ab}\) and \(\tilde{p}^{ab}\) by

\[
g_{ab} = \phi^4 \tilde{g}_{ab},
\]

\[
p^{ab} = \phi^{-4} \tilde{p}^{ab},
\]

(1.1)

(1.2)

using the conformal factor \(\phi = \phi(x)\). Throughout this work, we shall use a bar over a quantity to indicate that the quantity has been obtained with a rescaling using a power of \(\phi\). The strategy is to turn \(\tilde{g}_{ab}\) and \(\phi\) into new configuration variables. If the momentum of \(\tilde{g}_{ab}\) is \(\tilde{p}^{ab}\), then the momentum of \(\phi\) must be identified. In addition, there should be an additional constraint to compensate the conformal redundancy in (1.1) and (1.2). To this end, let us calculate the following:

\[
p^{ab} \dot{g}_{ab} = \phi^{-4} \tilde{p}^{ab} \left[ \phi^4 \dot{\tilde{g}}_{ab} + 4\phi^3 \dot{\phi} \tilde{g}_{ab} \right] = \tilde{p}^{ab} \dot{\tilde{g}}_{ab} + \pi \dot{\phi},
\]

(1.3)

where

\[
\pi := 4\phi^{-1} \tilde{g}_{ab} \tilde{p}^{ab} = -8\phi^{-1} \mu K,
\]

(1.4)

using the mean extrinsic curvature \(K\) and the scale factor \(\mu := \sqrt{\det g_{ab}}\). This implies that the variables \((\tilde{g}_{ab}, \tilde{p}^{ab}, \phi, \pi)\) can be treated as a canonical set if the constraint \(C\) given by

\[
C := \tilde{g}_{ab} \tilde{p}^{ab} - \frac{1}{4} \phi \pi
\]

(1.5)

\(^1\) As a historic note, H. Weyl attempted to unify general relativity with electromagnetism in the early 1990s by assuming an additional invariance (Eichinvarianz) under the local change of scale, i.e. gauge. Although this approach was later found to be unviable as a physical theory, the concept and the term ‘gauge’ was retained in the later development of field theories, in terms of different invariant properties. It was then realized that Maxwell electromagnetics has a \(U(1)\) gauge symmetry. As in Weyl’s original gauge theory, the extended conformal symmetry in this paper is a scaling symmetry. However it applies to the extended phase space rather than the position space postulated by Weyl.

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vanishes weakly. We have therefore identified \((\tilde{g}_{ab}, \tilde{p}^{ab}; \phi, \pi)\) as the new canonical variables together with \(C\) as the new (conformal) constraint for GR. In terms of these variables, the ADM diffeomorphism and Hamiltonian constraints become

\[
\mathcal{H}_a = -2 \tilde{\nabla}_b \tilde{p}^b_a + \pi \phi, a + 4(\ln \phi), a C, \tag{1.6}
\]

\[
\mathcal{H}_\perp = \phi^{-6} \tilde{\mu} \tilde{g}_{abcd} \tilde{p}^{ab} \tilde{p}^{cd} - \phi^2 \tilde{\mu} \tilde{R} + 8 \tilde{\mu} \phi \tilde{\Delta} \phi, \tag{1.7}
\]

respectively. Here we have used the scale factor \(\tilde{\mu} := \sqrt{\text{det} \tilde{g}_{ab}}\); Levi–Civita connection \(\tilde{\nabla}\); Ricci scalar curvature \(\tilde{R}\); and Laplacian \(\tilde{\Delta} := \tilde{g}_{ab} \tilde{\nabla}_a \tilde{\nabla}_b\), associated with the conformal metric \(\tilde{g}_{ab}\). In (1.6), the last term \(4(\ln \phi), a C\) can be dropped to define an effective diffeomorphism constraint (Wang 2006b).

### 2. Triad variables for gravity

The canonical framework in §1 will be applied to obtain a triad formulation of GR with extended conformal symmetry. Before we carry that out, however, it is useful to briefly review the standard triad formulation of GR with the canonical variables \((E^i_a, K^i_a)\). Here \(E^i_a\) is the densitized triad and \(K^i_a\) the extrinsic curvature. In terms of these variables, the ADM variables take the following forms:

\[
g_{ab} = \mu^2 E^i_a E^i_b, \tag{2.1}
\]

\[
p^{ab} = \frac{1}{2} \mu^{-2} K^i_a \left[ E^c_j E^i_a E^b_j - E^c_j E^i_a E^b_j \right], \tag{2.2}
\]

where \(E^i_a\) is the inverse of \(E^a_i\). It follows that

\[
p^{ab} g_{ab} = -E^a_i K^i_a \tag{2.3}
\]

and

\[
p^{ab} \dot{g}_{ab} = -E^i_a K^i_a - \frac{1}{\mu^2} E^a_i E^b_j K_{ab}, \tag{2.4}
\]

with the constraint

\[
K_{ab} := \mu K^j_a e^j_b = \mu^2 K^i_a E^i_b, \tag{2.5}
\]

to vanish weakly. This justifies \((K^i_a, E^a_i)\) as canonical variables. Instead of working with the constraint \(K_{ab}\), it is more convenient to adopt the equivalent constraint

\[
C_k := \epsilon_{kij} K_{al} E^a_j = -\frac{1}{\mu^2} \epsilon_{kij} K_{ab} E^a_i E^b_j, \tag{2.6}
\]

since it generates the rotation of the triad. We shall therefore refer to \(C_k\) as the ‘spin constraint’.

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In terms of the canonical variables \((K^i_a, E^a_i)\), the ADM diffeomorphism and Hamiltonian constraints then become

\[
\mathcal{H}_a = 2E^b_k \nabla_{(a}K_{b)}^k - \frac{1}{2} \epsilon_{ijk}E^i_aE^b_j \nabla_b C_k \tag{2.7}
\]

and

\[
\mathcal{H}_\perp = -\frac{1}{2\mu} K^i_a K^j_b E^i_a E^b_j - \mu R + \frac{1}{8\mu} C_k C_k \tag{2.8}
\]

respectively.

3. Conformal treatment of the triad variables

The rescaling relations (1.1) and (1.2) give rise to the conformal triad variables \((\tilde{K}^i_a, \tilde{E}^a_i)\) satisfying

\[
E^a_i = \phi^4 \tilde{E}^a_i, \tag{3.1}
\]

\[
K^i_a = \phi^{-4} \tilde{K}^i_a. \tag{3.2}
\]

Using (1.4) and the identities

\[
K^i_a E^a_i = \tilde{K}^i_a \bar{E}^a_i = 2\mu K, \tag{3.3}
\]

we can calculate that

\[
E^a_i \dot{K}^i_a = \phi^4 \tilde{E}^a_i \left[ \phi^{-4} \dot{\tilde{K}}^i_a - 4\phi^{-5} \dot{\phi} \tilde{K}^i_a \right] = \tilde{E}^a_i \dot{\tilde{K}}^i_a - 4\phi^{-1} \tilde{E}^a_i \tilde{K}^i_a \dot{\phi} = \bar{E}^a_i \dot{\tilde{K}}^i_a + \pi \dot{\phi}. \tag{3.4}
\]

This establishes the variables \((\tilde{K}^i_a, \tilde{E}^a_i; \phi, \pi)\) as canonical variables. By using (2.3), we see that the conformal constraint \(C\) defined in (1.5) now takes the form

\[
C = -\tilde{K}^i_a \bar{E}^a_i - \frac{1}{4} \phi \pi. \tag{3.5}
\]

The spin constraint defined in (2.6) then becomes

\[
C_k = \epsilon_{kij} \tilde{K}^i_a \bar{E}^a_j. \tag{3.6}
\]

In terms of the variables \((\tilde{K}^i_a, \tilde{E}^a_i; \phi, \pi)\), the conformal metric and its momentum are given by

\[
\bar{g}_{ab} = \bar{\mu}^2 \tilde{E}^i_a \bar{E}^i_b, \tag{3.7}
\]

\[
\bar{p}^{ab} = \bar{\mu}^{-2} \tilde{K}^i_c \left[ \tilde{E}^c_i \bar{E}^a_j \bar{E}^b_j - \tilde{E}^c_i \bar{E}^a_j \bar{E}^b_j \right]. \tag{3.8}
\]

By substituting (3.7), (3.8) into (1.6), (1.7) and making use of the expressions in (2.7) and (2.8) with the replacement \((E^a_i, K^i_a) \rightarrow (\bar{E}^a_i, \tilde{K}^i_a)\), we get the diffeomorphism constraint and Hamiltonian constraint, respectively, in the following
forms:
\[ \mathcal{H}_a = 2\tilde{E}^b_k \nabla_a \tilde{K}^b_k + \pi \phi,_{a} - \frac{1}{2} \epsilon_{ijk} \tilde{E}^i_a \tilde{E}^b_j \nabla_k C + 4(\ln \phi),_{a} C, \]  
\[ \mathcal{H}_\perp = - \frac{1}{2} \phi^{-6} \bar{\mu}^{-1} \tilde{K}^i_a \tilde{K}^j_a \tilde{E}^a_i \tilde{E}^b_j - \phi^2 \bar{\mu} R + 8\mu \phi \Delta \phi + \frac{1}{8} \phi^{-6} \bar{\mu}^{-1} C K. \]  
(3.9)  
(3.10)

4. Standard spin-gauge formalism

We now briefly review the existing real spin-gauge formalism for gravity (Ashtekar 1986, 1987; Barbero 1995; Immirzi 1997) before moving on to our final form of the spin-gauge formalism by assimilating the conformal symmetry. For any positive constant \( \beta \), introduce the spin connection
\[ \tilde{A}^i_a := \Gamma^i_a + \beta K^i_a := \Gamma^i_a + \tilde{K}^i_a, \]  
(4.1)
where \( \tilde{K}^i_a := \beta K^i_a \). Further, introduce the scaled triad
\[ \tilde{E}^a_i := \beta^{-1} E^a_i. \]  
(4.2)

Here and below, we use the tilde to emphasize a quantity’s dependence on the \( \beta \). This constant \( \beta \) is called the Barbero–Immirzi parameter. It can be shown that the variables \( (\tilde{A}^i_a, \tilde{E}^a_i) \) are canonical. They are used for the existing spin-gauge formalism for gravity. The curvature of the spin connection \( \tilde{A}^i_a \) is given by
\[ \tilde{F}^k_{ab} := 2\partial_{[a} \tilde{A}^k_{b]} + \epsilon_{kij} \tilde{A}^i_a \tilde{A}^j_b. \]  
(4.3)

Denoting by \( \tilde{D}_a \), the covariant derivative associated with \( \tilde{A}^i_a \), we can express the spin constraint in the form of the ‘Gauss law’ as
\[ \tilde{C}^k := \tilde{D}_a \tilde{E}^a_k = \tilde{E}^a_k + \epsilon_{kij} \tilde{A}^i_a \tilde{E}^a_j = \nabla_a \tilde{E}^a_k + \epsilon_{kij} K^i_a E^a_j = C^k. \]  
(4.4)

In the above equation, the torsion-free condition \( \nabla_a \tilde{E}^a_k = 0 \) for the Levi–Civita spin connection \( \nabla_a \) associated with the metric \( g_{ab} \) has been used. Using the relations
\[ \tilde{F}^k_{ab} \tilde{E}^b_k = 2\nabla_{[a} \tilde{K}^b_k \tilde{E}^b_k + \tilde{K}^k_{ab} \tilde{C}^k \]  
(4.5)
and
\[ \left[ \beta^2 \epsilon_{kij} \tilde{F}^k_{ab} - 2\beta^2 \tilde{K}^i_{[a} \tilde{K}^j_{b]} \right] \tilde{E}^a_i \tilde{E}^b_j = -\mu^2 R - 2\beta^2 \tilde{E}^c_k \nabla_c \tilde{C}^k, \]  
(4.6)
we see that (2.7) and (2.8) become
\[ \mathcal{H}_a = \tilde{F}^k_{ab} \tilde{E}^b_k - \tilde{A}^k_a \tilde{C}^k_a + \Gamma^k_a \tilde{C}^k - \frac{1}{2} \epsilon_{ijk} \tilde{E}^i_a \tilde{E}^b_j \nabla_k \tilde{C}^k \]  
(4.7)
and
\[ \mathcal{H}_\perp = \beta^{1/2} \mu^{-1} \left[ \epsilon_{ijk} \tilde{F}^k_{ab} - \frac{4\beta^2 + 1}{2\beta^2} \tilde{K}^i_{a} \tilde{K}^j_{b} \right] \tilde{E}^a_i \tilde{E}^b_j + 2\beta^{1/2} \mu^{-1} \tilde{E}^c_k \nabla_c \tilde{C}^k \]  
+ \frac{1}{8} \beta^{-3/2} \mu^{-1} \tilde{C}^k \tilde{C}^k. \]  
(4.8)
Here the scale factor of the metric defined using the triad $\tilde{E}_i^a$ is denoted by $\tilde{\mu} = \beta^{-3/2} \mu$. As per Ashtekar & Lewandowski (2004), we summarize the standard spin-gauge formalism of gravity by listing its effective Hamiltonian constraint

$$\tilde{C}_\perp = \beta^{1/2} \tilde{\mu}^{-1} \left[ e_{ijk} \tilde{F}_{ab}^k - \frac{4\beta^2 + 1}{2\beta^2} \tilde{K}_a^i \tilde{K}_b^j \right] \tilde{E}_i^a \tilde{E}_j^b$$

(4.9)

and diffeomorphism constraint

$$\tilde{C}_a = \tilde{F}_{ab}^k \tilde{E}_k^b - \tilde{\Lambda}_a^i \tilde{C}_i^a,$$

(4.10)

together with the spin constraint

$$\tilde{C}_k := \tilde{D}_a \tilde{E}_k^a.$$  

(4.11)

5. Parameter-free approach to the spin-gauge formalism using the conformal method

The parameter dependence of the standard spin-gauge formalism discussed above is due to the inequality

$$\mathcal{H}_\perp [K_a^i, E_i^a] \neq \mathcal{H}_\perp [\beta K_a^i, \beta^{-1} E_i^a],$$

(5.1)

for any non-unity $\beta$ and the fact that the kinetic and potential terms in $\mathcal{H}_\perp$ are rescaled differently under a constant conformal transformation (Kuchar 1981). When the phase space of GR is extended by conformal symmetry, the situation is completely changed. From the expression of $\mathcal{H}_\perp$ in (3.10), it is clear that we do have the following equation:

$$\mathcal{H}_\perp [K_a^i, E_i^a, \phi, \pi] = \mathcal{H}_\perp [\beta K_a^i, \beta^{-1} E_i^a, \beta^{1/4} \phi, \beta^{-1/4} \pi].$$

(5.2)

Therefore, if we introduce the alternative spin connection variable

$$\tilde{A}_a^i := \tilde{\Gamma}_a^i + \tilde{K}_a^i = \tilde{\Gamma}_a^i + \phi^4 K_a^i$$

(5.3)

and consider

$$\tilde{E}_i^a = \phi^{-4} E_i^a$$

(5.4)

as its conjugate momentum, then, unlike the construction of $\tilde{A}_a^i$ and $\tilde{E}_i^a$ in (4.1) and (4.2), any similar multiplicative constant can be absorbed into the conformal factor $\phi$ together with an inverse rescaling of $\pi$. As with the spin-gauge treatment in §4, the curvature of the spin connection $\tilde{A}_a^i$ is given by

$$\tilde{F}_{ab}^k := 2\partial_{[a} \tilde{A}_{b]}^k + \epsilon_{kij} \tilde{A}_a^i \tilde{A}_b^j,$$

(5.5)

Denoting by $\tilde{D}_a$, the covariant derivative associated with $\tilde{A}_a^i$, we can also express the spin constraint in the form of the Gauss law as

$$\tilde{C}_k := \tilde{D}_a \tilde{E}_k^a = \tilde{E}_{k,a}^a + \epsilon_{kij} \tilde{A}_a^i \tilde{E}_j^a = \tilde{\nabla}_a \tilde{E}_k^a + \epsilon_{kij} \tilde{K}_a^i \tilde{E}_j^a = C_k.$$  

(5.6)

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Here the torsion-free condition $\nabla_a \tilde{E}_k^a = 0$ for the Levi–Civita spin connection $\nabla_a$ associated with the conformal metric $\tilde{g}_{ab}$ (3.7) has been used. Using the relations

$$\tilde{F}_{ab}^k \tilde{E}_k^b = 2 \nabla_a [\tilde{K}_b^k] \tilde{E}_k^b + \tilde{K}_a^k \tilde{C}_k$$  \hspace{0.5cm} (5.7)$$

and

$$\left[ \epsilon_{ijk} \tilde{F}_{ab}^k - 2 \tilde{K}_a^i \tilde{K}_b^j \right] \tilde{E}_i^a \tilde{E}_j^b = -\mu^2 R - 2 \tilde{E}_k^c \nabla_c \tilde{C}_k,$$  \hspace{0.5cm} (5.8)$$

analogous to (4.5) and (4.6), we see that (3.9) and (3.10) become

$$\mathcal{H}_a = \tilde{F}_{ab}^k \tilde{E}_k^b - \tilde{A}_a \tilde{C}_k + \pi \phi_{,a} + \tilde{\Gamma}_a^k \tilde{C}_k - \frac{1}{2} \epsilon_{ijk} \tilde{E}_i^a \tilde{E}_j^b \nabla_b \tilde{C}_k$$  \hspace{0.5cm} (5.9)$$

and

$$\mathcal{H}_\perp = \phi^2 \tilde{\mu}^{-1} \left[ \epsilon_{ijk} \tilde{F}_{ab}^k - \frac{4\phi^8 + 1}{2\phi^8} \tilde{K}_a^i \tilde{K}_b^j \right] \tilde{E}_i^a \tilde{E}_j^b + 8 \tilde{\mu} \phi \tilde{\Delta} \phi + 2 \phi^2 \tilde{\mu}^{-1} \tilde{E}_k^c \nabla_c \tilde{C}_k$$

$$+ \frac{1}{8} \phi^{-6} \tilde{\mu}^{-1} \tilde{C}_k \tilde{C}_k.$$  \hspace{0.5cm} (5.10)$$

This leads us to the final expressions for the effective Hamiltonian constraint

$$\tilde{C}_\perp = \phi^2 \tilde{\mu}^{-1} \left[ \epsilon_{ijk} \tilde{F}_{ab}^k - \frac{4\phi^8 + 1}{2\phi^8} \tilde{K}_a^i \tilde{K}_b^j \right] \tilde{E}_i^a \tilde{E}_j^b + 8 \tilde{\mu} \phi \tilde{\Delta} \phi,$$  \hspace{0.5cm} (5.11)$$

effective diffeomorphism constraint

$$\tilde{C}_a = \tilde{F}_{ab}^k \tilde{E}_k^b - \tilde{A}_a \tilde{C}_k + \pi \phi_{,a},$$  \hspace{0.5cm} (5.12)$$

spin constraint

$$\tilde{C}_k = \tilde{D}_a \tilde{E}_k^a$$  \hspace{0.5cm} (5.13)$$

and conformal constraint

$$\tilde{C} = -\tilde{K}_a^i \tilde{E}_i^a - \frac{1}{4} \phi \pi.$$  \hspace{0.5cm} (5.14)$$

6. Concluding remarks

It is readily seen that, apart from ‘small’ extra terms involving $\phi$ and $\pi$, the structures of the first three constraints in (5.11), (5.12) and (5.13) above resemble very closely those in (4.9)–(4.11). In fact, the constraints $\tilde{C}_\perp, \tilde{C}_a$ and $\tilde{C}_k$ reduce to $\tilde{C}_\perp, \tilde{C}_a$ and $\tilde{C}_k$ on substituting $\phi \rightarrow \tilde{\phi}^{1/4}$. However, it is the introduction of the canonical variable $\phi$ that makes our spin-gauge formalism parameter free. It is interesting to observe that the constraint $\tilde{C}_\perp$ is independent of $\pi$. Consequently, the evolution of $\phi$ is like a gauge effect and is dictated by the effective diffeomorphism constraint $\tilde{C}_a$ and conformal constraint $\tilde{C}$. This is consistent with one of our motivating ideas that the true dynamics of GR is in the conformal three-geometry, rather than the conformal factor. An important technical implication of this is related to the regularization of the
Hamiltonian operator. It is envisaged that the \( (\vec{A}_a^k, \phi) \) representation is to be used for quantization. While the spin connection \( \vec{A}_a^k \) can be treated with spin networks, the conformal factor \( \phi \) will be treated similar to a coupled scalar field. The appearance of \( \phi \) in the first term in (5.11) should not spoil the regularization schemes developed by Thiemann (1996, 1998), since \( \phi \) commutes with all operators there just like the Barbero–Immirzi parameter in the existing loop quantum gravity. It remains to solve the quantum conformal constraint equation

\[
\mathcal{C}_\Psi[\vec{A}_a^k, \phi] = 0. \tag{6.1}
\]

Addressing this problem will require the construction of ‘conformally related spin networks’, which will form a subject for future investigation.

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References


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