Something old, something new

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Some thoughts on the development of theoretical fluid dynamics in this century and some reflections on the developments of the past century were provided as an introduction to a broad-ranging conference. A written synopsis of these remarks is provided here.

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1. Introduction

The title of this paper and the presentation that gave rise to it are derived from the well-known English poem,

Something old, something new,
Something borrowed, something blue,
And a silver sixpence in her shoe.

apparently originating in the Victorian era and used in conjunction with weddings. Each item in the poem represents a good luck token for the bride. If she carries all of them on her wedding day, her marriage will be happy. ‘Something old’ symbolizes continuity with the bride’s family and the past. ‘Something new’ means optimism and hope for the bride’s new life ahead. The first line of the poem seems appropriate as I consider the past, present and future of theoretical fluid dynamics at the juncture of the twentieth and twenty-first centuries. As with the new bride, fluid dynamics has deep continuity with the past, yet, I shall argue, remains a vibrant subject with appetite for life and new challenges.

I begin by asking the following three questions.

(i) Why is a ‘classical’ ‘mature’ field of science, such as fluid dynamics, as relevant today as at any prior time and, therefore, likely to survive the twenty-first century as well?
(ii) What are the peculiar features of this subject that continue to make it appealing and attractive to some of us and, therefore, to our students, who are likely to be active well into the twenty-first century?
(iii) What are some problems in theoretical fluid dynamics that have survived for centuries already and, therefore, are likely to be with us also at the end of the twenty-first century?

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One contribution of 10 to a Theme Issue ‘Theoretical fluid dynamics in the twenty-first century’.
I discuss the two first ‘philosophical’ questions in §2. Then, in §3 in response to the third question, I have chosen, somewhat at random and flavoured by personal interests, some problems that have a long history and tradition, yet continue to inspire new discoveries, analyses and developments.

My choices are:

(i) the link between flow kinematics and chaos in dynamical systems,
(ii) instances of relative equilibria and simple motions of particle-like entities, and
(iii) the phenomenon of splashing.

I could have chosen a multitude of other problem areas, a fact that simply amplifies some of the points made in the philosophical discussion on the relevance and sustainability of fluid dynamics as a field of scientific enquiry. I am sure many readers would expect turbulence to be on the list, and I suspect turbulence does indeed belong to the category of problems that will be with us for a long time. Let me simply reiterate the quote ascribed to Sir Horace Lamb: ‘I am an old man now, and when I die and go to Heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics and the other is turbulence. About the former, I am really rather optimistic.’

In the final section I discuss some potentially troubling concerns about the future of our subject and the interests and attitudes of the new generation of researchers and students entering the field.

Before commencing, it is appropriate to pay tribute to Prof. J. T. Stuart, FRS, whom I am proud to list as a mentor of my faculty career since its inception. In particular, Trevor’s keen interest in point vortex dynamics and the notions of integrability and chaos in that problem were important encouragements in the early 1980s when such work was not particularly in favour with the mainstream fluid dynamics community. Several visits to Imperial College over the years, capped off by the conference in Trevor’s honour for which this material was originally prepared, are now fond memories.

2. Sustained significance

Why would I state that fluid dynamics is as relevant today as at any prior time? In the present era, when most research funding is being directed at subjects that have the syllable ‘bio’, ‘info’ or ‘nano’ embedded—with ‘securo’ soon to be added to this list—is it not somewhat naive and misguided to promote a broad classical field of science, such as fluid dynamics? Has the heyday of fluid dynamics, with aviation as the driving force for much of the twentieth century, not long since passed, and is this mature subject now not simply something students can read in a book or buy an off-the-shelf computer program to assist with?

I argue, first of all, that the basics of fluid dynamics have survived all the much-heralded revolutions in physics, such as relativity and quantum theory. This fact in itself suggests a certain staying power of the subject. One might see this as simply another instance of Bohr’s correspondence principle that, when a new theory enters the arena, it does not invalidate older theories but simply restricts their domain of validity, and the older theory remains valid in those
areas where it was valid prior to the introduction of the new theory. But there is more to be said. For example, we can now quite readily consider relativistic fluid dynamics, maybe not immediately applicable to terrestrial experiences but surely of interest in astrophysics. Just like the marriage of electromagnetic theory and fluid dynamics gave birth to magnetohydrodynamics and electrohydrodynamics, so too has the merger of the theory of relativity and fluid dynamics given rise to a new subfield of fluid dynamics called relativistic hydrodynamics.

Furthermore, one can view the basic equation of non-relativistic quantum mechanics, Schrödinger’s equation, as part of an extended family of field equations of which the equations of classical fluid dynamics are simply one of the first examples. The hydrodynamic interpretation of the Schrödinger equation, wherein the amplitude and phase of the complex wave function are treated as coupled scalar fields, provides an interesting and provocative interpretation of many phenomena covered by non-relativistic quantum mechanics. The so-called nonlinear Schrödinger equation belongs rather squarely to fluid dynamics and has arisen in many contexts.

Rather than retreat into a confining shell, fluid dynamics has expanded its range of applications to meteorology and oceanography—resulting *inter alia* in a key societal problem such as global climate change coming under its purview—into micro- and nanotechnology and into biomechanics, at least those many portions of that field which deal with the motion of material in the fluid state. The somewhat applied subfield of fluid dynamics known as *fluidics* has taken on new life in the context of microfluidics and nanofluidics. This is impacting both the research and the textbook literature, including elaborations of the theoretical side of the subject, such as Karniadakis *et al.* (2005), Bruus (2007) and others.

New subfields of fluid dynamics such as computational fluid dynamics, or CFD as it is commonly known, have sprung into being. The subject of ‘complex fluids’, including the flow physics of granular materials, foams, all manner of ‘rheological fluids’ and even superfluids, has seen tremendous development and growth in its range of applications. Even in fields such as materials processing, which are primarily the domain of solid mechanics, we have witnessed increasing interest from and challenges to fluid dynamicists whenever the processing involves a fluid state.

It is, in my view, not an exaggeration to say that fluid dynamics today touches almost every aspect of human endeavour. Maybe precisely because it is so mature, fluid dynamics has become a fertile ‘testing ground’ for virtually every analytical, experimental and computational technique known. In some instances, fluid mechanical problems have even become drivers of technology, e.g. in the construction of ever more powerful computers such as the Earth Simulator.

Let us turn next to the question of what is unique about the field of fluid dynamics. I believe that the strong traditions of the field must be mentioned. After all, what fields can really claim among their founding fathers the likes of Newton, Euler, Bernoulli, Laplace, Lagrange, Helmholtz, Kelvin, Kirchhoff, Stokes, Maxwell and many other major figures? There was a time when fluid dynamics was a central discipline in physics, as is evident from several of the great textbook series such as Sommerfeld’s lectures, and Kirchhoff’s earlier, the books by Landau and Lifshitz and many more. This strong tradition has manifested itself in several elite journals catering to the subject such as *Journal of Fluid Mechanics*, *Physics of Fluids* and *Annual Review of Fluid Mechanics*.

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These journals generally have maintained high standards, primarily through rigorous review of the materials they publish. While this may sometimes have led them to become a bit ‘stodgy’, the benefits by and large have offset the drawbacks. Quantitative measures such as ‘impact factor’ (for whatever it is worth) tend to rate these journals and serials highly. Their standing within the fluid dynamics community and their importance to the development of the field are beyond discussion. In this connection, I would also mention the International Union of Theoretical and Applied Mechanics (IUTAM) which for several decades has upheld high standards of peer-reviewed meetings and papers not only in fluid dynamics but also in the sister subjects of solid mechanics and dynamics.

The problems of fluid dynamics are generally difficult, and so on the theoretical side they have, over time, challenged and stimulated the latest progress in mathematics just as they today challenge the frontiers of computing. As we consider various examples, we shall see repeatedly that progress in mathematics must wait decades, and sometimes a century or more, before it finds application in some fluid dynamics problem. Conversely, fluid dynamics is often a driver of mathematical development as in the theory of stochastic differential equations necessitated by the theory of turbulence or the discovery of non-integrable (chaotic) behaviour leading to the codification of the strange attractor. The theory of singular perturbations in ordinary and partial differential equations (ODEs and PDEs) was largely stimulated by the problem of the boundary layer in fluid dynamics. Similarly, much of what we know in general terms about weak solutions to PDEs derives from studying problems such as shock formation in fluids and other singular structures. On a more modest scale, in the context of the examples I have chosen, we shall see results about soap bubbles being the applications of earlier results in projective geometry, and the basic equation of relative equilibria of point vortices leading to a non-trivial problem in algebraic geometry that is currently unsolved.

Coupled with its high standards of scientific enquiry and its broad reach in applications, the field of fluid dynamics has always had an intrinsic aesthetic appeal even to the non-expert and layperson. There are many examples of this, from the great waterfalls that we migrate to as tourists, to the famous fountains that adorn city squares, to the fluid dynamics-based executive toys that abound. Sports and activities such as swimming, sailing, waterskiing—not to mention ordinary skiing on a complex granular material called snow!—diving, surfing, and even storm chasing reveal an innate desire to observe and interact with the fluid elements surrounding us, and we find beauty in many facets of these endeavours. Thus, intellectual appeal, cemented by high standards and strong traditions, is coupled with aesthetic appeal. One scientific outlet for this combined focus has been the Gallery of Fluid Motion, which on its website has the byline ‘When Science Creates Art’ (figure 1). In a different genre, the poster for the 20th International Congress of Theoretical and Applied Mechanics, held in the year 2000 in Chicago, based on a watercolour by Billy Morrow Jackson (see http://www.iutam.net/iutam/History/2000/index.php), attempted to summarize some of the greats of mechanics and their seminal contributions in an artistic format (figure 2). The proceedings of this congress carried the byline ‘Mechanics for a New Millennium’, but the strong traditions were clearly in evidence.
To conclude this discussion, we mention that although the governing equation of fluid motion, the Navier–Stokes equation, has been known and studied for at least a century, basic regularity properties of this PDE remain a mystery. For smooth initial data, do solutions to the Navier–Stokes equation remain smooth for all time? The answer to this basic question is unknown early in the twenty-first century. This unfortunate state of affairs was recently recognized by the question of infinite-time regularity of solutions to the Navier–Stokes equation for smooth initial data being included as one of the Clay Mathematics Institute ‘millennium problems’. To quote from the Institute website:

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the

Figure 1. From the introductory screen of the website of Gallery of Fluid Motion http://pof.aip.org/pof/gallery/. (Reprinted with permission from the American Institute of Physics.)

Figure 2. The watercolour Meters of Motion by Billy Morrow Jackson used for the poster announcing the 20th International Congress of Theoretical and Applied Mechanics held in Chicago, USA, in 2000. Archimedes, Galileo, Newton, Euler, Lagrange, Prandtl, Taylor and von Kármán are the main figures along with symbols of their work. (Reprinted with permission.)

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19th Century, our understanding of them remains minimal. The challenge is to make substantial progress towards a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

In summary, at the turn of the century, the mathematical analysis of a key fluid-dynamical equation is included among the major unsolved challenges in mathematics. As Hilbert put it at the turn of the nineteenth to the twentieth century, ‘We must know, We shall know.’

3. Case studies

(a) Flow kinematics and chaos

One of the major advances in theoretical physics and applied mathematics of the twentieth century was the realization that systems described by a small number of ODEs may exhibit non-integrable or chaotic behaviour. Poincaré’s seminal work on the three-body problem of celestial mechanics at the turn of the nineteenth to the twentieth century is the first example of work in this genre. Later important entries include the so-called KAM theorem of Kolmogorov, Arnold and Moser (cf. Tabor 1989), essentially a stability result for non-integrable systems, the identification of the strange attractor by Lorenz (1963), the notion of a horseshoe map due to Smale (1967), the repeated subharmonic bifurcation route to chaos quantified by Feigenbaum (1979) and several others.

In the context of fluid dynamics, the systems of equations that were studied initially were truncations of the full expansion of the equations of motion in a set of Fourier modes. Thus, the well-known Lorenz system,

\[
\begin{align*}
\dot{X} &= -\sigma X + \sigma Y, \\
\dot{Y} &= -XZ + rX - Y, \\
\dot{Z} &= XY - bZ,
\end{align*}
\]

arises by truncating the equations describing thermal convection to a set of three modes, two streamfunction modes and one mode of the temperature field. Here, $\sigma$ is the Prandtl number and $r$ is the Rayleigh number of the flow, while $b$ is a geometric parameter. Usually (for physical reasons), one sets $\sigma=10$, $b=8/3$ and allows $r$ to vary. Clearly, such truncations are at best caricatures of what the real flow does or, alternatively, they are severely constrained flow fields where the spatial variability has somehow been frozen out and one simply looks at the temporal dynamics of the various amplitudes. One may say that, whereas the original identification of chaos in dynamical systems in Poincaré’s work employed a particle-based Lagrangian representation, the later work, and the work that gave rise to most of the excitement in fluid dynamics in the 1960s and 1970s, worked within the usual Eulerian representation.

The subject of flow kinematics brings the two viewpoints together. The basic idea is quite straightforward. One imagines a particle moving in a fluid velocity field, $\mathbf{V}(x, y, z, t)$, in such a way that, at every point in space and every instant in time, it follows the flow. The particle must be sufficiently light and otherwise inert that this is plausible, but such particles exist, if nothing else as tagged

1 In keeping with the theme of the conference, one might speculate on how many nineteenth-century researchers saw non-integrable behaviour as a potentially major theme of the twentieth century.
particles of the fluid flow itself. In terms of the Lagrangian positions of such a particle, which we again denote \((X, Y, Z)\), the equations expressing that its velocity agrees with the fluid velocity are

\[
\frac{dX}{dt} = u(X, Y, Z, t), \quad \frac{dY}{dt} = v(X, Y, Z, t), \quad \frac{dZ}{dt} = w(X, Y, Z, t). \tag{3.2}
\]

Here \((u, v, w)\) are the Cartesian components of \(V\). We refer to equations (3.2) as the advection equations. The restriction to three ODEs here is the result not of a somewhat arbitrary truncation to a small and manageable number of Fourier modes, but rather of the quite physical picture of passive advection coupled with the common restriction of fluid dynamics to three-dimensional space.

We have written equations (3.1) and (3.2) provocatively in terms of three dependent variables that are in both cases called \(X, Y, Z\). However, the physical significance of these variables, and the underlying reasons for the restriction to just three variables, could not be more different. Were we for a moment, nevertheless, to interpret equations (3.1) as an example of a set of advection equations in three dimensions, the (fictitious) flow field would have divergence \(-\sigma - 1 - b\) and thus be contracting everywhere.

Viewed from the perspective of dynamical systems theory, equations (3.2) are easily rich enough to yield non-integrable behaviour. This can occur for time-dependent plane flow, when \(w=0\) and \(u\) and \(v\) depend only on \(X, Y\) and \(t\), and for both steady and unsteady three-dimensional flows. If the advection equations (3.2) are integrable, as they are in the various examples commonly given in textbooks, including steady plane flow, we speak of regular advection. On the other hand, if the advection equations are non-integrable (in the sense of dynamical systems theory), we have the phenomenon of chaotic advection introduced and named by Aref (1984; see also Aref (2002) for a more detailed historical review of the development of the idea). Many examples of chaotically advecting flow have been studied during the past 25 years and the subject promises to remain vibrant for at least two reasons. On the one hand, chaotic advection has been embraced by the practitioners of microfluidics, essentially as a design tool for microfluidic configurations that mix well. For an example of the application to microarrays, see Stremler et al. (2004), but there are now many others. On the other hand, new theoretical developments have become possible with the realization that basic elements of the topology of the flow will enforce chaotic advection, indeed will lead to a form of ‘maximally chaotic’ advecting flow known as pseudo-Anosov. This line of investigation began with the work of Boyland et al. (2000), where the application of what is known to the mathematician as Thurston–Nielsen theory for continuous, two-dimensional mappings is applied to fluid kinematics and stirring. The approach has later been considerably elaborated in the work of Thiffeault and collaborators (cf. Thiffeault & Finn 2006; Gouillart et al. 2006) done primarily at Imperial College (figure 3).

The subject of chaotic advection has several of those features that were called out as being a key to sustained significance earlier. It deals with basic aspects of fluid dynamics, in this case flow kinematics and fluid stirring and mixing. It invokes important new developments in mathematics and mathematical physics, such as chaos in a dynamical system and advances in the topological classification of continuous mappings. It has significant areas of application, from slow, large-scale mixing processes that are predominantly laminar, to the
high technology of micro- and nanofluidics. Finally, it has aesthetic appeal: chaotic advection produces quite spectacular images with relatively simple flow visualization techniques. Thus, apart from the intellectual and practical appeal, we see, once again, the visual appeal of the subject. The subject of fluid stirring and mixing is, at one level, intuitive and elementary. Nevertheless, it challenges our analytical capabilities to their limits.

In the case of advection by an incompressible, two-dimensional flow, where we have a streamfunction \( \psi(x, y, t) \), the advection equations take the form

\[
\frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}.
\]

These equations are in Hamilton’s canonical form for a system with one degree of freedom. The canonically conjugate generalized ‘coordinate’ and ‘momentum’ are the \( x \)- and \( y \)-coordinates of the advected particle. The streamfunction \( \psi \) plays the role of the Hamiltonian. Configuration space is phase space. As we view the stirring in real space, we are, in fact, peering into the phase space of a Hamiltonian system. When the flow is unsteady, we have a time-dependent, single degree-of-freedom Hamiltonian system, sometimes said to be a ‘system of 1.5 degrees of freedom’. Such a system is at the edge of our analytical capabilities.\(^2\)

As effects are added beyond flow kinematics, i.e. as the particles or substance being stirred becomes less passive, a plethora of new problems arise. These problems ‘live’ in that intriguing subspace where the evolution is predominantly determined by the set of ODEs (3.2), yet a full description requires an extension of the description, which usually invokes the appropriate PDEs. For example, stirring a blob of liquid into a resident liquid with which it does not mix.

\(^2\)We recall that Arnold (1978; ch. 2 §5), somewhat provocatively, wrote that ‘analysing a general potential system with two degrees of freedom is beyond the capability of modern science.’
introduces surface tension along the interface between the two liquids in addition to the advection. As the stirred fluid is drawn out into long filaments, they become subject to the Rayleigh–Plateau instability and bead up into a series of droplets. The further interaction of the series of beads with the advecting flow is, of course, different from what one obtains by passive advection alone.

(b) Relative equilibria of particle-like entities

Fluid dynamics abounds with particle-like entities that a condensed matter physicist might call ‘quasi-particles’: solitary waves, shock waves, drops, bubbles and vortices of many varieties, including the coherent structures of turbulent shear flows, are all temporarily well-defined particle-like entities in terms of which one can describe the flow physics. They may have their own equations of motion, derivable from the Navier–Stokes equations when certain assumptions are made, yet these equations are distinct in character and content from the overarching PDEs. Like the quasi-particles of condensed matter physics, these entities form, propagate, interact and ultimately disperse, dissolve or dissipate. They may not be there in the initial state of a fluid motion, and they may have disappeared by the time we reach the final state. However, in between they govern the flow, and the flow becomes intelligible and describable in terms of them. Their dynamics provides, in effect, a reduced description of the flow, a projection onto a lower dimensional system, often a system determined by a set of ODEs (or a reduced set of PDEs) rather than the full PDEs governing the fluid dynamics.

The vortex ring emanating from a ‘vortex gun’, propagating across an expanse of still air and ultimately reaching and extinguishing a candle, may be the best known example. The vortex ring—a quasi-particle with clear parallels in the rotons of superfluids—organizes the discussion of this flow and provides a clear and intuitive explanation of the process. In particular, the velocity of propagation of the vortex ring and the separation between vortex generator and candle sets the time-scale of the entire chain of events. This velocity is not simply related to the material properties of the fluid, such as the speed of sound. Indeed, the process comfortably takes place in an incompressible fluid! Furthermore, an explanation of the process in terms of the so-called ‘primitive fields’ of pressure and velocity will almost surely be more cumbersome and considerably less ‘physically intuitive’. The introduction of the vortex ring makes the explanation of the process physically clear and it suggests the need for developing a dynamics of vortex rings.

Among the several examples of quasi-particles in fluids that I have been particularly intrigued by are the simple point-like vortices that Helmholtz introduced towards the end of his seminal paper (von Helmholtz 1858). Equivalently, we may think of them as parallel line vortices in three dimensions. Relative equilibria of point vortices are of interest in meteorology (Kossin & Schubert 2004), plasma dynamics (Durkin & Fajans 2000), superfluids of various kinds (Yarmchuk et al. 1979), including the recently much-studied Bose–Einstein condensates (BEC; cf. Ketterle 2001), and, by analogy, in other systems such as interacting droplets (Voth et al. 2002) or the ‘self-assembly’ of floating magnets (Mayer 1878; Grzybowski et al. 2000). Figure 4 contains a collage of related images of relative equilibrium patterns seen in these systems.

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The equations to be solved for relative equilibria of identical vortices are (Aref et al. 2002)

\[ z_\alpha^* = \sum_{\beta=1}^{N} \frac{1}{z_\alpha - z_\beta}, \quad \alpha = 1, \ldots, N. \tag{3.4} \]

Here, the \( z_\alpha \) are the positions of the interacting vortices in the plane, thought of as the complex plane. The asterisk denotes complex conjugation. The prime on the sum on the r.h.s. reminds us to skip over the singular term \( \beta = \alpha \). The angular frequency of rotation, \( \omega \), and the value of the circulation of the vortices, \( \Gamma \), have been scaled such that \( 2\pi \omega/\Gamma = 1 \). Equation (3.4) expresses a problem in algebraic geometry, a branch of mathematics that is not particularly well represented in fluid dynamics, yet one that has seen considerable development in the twentieth century. Finding the complete set of solutions to a system of equations such as (3.4) is a non-trivial task even for moderate values of \( N \). The solution space for (3.4) includes single, double (Havelock 1931) and triple (Aref & van Buren 2005) rings of nested regular polygons, both open and centred. For each \( N \), the vortices may be placed on a line such that this collinear configuration rotates rigidly (Stieltjes 1885). In that case, the vortices must be placed at the zeros of the \( N \)th Hermite polynomial. Presumably, one can nest an arbitrary number of regular

Figure 4. Various relative equilibrium patterns: (a) pattern of electron density peaks in a plasma (Durkin & Fajans 2000), (b) particle cluster in vibrated fluid (Voth et al. 2002), (c) seven point vortices (Aref & Vainchtein 1998), (d) hurricane core with centred pentagon of mesovortices (Kossin & Schubert 2004), (e) analytically known relative equilibrium of 19 identical point vortices (Aref & van Buren 2005), (f) self-assembly of magnetized discs (Grzybowski et al. 2000), (g) a configuration sketched by Mayer (1878), (h) vortex patterns in rotating He II (Yarmchuk et al. 1979), (i) vortex pattern in BEC (Ketterle 2001), (j) asymmetric 10-vortex pattern (Aref & Vainchtein 1998) and (k) vortex tripole (Kloosterziel & van Heijst 1991).
polygons with the same number of vertices. Indeed, the collinear relative equilibria represent a system of nested digons symmetrically arranged! However, there are more exotic patterns that have only been determined numerically (albeit to high precision), including the patterns that appear to lack any symmetry whatsoever (Aref & Vainchtein 1998). Figure 4 provides a minute sampling.

One useful device in studying such patterns appears to be the generating polynomial, by which we mean the polynomial $P(z)$ of degree $N$ that has the $z_\alpha$ from equation (3.4) as its roots, \textit{viz},

$$P(z) = (z - z_1)(z - z_2)\ldots(z - z_N).$$

The discriminant of this polynomial is

$$\prod_{\alpha, \beta=1}^{N} (z_\alpha - z_\beta) = \prod_{\alpha=1}^{N} P'(z_\alpha).$$

Here, the prime on the product sign on the l.h.s. means $\alpha \neq \beta$, whereas the prime on $P$ on the r.h.s. means its derivative with respect to $z$. The formula (3.6) is known in the literature as the expression for the discriminant as a symmetric function of the roots. It is a classical formula in the theory of polynomials (cf. Cajori 1904, §77).

The kinetic energy of the flow set up by a configuration of identical point vortices (less a ‘self-energy’ term) is given by the Hamiltonian

$$H = -\frac{I^2}{4\pi} \sum_{\alpha, \beta=1}^{N} \log l_{\alpha\beta},$$

where $l_{\alpha\beta} = |z_\alpha - z_\beta|$. This Hamiltonian is proportional to the logarithm of the product of intra-vortex separations

$$\Theta = \prod_{\alpha, \beta=1}^{N} l_{\alpha\beta}.$$  

Comparing (3.8) and (3.6) we see that instead of calculating $N(N-1)/2$ separations, the energy may be calculated as the product of $N$ terms of the form $P'(z_\alpha)$. This application of a classic formula from the theory of polynomials to vortex dynamics at least a century later appears to be new (Aref 2007). In its own small way, this encapsulates the fitful progress of our science where increasing specialization leads to the compartmentalization of results and researchers, and a lack of crosstalk about common problems and techniques.

Two-phase flows represent another area of fluid dynamics that displays particle-like entities. In such flows bubbles or droplets of one phase form in and are embedded in the other phase. The evolution and statistics of such a flow can, in principle, be followed by developing the dynamics of interacting bubbles and proceeding to what is, in some ways, a variant of kinetic theory. The limit of zero liquid, as in dry foam, has been studied extensively. Plateau was the first to formulate the laws of bubble statics in terms of the geometric configuration of liquid films (bubble faces) and edges within the foam.

Plateau’s theorem is an intriguing result concerning the situation of the centres of the intermediate liquid films that are produced when three spherical bubbles merge to form a bubble cluster. In this case, all liquid films are spherical.
His result that the three centres of the intermediate spherical surfaces will be collinear is essentially an application of ideas from projective geometry, which was developed by Desargues decades earlier. Extension to four bubbles is possible (Herdtle & Aref 1991). It is apparently still an open problem as to when the first non-spherical, internal surface appears in a few-bubble cluster. Based on numerical computations, John Sullivan conjectured that the minimal such cluster has six bubbles (see http://torus.math.uiuc.edu/jms/images/obub.html).

In two dimensions, the situation is much simpler because the Young–Laplace law guarantees that all interfaces in a foam are circular arcs! The proof that the minimum surface area solution for two bubbles is the familiar ‘dumb-bell’ configuration is, maybe surprisingly, of relatively recent vintage (see Hutchings et al. 2002). The intriguing analogy between minimum interface clusters of equal-volume bubbles and observed patterns of fruitfly ommatidia, documented by Hayashi & Carthew (2004), is mentioned as an example of how these age-old problems continue to find new applications.

We also mention the problem of the evolution of a dry foam as the gas in the bubbles is allowed to diffuse from bubble to bubble, driven by the pressure difference across interfaces between bubbles (cf. figure 5; Fetterman et al. 2000). Von Neumann (1952) established the evolution equation for dry two-dimensional foam that today bears his name. If $A_n$ represents the area of any $n$-sided bubble, von Neumann’s law is

$$\frac{dA_n}{dt} = k(n-6), \quad (3.9)$$

where $k$ is a constant, independent of the area of the bubble and of $n$, that contains such quantities as the permeability of the foam interface to gas flow and the surface tension. Equation (3.9) says that all six-sided bubbles retain their area. All five-sided bubbles shrink their area at the same rate, and their rate of area shrinkage is exactly the rate of area expansion of seven-sided bubbles. Four-sided bubbles shrink twice as fast as five-sided, and eight-sided bubbles grow twice as fast as seven-sided. The rate of expansion of the area of an eight-sided bubble equals the rate of contraction of a four-sided bubble and so on.

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The question of an exact generalization of this dynamics to three dimensions, assuming the same basic physics, has been an open problem at least since 1952. This problem was solved recently by MacPherson & Srolovitz (2007). Their solution involves the so-called Hadwiger measure of the bubble volume, which, to quote Kinderlehrer's (2007) commentary, was 'hiding in plain sight' and, in fact, allows generalization to an arbitrary number of spatial dimensions. To pursue what is becoming something of a theme for this discussion, Hadwiger's theorem, in which his unique volume measure is established, dates from the 1950s and provides yet another example of a well-known mathematical result that lay dormant insofar as application to fluid dynamics is concerned for more than half a century. In three dimensions, the coarsening law is not topological, as it is in two dimensions, but the rate of contraction or expansion of a bubble in three dimensions depends on both its topology and geometry. Whether this is important for issues such as scaling and universality is unknown. Large-scale numerical experiments on the coarsening of three-dimensional foams have not been performed, and the experimental evidence is cumbersome to obtain and ascertain.

Both point vortices and the evolution of foams bring up another subject that, possibly, could become a bit of a theme in early twenty-first century theoretical fluid dynamics. This is the notion of a punctuated dynamical system. If, say, one is trying to model the dynamics of two-dimensional turbulence using point vortices, it seems essential to allow two close like-signed vortices to merge into one from time to time. These mergers would be arranged such that if the circulations of the two vortices were \( \Gamma_1 \) and \( \Gamma_2 \), respectively, the merged single vortex would have circulation \( \Gamma_1 + \Gamma_2 \). Similarly, if the vortices prior to merger were located at positions \( z_1 \) and \( z_2 \), respectively, the merged vortex would be located at the centre of vorticity of the two vortices, i.e. at

\[
z = \frac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2}.
\]

This rule preserves the linear impulse of the flow. It is not possible rigorously to preserve the flow Hamiltonian since one is transferring energy from ‘flow energy’ to ‘self-energy’ of the configuration. However, except for the divergent term associated with the two merging vortices, the flow energy before and after the merger for the system at large should be essentially unchanged. Regardless, the idea of ‘punctuation’ is to stop the evolution an instant before the merging, perform the merging and then continue the evolution with the merged vortex in place and the two original vortices removed. There is a change in degrees of freedom of the system (often a reduction) during the merger process. The merger is supposed to mimic the effects of viscosity. Owing to Kelvin's circulation theorem, vortex mergers cannot occur in inviscid incompressible flow. See Carnevale et al. (1991) for an attempt along these lines.

Similarly, in the foam-coarsening problem, von Neumann’s law, equation (3.9), or its three-dimensional generalization govern the evolution with invariant foam topology. When a small bubble shrinks, or an interface becomes very small, one observes that a reconnection event takes place. This event, which is not intrinsically captured by the evolutionary law, needs to be effected ‘by hand’ in a numerical simulation or theoretical consideration. It entails a change in the...
nature and number of degrees of freedom of the system. It is, once again, an example of punctuation. And, as in the vortex merger, it implies a change in the topology at (essentially) fixed energy.

The transitions at the instants of punctuation play an important role in setting up the statistics of the state of the system. In the vortex case, this is measured by the energy spectrum of the flow as a function of wavenumber. In the case of the relaxing foam, the distribution function for the number of faces of the constituent bubbles assumes a universal form. Numerical experiments (Herdtle & Aref 1992) suggest that this distribution is lognormal. 3

(c) Splashes

Splashes have fascinated humans forever.Splashes probably received their first comprehensive scientific discussion in the book by Worthington (1908) summarizing work, begun in 1876, on high-speed photography of drops of liquid falling onto a horizontal surface. Figure 6 shows one of the Worthington’s pictures alongside a frame from a recent numerical simulation using level set methods by K. Yokoi (2006, private communication). As part of the lecture, the tremendous development in technique that had taken place in the ensuing century and a quarter, in large measure through the work of Harold ‘Doc’ Edgerton at MIT, was noted, as was the pure aesthetics of the phenomenon, wonderfully celebrated in the images by Martin Waugh that may be found on his website http://www.liquidsculpture.com/.

The variety of images and phenomena is staggering and somewhat intimidating to the theoretical fluid mechanician! Even for nominally the same (macroscopic) initial conditions, the sensitivity of the ensuing splash pattern is quite astounding. Slight changes in surface tension and other parameters can produce tremendous variability of the ensuing pattern. Waugh’s interest is, in part, to exploit this variability for artistic purposes. The fluid mechanician wishes to understand the effect of competing parameters and thereby to codify the behaviour and eventually describe it analytically. Is the sensitivity that intrigues us similar to—albeit more ‘spatio-temporal’ than—that which makes

3 This observation is due to G. I. Barenblatt during a discussion of the paper just cited. He ascribes the origins of the lognormality to a result of Kolmogorov (1941). See also the review by Villermaux (2007), which outlines a research agenda in this general area.
fractal patterns so attractive to us? And is this sensitivity, maybe, a manifestation of some kind of ‘strange saddle’ in the phase space of the splash?

Describing theoretically a phenomenon that shows great sensitivity to initial conditions, e.g. the drop release height and the depth of the layer into which the drop is splashing, and to material parameters, such as surface tension, wetting properties of surfaces, etc., appears as a very big challenge. The distributions of droplet sizes, of the number of fingers produced on impact, and other details of the dynamics are currently without much explanation from the fundamental equations. As a theoretician and occasional numerical simulator, I marvel at the number of potential PhD theses displayed in the array of splatters on the floor between my office and the coffee machine down the hallway!

The surprisingly decisive role of the ambient air in determining the properties of the splash (Xu et al. 2005) has opened up this classical subject anew and promises to keep the research—and the art photography—community busy for quite some time to come. Every increase in speed of high-speed photography seems to reveal new mechanisms and processes as the drop hits the surface below it.

The amazing network of liquid bridges into which the ‘bowl’ breaks up (figure 7), presumably due to Marangoni stresses as a viscous drop impacts a thin layer of liquid having lower surface tension, displayed in the video by Thoroddsen et al. (2006), again shows the multitude of paths this exploration can take. The liquid network seen is superficially reminiscent of the two-dimensional foams mentioned earlier, although the holes are genuine holes and not liquid films stretched between liquid edges.

4. Theoretical fluid dynamics in the twenty-first century

The preceding remarks, sketchy as they may be, have hopefully established that as we enter the twenty-first century, theoretical fluid dynamics is a vibrant field, with noble traditions, high standards, challenging problems, relevant
applications and aesthetic appeal. In this favourable environment, what possible concerns might one have for the future? I list a few concerns below.

(i) Waning interest among students in demanding analytical work.
(ii) Short-term view of funding agencies in terms of theoretical development.
(iii) Quantity versus quality of journal publications.
(iv) Proliferation of congresses, conferences and other scientific meetings.

Re (i): This, I am sure, depends on one’s academic environment. Yet, at least in the USA, much fluid dynamics research is done in departments of engineering, primarily mechanical and aerospace engineering. There is some, although markedly less, fundamental work on fluid dynamics done in physics departments, in spite of the aforementioned tradition that includes contributions by many of the greatest physicists. There is some activity in departments of mathematics, particularly if and when they have a strong applied mathematics group. There has been an irreversible tendency for universities to shut down programmes and departments of theoretical and applied mechanics, engineering mechanics and the like, or to merge them with professional engineering programmes. The number of such departments in the USA is now at an all-time low. The Department of Theoretical and Applied Mechanics at University of Illinois, Urbana-Champaign, which the author had the honour of heading for a decade, has been the latest victim of this unfortunate trend.

Insofar as the engineering departments are concerned, the motivation for studying fluid dynamics tends to be driven by applications and, in turn, by those areas of engineering involving fluid flow that currently can compete for research funding. Engineering researchers and their students thus tend to view fluid dynamics as an enabling subject, a tool among several others that must be mastered in order to complete the research. These researchers and their students tend to be consumers rather than producers of basic insights into fluid dynamics. They perform the very valuable function of enlarging the range of applications of the subject that we considered in §1. However, they do not, typically, advance the boundaries of the fundamentals of the subject. Mechanics, both fluid dynamics and solid mechanics, viewed as disciplines of science thus lose their intellectual identity and their ‘academic home’ on the campus and, over time, are inevitably de-emphasized.

This troubling pattern is not entirely new, but as funding has become harder and harder to come by, and more and more mission oriented, the possibility of sustaining a curiosity-driven research programme in theoretical fluid dynamics has declined. (Maintaining an experimental programme aimed at basic fluid dynamics is also rather difficult, and the number of such programmes has

4 According to a tally by I. K. Puri (2007, private communication) of undergraduate programmes in engineering mechanics, ‘[t]here are perhaps 58 such programs of which at least six in Engineering Mechanics share strong similarities with one another.’ On a more hopeful note, he adds that ‘[p]rograms in Engineering Mechanics, Engineering Science and Engineering Physics continue to be added to the ABET-accredited list in the 21st century.’ ABET, Inc., formerly the Accreditation Board for Engineering and Technology, is a non-profit organization that serves US post-secondary degree programmes in applied science, computing, engineering and technology. Accreditation is intended to certify the quality of these programmes. There are over 2700 programmes accredited at over 550 colleges and universities in the USA.
declined as the costs of building and maintaining state-of-the-art facilities have continued to rise.) Engineering departments infrequently admit students whose main motivation is fundamentals-oriented research in fluid dynamics, and the number of engineering students applying with this inclination is small in any event. At the same time, since the subject is usually represented by a subcritical number of faculty in any given science department (including mathematics), the number of students attracted to fundamental fluid dynamics in such departments is also modest. Overall, the number of students at the graduate level from whatever background seeking to specialize in basic fluid dynamics is relatively small, and from this group the number seeking the kind of heavy analytical work needed for cutting-edge research in theoretical fluid dynamics, and who are able to make an important contribution in this realm, is small indeed. It is not unreasonable to say that, as we enter the twenty-first century, theoretical fluid dynamics has something of a ‘pipeline problem’. If we wish theoretical fluid dynamics to flourish in the present century as it did in the twentieth century, efforts to address the pipeline are essential.

One may think that these lamentations are unique to fluid mechanics. One may ascribe them to the absence of the subject from most school science curricula, to the reduced ‘visibility’ of the subject as a specialization for students at university level, to the ‘maturity’ of the field in general, to the lack of academic departments devoted to the subject, to the absence of major awards such as Nobel prizes being awarded to prominent fluid mechanicians, and so on. It is thought provoking, then, to read that chemistry, a subject that seems to have all these bases covered, feels it suffers from similar pipeline issues (Adam 2001).

Large-scale computing still tends to attract students, and while the development of CFD was clearly one of the most important developments in our science in the twentieth century, computer simulation is no more a ‘theory’ than is a well-conducted laboratory experiment. Compare figure 6a, b for example. While one can marvel at the artistry and technique involved, the outcome in terms of understanding of, say, how many droplets are produced in the crown of the splash is roughly equivalent. There is, unfortunately, often a tendency for students—and sometimes their supervisors!—to run to the computer prematurely only to demonstrate something numerically that a brief period of analytical contemplation would have rendered obvious. Lumley (1992) voiced similar thoughts and, at least for the US context, placed them in a sociological perspective:

I would like to close with a few words about being a theoretician in the United States towards the close of the 20th century. The United States is a curiously unsympathetic environment for a theoretician, or any scientist interested in fundamental work. We have a sociocultural/historical myth with which those of us who were children here grew up, of egalitarianism, practicality, inventiveness. An American, in this myth, is a man who rolls up his sleeves and pitches in, solving the problem at hand in a clever, simple, practical way (often involving bailing wire and a wad of chewing gum), usually saying over his shoulder that he does not hold with book learning. Edison is often suggested as an example. Many of our heroes had trouble in school. We tend to regard too much faith in what is written as being a foreign invention. In this environment, the theoretician is viewed with alarm, and felt to be irrelevant. He is regarded as impractical, pie in the sky...

Despite all that, theory is what gives meaning to observation. Understanding is the process of constructing simple models that explain the observations, and permit predictions. What the theoretician does is a vital part of the loop, and does not receive enough credit
here. Our typical reaction to a theory is ‘Let’s see some computations. How does that compare with the data?’ Those pragmatic questions are legitimate, and of course, any theory must rush to answer them. However, first the theory exists alone, as an entity in and of itself, and deserves to be appreciated on its own merits. Is it internally consistent, does it connect all the known behavior in a minimalist way? Does it patch smoothly to previously accepted theories? A theory that does all that in an effortless way is often called elegant. Tomorrow, it may be wrong. Even so, it deserves to be regarded as one of the better things of which man is capable.

Re (ii): We have already alluded to the funding issue. The mission-oriented agencies quite logically direct their resources towards those aspects of fluid dynamics that serve their immediate, often relatively short-term purposes. While we have argued that fluid dynamics touches almost every aspect of human endeavour, a problem such as the fluid dynamics of dispersal of a pathogen used in bioterrorism is likely to consist mostly of the application of rather well-known theories and techniques (e.g. in numerical simulation) applied in a new context and is unlikely to engender much new fundamental insight. There is again nothing wrong with this but it is unlikely to produce breakthroughs on a par with Prandtl’s boundary-layer theory or Kolmogorov’s scaling theory of fully developed turbulence. More importantly, it is unlikely to train students to think in such overarching terms.

This is not an ironclad causal argument, of course, since key fundamental insights have been produced under direct pressure from applications. Several instances of the work of G. I. Taylor, T. von Kármán or J. von Neumann are remarkable in this regard, so the more appropriate statement may be that the number of researchers who can take a rather applied problem setting and ‘turn it into gold’ is at any one time rather small.

Stated from a resource allocation standpoint, however, the investment necessary to acquire data is often quite different and even distinct from the investment needed to achieve understanding. The short-term view of most funding agencies and their officers, coupled with the increasing pressure on university faculty to procure research funds, tend to conspire to produce a large number of quite applied projects without long-term advances of the subject.5

Maybe surprisingly, one very positive development for the field has been the technological interest in microfluidics. Initially, this subject started as a quite applied one. The term ‘widget’ was frequently heard. Researchers built devices not because they served much purpose or furthered understanding but because they could be built! A large amount of work in microfluidics was done by the same electrical engineers who taught us how to make the chips needed. These individuals were often quite innocent about fluid dynamics and were rather exclusively users of the subject. This is all changing, partly because the problems have become harder and require more sophisticated understanding and partly because researchers with a solid founding in fluid dynamics have assimilated what they need to know from the electrical engineering and manufacturing side and are proceeding apace. Microfluidics has revived research in laminar flows,

5 These remarks are again conditioned on the current US funding environment with which the author is most familiar. The grant through which my activities in Denmark are currently supported is, refreshingly, entitled ‘Fundamental Problems in Fluid Dynamics’!

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and nanofluidics promises to inject new life into the study of rarefied gas dynamics as the device dimensions become comparable with the mean free path of motion of the constituent particles of the fluid.

Re (iii): Writing with the perspective and biases of a fluid dynamics researcher, Gad-el-Hak (2004) has addressed this problem in general and with emphasis on our field. He writes:

Academic institutions in the US have made it imperative for faculty members to publish in order to survive and prosper. The publish-or-perish mantra became a household motto for faculty. There is nothing wrong with that principle if it emphasizes quality rather than quantity. For the most part, that emphasis on publishing has worked for many decades. The number of publications was reasonable, and tenure and promotion decisions in research universities were largely based on the impact of a candidate’s scholarly work, as measured by the number of citations and, less quantitatively, by expert opinions. The number of journals and consequently the number of requests for refereeing were both manageable...

So far so good. But he continues:

Unfortunately, today we witness a different environment from that of a generation ago. The publish-or-perish emphasis for some, but not all, institutions has deteriorated into bean counting, and the race is on to publish en masse. Demand spurs supply. Mostly for-profit publishers of books and journals have mushroomed, and mediocrity has crept into both venues. Journal pages have to be filled, and library shelves have to be stacked with books.

He cites the following frightening statistics:

The number of periodicals worldwide currently stands at 169,000 and the number of books published in the US alone in 2001 is 56,364. Of course, not all of these are academic publications, but the sheer numbers are frightening enough. Currently, more journals in a particular research field are published than anyone can reasonably keep up with.

Even in our relatively small corner of research activity there are, according to Gad-el-Hak (2004), at least 250 journals published in English on fluid dynamics. He asks the obvious if somewhat rhetorical question: ‘Who can keep up with 250 journals?’ More sinister conclusions follow, such as what the possible quality of refereeing for all these journals might be and, consequently, what the quality of the papers published in them is. Most published papers today are read by a very small number of people, a number that may in many cases fall to zero if the author and his/her immediate collaborators are excluded. Many papers are never cited and thus, effectively, have no impact on the field. The facility of ‘cutting and pasting’ in electronically produced documents promises, if anything, to increase the pace of publication. One is reminded of the concern expressed by a physicist that if the rate of publication of Physical Review continued to increase at its current pace, the speed of propagation as volumes are added along the shelves in the library would eventually reach the speed of light. The humorous resolution of this apparent contradiction with the theory of relativity is that the information content would by then have dropped to zero! For a perspective from the ‘high end’ of the field on journal quality and impact, see the still very readable article by Batchelor (1981) and the recent survey by H.E. Huppert (2006, unpublished manuscript, http://en.wikipedia.org/wiki/Journal_of_Fluid_Mechanics).

Re (iv): Much of what was said about journal publication can, unfortunately, be repeated with slight edits for the activity of holding congresses, conferences, workshops and the like. Such scientific meetings are today scheduled with
abandon and quality control is often light to non-existent. Contributions are accepted without any review whatsoever\textsuperscript{6} and sessions are reorganized ‘on the fly’. Many of these meetings are marked by serious inconsistency of content. IUTAM has succeeded in maintaining a remarkably consistent series of symposia and does so through serious peer review of both proposals and contributions, and by making all symposia invitation-only gatherings. The conduct of a quality scientific meeting is something of an art that depends not only upon the scientific quality of the contributions but also on thematic definition and attention to organizational detail. Many meeting organizers would benefit from reading the study by Epple (1997), which contains some innovative and provocative recommendations as well. The general malaise, however, is simply that there are too many meetings, far more than the acknowledged experts can or want to attend, and the quality consequently declines. Most of these meetings have no discernible impact on the progress of the field and as such are a waste of time and resources. Various competitive games with intellectual content, such as chess, have developed an elaborate ranking system for players and tournaments. If we draw the crude analogy between a chess player and a researcher and between a chess tournament and a conference, one could imagine ranking conferences by the quality of participants and publishing such rankings. In order for a researcher to advance in rank, he or she would have to participate and present in a conference of a suitable rank! This sounds Orwellian, but some kind of order needs to be brought to the systems of scientific communication that today are growing out of control, or they will lose all credibility.

Happily, the meeting at which this paper was presented was a notable exception. In spite of its general lack of thematic focus, the level of the presentations was uniformly high. Many of the senior contributors spoke about problems that had plagued them for a lifetime, which provided rare glimpses on problem approaches and motivations to tackle various areas.

The dilemma of a discourse such as the one given here was aptly captured by Sir Michael Atiyah in his Fields Lecture at the World Mathematical Year 2000 Symposium, held in Toronto, 7–9 June 2000, entitled ‘Mathematics in the 20th Century’ (transcribed in Shenitzer & Stillwell 2002). He started by saying:

\begin{quote}
If you talk about the end of one century and the beginning of the next you have two choices, both of them difficult. One is to survey the mathematics over the past hundred years; the other is to predict the mathematics of the next hundred years. I have chosen the more difficult task. Everybody can predict and we will not be around to find out whether we were wrong. But giving an impression of the past is something that everybody can disagree with.
\end{quote}

Within the much more restricted scope of this article, I can at best hope to have given some material for contemplation and discussion.

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\textsuperscript{6}This practice includes the annual meetings of the American Physical Society, Division of Fluid Dynamics, which, somewhat miraculously, maintain reasonable quality simply by attracting a substantial fraction of the leading researchers in the field, particularly from the USA, each year. Again, this is an example of the high end of the enterprise that is subject to its own dynamics.
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