On the linear stability of Stokes layers

BY P. J. BLENNERHASSETT1,* AND ANDREW P. BASSOM2

1School of Mathematics and Statistics, University of New South Wales,
Sydney, NSW 2052, Australia
2School of Mathematics and Statistics, University of Western Australia,
Crawley, WA 6009, Australia

Oscillatory flows occur naturally, with applications ranging across many disciplines from engineering to physiology. Transition to turbulence in such flows is a topic of practical interest and this article discusses some recent work that has furthered our understanding of the stability of a class of time-periodic fluid motions. Our study starts with an examination of the linear stability of a classical flat Stokes layer. Although experiments conducted over many years have demonstrated conclusively that this layer is unstable at a sufficiently large Reynolds number, it has only been relatively recently that rigorous theoretical confirmation of this behaviour has been obtained. The analysis and numerical calculations for the planar Stokes layer were subsequently extended to flows in channels and pipes and for the flow within a torsionally oscillating circular cylinder. We discuss why our predictions for the onset of instability in these geometries are in disappointingly poor agreement with experimental results. Finally, some suggestions for future experimental work are given and some areas for future theoretical analysis outlined.

Keywords: linear stability; oscillatory flows; channel flows; cylinder flows; unsteady flows

1. Introduction

The stability of time-periodic laminar flows is a topic of long-standing theoretical and practical interest. The paradigm for such flows is provided by the flat Stokes layer; that is the motion induced within a semi-infinite layer of viscous fluid when the bounding plate oscillates in its own plane. Some form of Stokes layer appears on the bounding surfaces of many high-frequency oscillatory flows and, in particular, purely oscillatory flow in pipes or channels can be decomposed into an inviscid flow away from the walls with a Stokes layer adjacent to the boundaries, provided that the frequency of oscillation is sufficiently large. Periodic flows consisting of a steady velocity together with a zero temporal mean oscillatory component can be subject to an instability mechanism associated with the mean part. The stability properties of such flows can usually be examined using a perturbation-type analysis (Hall 1975) when the oscillatory part of the flow is small compared with the mean. However, if the periodic component is larger, or in the extreme case when the mean
is absent altogether, such methods fail. Clearly, other techniques are then needed to determine the underlying stability characteristics.

There is overwhelming experimental evidence, gathered over a considerable time, that suggests purely oscillatory flows are likely to be unstable if the fluid velocity is sufficiently large. There are inherent technical difficulties in conducting careful practical work with oscillating profiles for it is not at all straightforward to even produce an accurate basic flow. Various devices have been used to generate temporally oscillating flows; for example, Merkli & Thomann (1975) conducted their experiments in a pipe with one end sealed, Clamen & Minton (1977) oscillated the pipe containing their fluid, and others, including Eckmann & Groebler (1991), generated their flow via a piston oscillating in a long pipe. It can be shown that while these various methods of generating the flow should not affect the stability observations, they do alter the corresponding disturbance structures and thus how any perturbations might be described. An indication of the difficulties arising in this practical work is that there is no agreed value of the Reynolds number at which the flow first becomes unstable: some estimates differ from others by a factor of two or more.

Despite this wide discrepancy in the results reported by these various experiments, one common thread running through them is the conclusion that Stokes layers do become unstable under suitable conditions. However, much of the early theoretical work concerning the stability properties of Stokes layers could not detect any evidence of instability at all. Hall (1978) applied Floquet theory to the classical flat Stokes layer but was only able to locate decaying modes. An earlier study by von Kerczec & Davis (1974), who considered the oscillatory flow in a wide, but finite, channel with one wall oscillating, found similar behaviour. Numerical studies have also only identified decaying Floquet modes; for example, Akhavan et al. (1991a,b) used direct numerical simulations to examine both the linear and the nonlinear stability of oscillatory flow in a channel and found only decaying disturbances as solutions of the linear governing equations.

Several reasons for the discrepancies between theory and experiment have been proposed. Analytical and numerical studies by Blondeaux & Vittori (1994) and Vittori & Verzicco (1998) suggest that wall imperfections or other external sources may trigger turbulence in oscillatory flows. Blondeaux & Vittori (1994) used a two-dimensional analysis to prove that the flow deviates from the laminar regime due to the growth of perturbations during certain phases of the oscillation cycle; the origin of this growth lies with a resonant effect. By way of an alternative, Wu (1992) proved how a resonant triad interaction can generate a finite-time singularity. More recent calculations by Costamagna et al. (2003) throw light on the more strongly nonlinear aspects of transition. Here we do not consider which of these various alternative mechanisms might be the most relevant in practice, but instead just comment that the various ideas have probably been examined under the assumption that the flat Stokes layer is linearly stable. Indeed, this point of view has been put quite forcefully by Yang & Yih (1977) who state: ‘... the conjecture of von Kerczec & Davis that Stokes flows are stable at high Reynolds numbers and our conjecture that time-periodic flows in a circular pipe (with no steady component) are stable at high Reynolds numbers stand or fall together. We believe in the conjecture of von Kerczec & Davis, and therefore in ours’.
This brief account of the instability properties of oscillatory flows summarizes the rather unsatisfactory state of knowledge relating to the stability of such layers. There are two principal issues that were felt worthy of further examination.

— Why do experiments regularly find instability, but all the available Floquet theory-based calculations do not? Is the instability a genuine nonlinear effect that escapes detection by standard linearized analysis?
— If the instability is present, as indicated by the practical work, why is there such a wide variation in the estimates of the Reynolds number for the onset of transition?

It is with these questions in mind that we embarked on a systematic study of the linear stability of temporally periodic flows. Our first concern was with the classical flat Stokes layer, which is a particularly attractive flow for theoretical analysis as it is one of the relatively few exact solutions of the incompressible Navier–Stokes equations. Later studies have been extended to consider flows in channels and pipes. Subsequent sections of this paper describe how our investigations have advanced the knowledge of the stability of purely oscillatory flows, but space precludes a thorough discussion of all the salient points. Thus, where appropriate, the interested reader is directed to the extended descriptions of the studies, which may be found in Blennerhassett & Bassom (2002, 2006, 2007) and which will be referred to as BB02, BB06 and BB07, respectively.

2. Formulation of the stability equations

As the eventual aim is to examine oscillatory flows in a variety of geometries, it is convenient to derive the underlying equations in a fairly general setting which can then be specialized as necessary. To that end, consider orthogonal basis vectors $e_1$ and $e_2$ that are aligned so that the solid boundaries to the flow are parallel to the $e_1$ direction and, relative to these vectors, a general point has coordinates $(x_1^*, x_2^*)$. Suppose that a periodic basic flow is driven by the motion of a boundary on which the fluid velocity is $u^* = U_0 \cos \omega t^* e_1$, where standard notation has been used. Given such a forcing, elementary scaling arguments suggest that a Stokes layer of thickness $O(\sqrt{\nu})$ will be formed adjacent to the boundary; here $\nu$ denotes the kinematic viscosity of the fluid.

This configuration sets up a dimensionless basic viscous flow $U_B(x_2, t)e_1$ with $t=\omega t^*$. This velocity profile is unidirectional and purely oscillatory with the property that

$$U_B(x_2, t) = U_B(x_2, t + 2\pi),$$

where lengths are scaled on $\sqrt{2\nu/\omega}$ and velocity is scaled on $U_0$.

Consider perturbations to this flow that are two-dimensional, have a wavenumber $a$ in the $e_1$ direction and are sufficiently small for linearization of the governing equations to be valid throughout a complete cycle of the basic flow. This leads to a problem that, symbolically, can be written as

$$l(x_2, t; a, Re)\phi(x_2, t) = 0 \quad \text{with} \quad b(\phi) = 0 \quad \text{on the boundaries}.$$
Here \( l(x, t; a, Re) = l(x, t + 2\pi; a, Re) \) is a differential operator, \( b \) enforces the boundary conditions and

\[
Re \equiv \frac{U_0}{\sqrt{2\nu\omega}}
\]

is the Reynolds number of the flow. In what follows our interest is solely with Floquet-type solutions for which

\[
\phi(x_2, t) = \exp(\mu t) \psi(x_2, t) \quad \text{with} \quad \psi(x_2, t) = \psi(x_2, t + 2\pi),
\]

so that the net growth or decay experienced by the disturbance over the course of a cycle of the basic flow is subsumed entirely within the parameter \( \mu \). After this decomposition, what remains is a problem of the form

\[
\mathcal{L}(x_2, t; a, \mu, Re)\psi(x_2, t) = 0 \quad \text{with} \quad B(\psi) = 0 \quad \text{on the boundaries},
\]

which needs to be solved subject to the requirement that the eigenfunction is \( 2\pi \) periodic in time. By way of a specific example, for oscillatory flow in a channel of dimensionless width \( 2h \) and with \( x_2 \) above interpreted as the coordinate \( y \), the linear stability problem can be cast as

\[
\zeta_t = \frac{1}{2} (\zeta_{yy} - a^2 \zeta) - \mu \zeta - i a Re U_B \zeta + i a Re U_{Byy} \psi \quad \text{where} \quad \zeta \equiv \psi_{yy} - a^2 \psi,
\]

to be solved with \( \psi = \psi_y = 0 \) on \( y = \pm h \). Here subscripts denote partial derivatives in the usual way, and the basic flow is given by

\[
U_B = \text{Re} \left\{ \frac{\cosh((1 + i)y)}{\cosh((1 + i)h)} \exp(it) \right\}.
\]

The equivalent systems for the classical Stokes layer in a semi-infinite layer and for the flow induced within a torsionally oscillating cylinder are given in BB02 and BB07, respectively.

An eigenproblem of type (2.2) requires that the parameter \( \mu \) be computed as a function of disturbance wave number \( a \) and Reynolds number \( Re \). In general, \( \mu \) is complex valued, but for stability purposes it is the real part of \( \mu \) that is of most significance. Regions of stability and instability in parameter space are separated by curves defined by the requirement that \( \text{Re}(\mu) = 0 \); these neutral curves can only be found numerically.

(a) Numerical methods

The system (2.2) was solved by first decomposing the function

\[
\psi(x_2, t) = \sum_{-N}^{N} \psi_n(x_2) \exp(nt),
\]

where theoretically \( N = \infty \); this automatically ensures the temporal periodicity of \( \psi(x_2, t) \). The governing partial differential equation was thereby transformed into an infinite-dimensional system of ordinary differential equations for the unknowns \( \psi_n \). In practice, some sort of truncation is necessary and great care was taken to ensure that the value of \( N \) was sufficiently large for the omitted higher harmonics to be negligible. In all the calculations reported here, it was found that the truncation point \( N \) had to be quite large; values in the range 200–300 turned out to be necessary to find neutral curves.
Once the coupled ordinary differential equations had been derived the solution method of choice was largely dictated by the nature of the underlying basic flow. For the semi-infinite fluid layer, Hall (1978) demonstrated how the defining system of equations can be solved semi-analytically; by this, we mean that the individual unknowns $\psi_n(x_2)$ can be written as doubly indexed sums of exponential functions that satisfy the governing equations. The coefficients of these functions are then required to satisfy complicated recurrence relations and it can be shown that the linear stability equations admit a non-trivial solution if and only if a certain (formally infinite) matrix system $Ax = 0$ also has a non-zero solution. These steps enable the solution of the full system to be transformed from a problem of solving differential equations to the conceptually easier task of finding the zeros of an associated determinant. When the geometry of the flow is confined, rather than semi-infinite, the equations were solved using pseudo-spectral discretization following the procedures discussed by Trefethen (2000), with the computations performed using MATLAB. For non-axisymmetric flows in pipes, spectral methods were again used, but now a Petrov–Galerkin discretization was implemented (Meseguer & Trefethen 2003). More details of each of these numerical strategies may be found in BB02, BB06, and BB07.

3. Results

Our first calculations examined the stability characteristics of the semi-infinite Stokes layer using the determinant method pioneered by Hall (1978). Initial computations showed that, for a fixed wavenumber $a$, the growth rate $\mu_r = \text{Re}(\mu)$ of the perturbation is reasonably large but negative for Reynolds number $Re \leq 100$ indicative of quite strongly damped modes. As $Re$ was increased further, gradually the modes become increasingly less stable and, eventually, $\mu_r$ changes sign. As an example, with $a = 0.3$ it was estimated that $\mu_r = 0$ when $Re \approx 780$. This simple-minded approach enabled us to construct an approximate neutral curve sketched in

![Figure 1. (a) Smoothed form of the neutral curve obtained by interpolating through the neutral points $(a_j, Re_N(a_j))$, where $a_{j+1} - a_j = 0.01$. (b) Detailed form of the neutral curve in the range $0.368 \leq a \leq 0.385$ showing two of the fingers. The curve is constructed by interpolation through the marked points and shows that the critical disturbance is stationary with $a = a_c \approx 0.3746$ and $Re = Re_c \approx 707.84$.](image-url)
figure 1a, which suggests that the critical \( Re \) is approximately 708 and occurs at a wavenumber \( a \approx 0.38 \). The description of the curve as approximate is quite deliberate, for it turns out that its geometry is somewhat more involved than first appears. All the solutions on the main part of the neutral curve have the imaginary part of \( \mu \), denoted \( \mu_i \), non-zero. This means that the perturbation eigenfunction has a phase speed relative to the underlying basic flow, but the calculations described in BB02 uncovered a second set of standing wave solutions for which \( \mu_i = 0 \). In most cases, the propagating waves are the ones with the greater growth rates but, for certain wavenumbers, it is actually the standing waves that become unstable at the lower \( Re \). The implication in terms of the neutral curve is that, rather than being a familiar smoothed shape, it is actually punctuated by thin finger-like protrusions; on these all the modes are stationary. A small section of the modified neutral curve is shown in figure 1b, which shows the two fingers nearest the minimum of the main curve. It is noted that the fingers are quite short in terms of Reynolds number scalings as their length is less than unity; it should also be remembered that these features exist all the way around the approximate smooth neutral curve shown in figure 1a. Overall, it was found that the first mode to become unstable has \( a \approx 0.3746 \) at a critical \( Re = Re_c \approx 707.84 \) and that this is actually of the stationary type.

The principal result of these calculations was the first theoretical evidence that the semi-infinite Stokes layer can become linearly unstable at sufficiently large Reynolds numbers. This at least was a step in the right direction, for now there was some analytical work that was broadly in accord with experiments. However, there remained several unsatisfactory aspects of the problem, not least the fact that there was still a large disparity between the theoretical prediction and practical measurements of \( Re_c \). Despite the wide variation in measured values of \( Re_c \), it is generally accepted that experimentalists estimate the onset value of \( Re \) to be of the order of a few hundred in terms of the scalings assumed here; certainly much less than our theoretical prediction of 708. This unexpectedly large value also suggests why many of the previous theoretical attempts to find a neutral curve had been unsuccessful: workers had often not computed to sufficiently large values of \( Re \) for it was scarcely credible that the analytical predictions would so far exceed the experimental measurements. One possible explanation for the disagreement is that the practical generation of Stokes layers tends to use wide channels or large radius pipes; it is not easy to generate semi-infinite layers in the laboratory! Perhaps the experimental determination of \( Re_c \) is so different from the theoretical suggestion precisely because it is not the idealized semi-infinite flat Stokes layer that is studied in practice. Motivated by this, the analysis and numerical calculations were next extended to describe oscillatory flows in channels and pipes.

(a) Calculations for periodic flow through channels and pipes

Consider the motion induced within a Newtonian fluid by a pair of synchronously oscillating infinite flat plates separated by a dimensionless distance \( 2h \); the relevant linear stability equation has already been recorded in (2.3) and the base flow is defined by (2.4). For \( h \) greater than about \( 2\pi \), the vorticity generated by the movement of the walls is unable to diffuse to the centre of the channel and the basic flow is effectively zero around \( y = 0 \). Thus, as interest was mainly on the stability of oscillatory flows that have Stokes layers bounding an almost inviscid core flow, computations were restricted to values \( h \geq 5 \).

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Some sample neutral curves are shown in figure 2 for various channel half-widths in the range 5 ≤ h ≤ 8. Just as for the semi-infinite flow results described above, the main smoothed neutral curves are punctured by regularly spaced needle-like features on which the neutral mode is stationary. The symmetry of the channel flow means that the eigenfunctions divide naturally into two classes distinguished by whether the perturbation stream function is even or odd about the channel midline. Over the parameter ranges investigated, it turned out that it is the even modes that are always the less stable, and so it these that are shown in figure 2. These results show vividly that, over the range studied, as the channel width decreases so the critical Reynolds number falls. As h → ∞ all the details of the semi-infinite flat plate results of BB02 could be retrieved; recall that those calculations were performed using the determinant method but the present channel equivalents were executed using a spectral code. The fact that the two completely independent solution methods gave results that are consistent in appropriate limits provides some confidence that both techniques yield meaningful results. Our results also shed light on some of the earlier predictions of von Kerczek & Davis (1974). Those authors found that at values of Re up to a few hundred oscillatory channel flow is stable, but only weakly so; however, the damping rate increases with Re so it seemed probable that the flow is always stable. Our calculations indicated that in fact oscillatory channel flow admits essentially two classes of eigenmodes, which can be (loosely) divided into centre modes, which are concentrated near the middle of the channel, and wall modes, which are the natural analogues of the Stokes layer disturbances found by Hall (1978) and BB02. For relatively small Re, the wall modes are much more heavily damped than their centralized counterparts, but while the latter just become more stable with increasing Re, as found by von Kerczek & Davis (1974), it is the wall modes that eventually become unstable.

Figure 2. (a) Neutral curves for even disturbance modes in a channel for 5 ≤ h ≤ 8 in steps of one-half; the lowest curve is for h = 5 and h increases monotonically up the figure. Some fingers are shown explicitly on the upper four curves with the crosses indicating the location of the base of the finger on the others. (b) Neutral curves for even and odd disturbance modes in a channel and for axisymmetric perturbations in a pipe. The circle symbol labels the pipe neutral curves, the cross symbol the even channel modes with a plus sign attached to the odd channel neutral modes. The solid lines are for h = 16, the dotted lines for h = 10 and the dashed lines for h = 7.
Computations of the critical $Re$ for various channel half-widths $h$ suggested that the minimum is $Re_c \approx 640$ when $h \approx 5$. Thus, replacing a semi-infinite flat plate flow by a finite channel analogue does reduce the onset Reynolds number by approximately 10%, which brings the results of BB02 nearer to experimental measurements but the difference between the theory and practice is still quite considerable. In an effort to incorporate a geometry even closer to the majority of experiments, we next conducted a series of calculations relating to axisymmetric disturbance modes within a longitudinally oscillating pipe. It can be shown that the stability properties of disturbances are identical whether the tube is oscillated longitudinally, whether it is held fixed and a periodic pressure gradient moves the fluid, or whether the motion is induced by a reciprocating piston inside the pipe. However, in all three cases, the structure of the corresponding eigenmodes will differ. Neutral curves for the pipe flows are shown in figure 2b; once again the smallest $Re_c$ (as a function of pipe radius) falls by about a further 10%. So at least the use of a cylindrical geometry improves the correlation between theory and practical determinations of $Re_c$ but still the discrepancy is rather large. As an illustration of the size of this disagreement, it is noted that Akhavan et al. (1991a, b) examined the structure of an apparently turbulent, purely oscillatory flow in a pipe at a Reynolds number that, according to our calculations, should only be just linearly unstable.

Most experiments on purely oscillatory flow examine the structure of the flow via velocity traces either from laser Doppler devices or from hot-wire anemometers at a selection of fixed points inside the pipe. Figure 3a shows an arbitrarily scaled predicted axial velocity perturbation for the least damped mode at parameter values close to some used by Akhavan et al. (1991a, b) in their experiments using an oscillatory pressure gradient-driven flow. It is striking how similar the velocity profile resembles turbulent bursts in the flow even though theoretically the flow is linearly stable. It turns out that the extremely high-frequency components in the velocity profile are just an artefact of the choice of reference frame for the basic
flow. To see this, figure 3b shows the eigenfunction for exactly the same parameter choices but for the corresponding wall-driven flow. The numerical solution of the governing system gives the identical eigenvalue, as of course it should, but it is surprising just how much simpler the eigenfunction is.

A survey of the literature suggests that experiments designed to determine critical conditions for the onset of instability yield a wide range of values for the onset Reynolds number (e.g. Merkli & Thomann 1975; Hino et al. 1976; Clamen & Minton 1977; Eckmann & Grotberg 1991). In practice, critical conditions are often estimated by comparing measured velocity profiles with those associated to oscillatory flow in an infinitely long pipe. If there are significant differences then the flow is deemed to be unstable; of course, the underlying assumption is that any disturbances caused by a driving piston will decay away from the piston. It can be shown (see BB06 for details) that our findings that \( \mu = 0 \) at critical conditions (so in terms of the earlier figures, critical conditions occur on one of the ‘fingers’, where \( \mu_i = 0 \)) imply that any perturbations to the basic flow caused by imperfections in the apparatus will not decay spatially. Further, many of the experiments report significant levels of extraneous vibration associated with the production of linear oscillatory motion from rotational motion, often via some kind of scotch yoke. When these factors are combined with the knowledge that the eigenfunctions have a very high-frequency content as the bulk of the fluid is moving, it is easy to appreciate that finding critical conditions is fraught with problems and significant variation in the results of the various experiments is not surprising.

(b) Flow in torsionally oscillating cylinders

The findings thus far suggest that experimental studies of the stability of oscillatory flows can suffer from at least three inherent difficulties: inlet length problems; vibration of equipment; and disturbances with very high-frequency content. It would be convenient if some or all of these effects could be eliminated or minimized in some way. This might be possible if experiments were to be conducted with a confined flow in which the driving oscillation is due to a moving boundary rather than a pressure gradient. Given these requirements, it seemed that improvements for the accurate experimental generation of oscillatory shear modes might be achieved if such disturbances were studied in a fluid contained within a torsionally oscillating circular cylinder. This geometry forces the incorporation of the effects of curvature on the stability of flat oscillating layers and it also has the unattractive property that centripetal instability could also be present. However, it is quite easy to predict what might be expected from the associated linear stability calculations. Suppose a viscous fluid is contained in a long cylinder that oscillates about its axis. For a relatively large container radius, it might be anticipated that the Stokes layer on the wall is similar to that on a flat plate and thus its stability properties should be related closely to those described above. On the other hand, if the radius of the cylinder is small then the curvature of the streamlines is likely to be important so centripetal instabilities should arise and dominate. In this case, the instability appears as axisymmetric Taylor vortices that are periodic along the axis of the cylinder and the mechanism that excites the mode is similar to that operating in the problem studied by Papageorgiou (1987) concerning the stability of oscillatory flow through a curved pipe. Thus, a key issue for the feasibility of observing shear modes in a torsionally oscillating cylinder is
determining the switch-over point in terms of the radius of the cylinder at which the shear and centripetal modes exchange dominance and what this might imply for the design of practical apparatus.

Let us first consider the shear mode structure. Unlike the previous work on flat plates or in channels or longitudinally oscillating pipes, the wavenumber of shear modes in a torsionally oscillating cylinder is not a continuous variable but takes one of a set of discrete values. Relative to the usual \((r, \theta)\) polar coordinates, periodicity requirements demand that the perturbation mode is proportional to \(\exp(iq\theta)\) for integer \(q\). Then, using a suitably adapted numerical scheme, for each \(q\) it is possible to compute a neutral Reynolds number as a function of the dimensionless cylinder radius \(H\). An amalgamation of the results is shown in figure 4a, which illustrates parts of the curves tracing \(Re_N\) as function of \(H\). For each \(q\), there is an entire \(Re_N\) versus \(H\) curve, say \(Re_N(q, H)\), and what is shown by the solid line in figure 4a is the behaviour of the function \(\min_{k \in \mathbb{Z}} Re_N(2k, H)\). The points where \(Re_N(2k, H) = Re_N(2k+2, H)\) are shown explicitly \(c_{2k,2k+2}\) and we show explicitly \(c_{6,8}\) and \(c_{8,10}\) in figure 4a. These two points bound an interval on the \(H\)-axis where the most unstable even wavenumber is \(q=8\). In a similar way, the dashed curve indicates the critical curve taken over the odd integers \(q\), with the points \(c_{2k+1,2k+3}\) defined by the condition that \(Re_N(2k+1, H) = Re_N(2k+3, H)\). It is to be expected that each \(Re_N\) curve will have a minimum near where \(q/H\) is close to the critical wavenumber for the flat Stokes layer.

Neutral curves for axisymmetric centripetal modes were computed using spectral methods and the critical Reynolds number found as a function of \(H\). The governing equations for centripetal instability can be analysed in the large \(H\) limit and it can be shown (see BB07 for details) that \(Re_c \sim 77.2\sqrt{H}\) when \(H \gg 1\). Figure 4b shows the superposition of the shear mode curves together with the centripetal mode results. As was expected, the critical Reynolds number for

\[ \begin{align*}
\text{Figure 4. (a)} & \quad \text{Even and odd integer critical conditions as functions of } H. \text{ The even integer wavenumber conditions are shown by the solid line and the values of the wavenumber } q \text{ are shown for the first few parabola-like subsections of the critical curve. The dashed line gives the critical conditions for the odd integer wavenumber disturbances. (b) Critical conditions for instability to Stokes layer shear modes and centripetal modes. The calculated critical conditions are shown with the plus signs; the asymptote } Re \sim 77.2\sqrt{H} \text{ is denoted by the chain line. The shear mode critical results are indicated by the solid and dotted lines.}
\end{align*} \]
the centripetal mode is much lower at relatively small values of $H$ but there is a point at which the shear mode is the more dangerous instability. This swap-over occurs when the cylinder radius is approximately 86 Stokes layer thicknesses and it is worth translating this into physical values. Using water as the experimental fluid and a frequency of oscillation of 1 Hz, the requirement that $H > 86$ implies that the radius of the cylinder should exceed roughly 50 mm. However, a more stringent constraint on any physical apparatus is provided by the need that the Reynolds number be sufficiently large to ensure that shear mode instability is possible. If the cylinder is oscillated at 1 Hz through an amplitude of 1 rad, then with water as a working fluid this raises the minimum radius of the cylinder to approximately 40 cm. Although this sized apparatus is rather larger than ideal, the numerical solutions of the instability equations show, just as for the channel calculations, that the stability properties of the cylinder flow are essentially identical to those of an annular flow as long as the latter is more than approximately 16 Stokes layers wide. Thus, there is no need to conduct experiments on oscillating shear layers using a complete cylinder of fluid; if the fluid is confined by an inner solid cylindrical boundary and a concentric outer oscillating sleeve then the effect should still be observable.

4. Discussion

This article has summarized some of our recent studies on the linear stability of oscillatory shear layers. We have obtained the first theoretical evidence of the linear instability of the classical semi-infinite Stokes layer although the predicted critical Reynolds number is considerably larger than the generally accepted range arising from experimental investigations of the instability. Replacing the semi-infinite Stokes layer with an oscillatory flow in a channel or pipe lowers the onset Reynolds number somewhat, but not enough to properly explain the discrepancy between the theoretical predictions and the practical observations. We have suggested that oscillatory shear modes might be more easily studied in a torsionally oscillating cylinder as this configuration circumvents some of the difficulties our work has exposed associated with experiments in long channels or pipes. Of course, this cylindrical geometry introduces new problems, but it would be interesting to consider further the feasibility of using a torsionally oscillating cylinder as a new device for furthering our understanding of the stability properties of oscillatory shear layers.

There are several other directions in which our work could be extended. All the theoretical studies discussed in this paper have been based on a standard linearized stability theory: there remains the potentially complicated task of extending our analysis to incorporate nonlinear effects. This would seem to require an almost entirely numerical method of attack although some weakly nonlinear studies have already been completed. For oscillating flow in a channel, Reid (2006) has shown that the bifurcation is supercritical for wavenumbers in a range around the relevant critical conditions. Thus, equilibrium amplitude disturbances can be expected close to the neutral curve where the growth rate $\mu_0 < 0$, including the regions within the fingers described earlier. Of further interest would be the question of how nonlinear extensions of our Floquet modes might relate to some of the alternative theories mentioned earlier and which were proposed as mechanisms
that might explain the onset of turbulence in oscillatory flows. It is unlikely that these issues will be amenable to theoretical analysis but direct numerical simulations may well yield important information. Such calculations could also throw light on the possible significance of three-dimensional disturbance modes.

All the work described here has been restricted to a unidirectional basic flow. There are several instances where oscillatory shear layers occur in more complicated ways. To take just one example, chemical reactions in the analysis of medical or biological specimens can take a long time, due to the slow diffusion of the large molecules making up the reacting species. In the laboratory, this mixing is often enhanced by using an orbital shaker, which essentially is just a flat plate in orbital motion on which containers of the reacting species are placed. Idealizing a beaker of fluid to a semi-infinite layer, the fluid motion generated by the orbital movement is a generalization of the classical Stokes layer. The stability of the semi-infinite Stokes layer and this two-dimensional generalization can be related via two transformations directly analogous to a standard Squire transformation and similar in spirit to those used in the analysis of the three-dimensional crossflow instability (Gregory et al. 1955). The main result is that the two-dimensional Stokes layer is more stable than its corresponding one-dimensional counterpart (Blennerhassett & Bassom 2007b).

In conclusion, it is clear that despite over 30 years of experimental and theoretical investigations into the stability properties of oscillatory parallel shear flows, much work remains to be carried out before we can claim to have a deep understanding of the problem. A necessary first step is the resolution of the disagreement between theoretical predictions and experimental observations of the critical Reynolds number for instability in a Stokes layer. Our contribution in this direction has been to propose a possibly improved experimental configuration for future observational studies. Further theoretical work could follow the path laid out in the study of disturbances in spatially developing boundary layers, but guidance from experiments will be crucially important. Much of our current understanding of centripetal instabilities in oscillatory flows has come from the work of students and colleagues of Prof. Trevor Stuart; the aim for the future would be to acquire a similar level of knowledge of the stability properties of plane oscillatory flows.

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