On the stability of lumps and wave collapse in water waves

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In the classical water-wave problem, fully localized nonlinear waves of permanent form, commonly referred to as lumps, are possible only if both gravity and surface tension are present. While much attention has been paid to shallow-water lumps, which are generalizations of Korteweg–de Vries solitary waves, the present study is concerned with a distinct class of gravity–capillary lumps recently found on water of finite or infinite depth. In the near linear limit, these lumps resemble locally confined wave packets with envelope and wave crests moving at the same speed, and they can be approximated in terms of a particular steady solution (ground state) of an elliptic equation system of the Benney–Roskes–Davey–Stewartson (BRDS) type, which governs the coupled evolution of the envelope along with the induced mean flow. According to the BRDS equations, however, initial conditions above a certain threshold develop a singularity in finite time, known as wave collapse, due to nonlinear focusing; the ground state, in fact, being exactly at the threshold for collapse suggests that the newly discovered lumps are unstable. In an effort to understand the role of this singularity in the dynamics of lumps, here we consider the fifth-order Kadomtsev–Petviashvili equation, a model for weakly nonlinear gravity–capillary waves on water of finite depth when the Bond number is close to one-third, which also admits lumps of the wave packet type. It is found that an exchange of stability occurs at a certain finite wave steepness, lumps being unstable below but stable above this critical value. As a result, a small-amplitude lump, which is linearly unstable and according to the BRDS equations would be prone to wave collapse, depending on the perturbation, either decays into dispersive waves or evolves into an oscillatory state near a finite-amplitude stable lump.

Keywords: water waves; lumps; wave collapse; fifth-order Kadomtsev–Petviashvili equation

1. Introduction

Unlike their plane-wave counterparts, solitary waves localized in all directions, commonly referred to as lumps, arise in dispersive wave systems under rather special conditions. On physical grounds, for a lump to remain locally confined, its speed must be such that no linear wave can co-propagate with the main wave core. This is feasible when the linear phase speed features a minimum, so lumps propagate below this minimum.

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In the classical water-wave problem, in particular, the phase speed features a minimum only if both gravity and surface tension are important, and two distinct types of lumps arise: firstly, in the high-surface-tension regime (Bond number, $B>1/3$), where the phase-speed minimum occurs at zero wavenumber, lumps bifurcate at the linear-long-wave speed and are generalizations of the familiar solitary waves of the Korteweg–de Vries (KdV) equation in shallow water. Secondly, for $B<1/3$, on water of finite or infinite depth, where the phase-speed minimum is realized at a non-zero wavenumber, lumps bifurcate from linear sinusoidal waves of finite wavelength at this minimum and, in the near linear limit, behave like localized wave packets with envelope and crests moving at the same speed.

Although shallow-water lumps have been studied extensively, lumps of the wave packet type have been discovered only recently. Using an asymptotic approach, Kim & Akylas (2005) constructed small-amplitude wave packet lumps slightly below the minimum gravity–capillary phase speed. In this limit, lumps can be approximated in terms of a particular steady solution (ground state) of an elliptic equation system that governs the coupled evolution of the envelope along with the induced mean flow. We shall refer to this system as the Benney–Roskes–Davey–Stewartson (BRDS) equations, as the coupling of the envelope to the mean flow for a gravity wave packet with two-dimensional modulations was first studied by Benney & Roskes (1969), followed by Davey & Stewartson (1974); the effects of surface tension were included later (Djordjevic & Redekopp 1977). The asymptotic analysis of Kim & Akylas (2005) suggests that two branches of symmetric lumps, one of elevation and the other of depression, bifurcate at the minimum phase speed; they are fully localized counterparts of the two symmetric plane-solitary-wave solutions that bifurcate there as well (see Dias & Kharif 1999).

In an independent numerical study, Parau et al. (2005) computed symmetric elevation and depression gravity–capillary lumps on deep water with speed less than the minimum phase speed. In the small-amplitude limit, consistent with the asymptotic analysis, these lumps behave like wave packets and, as the steepness increases, they transform into fully localized disturbances whose centreline profiles resemble steep plane solitary waves on deep water computed in earlier studies (Longuet-Higgins 1989; Vanden-Broeck & Dias 1992). More recently, the same authors also presented computations of similar gravity–capillary lumps on water of finite depth (Parau et al. 2007). Finally, a rigorous existence proof of gravity–capillary lumps was given in Groves & Sun (2005).

The studies cited above deal exclusively with steady wave profiles, and little is known about the stability properties and dynamics of lumps that bifurcate at the minimum gravity–capillary phase speed for $B<1/3$. In an attempt to fill this gap, Kim & Akylas (2007) examined the stability of plane solitary waves of depression to long-wavelength transverse perturbations. Depression solitary waves, while stable to longitudinal perturbations (Calvo & Akylas 2002), turn out to be transversely unstable, suggesting that, similarly to shallow-water lumps, lumps of the wave packet type may arise from the transverse instability of plane solitary waves. This scenario was confirmed in Kim & Akylas (2006) using a model equation for weakly nonlinear long interfacial waves in a two-layer configuration. In this instance, according to unsteady numerical simulations, transverse instability leads to the formation of lumps, which propagate stably at a different speed, thus separating from the rest of the disturbance.
However, there is also reason to suspect that, close to their bifurcation point at least, gravity–capillary lumps of the wave packet type are unstable, since in the small-amplitude limit, they are related to the ground state of the BRDS equations. This equation system predicts the formation of a singularity at finite time, known as wave collapse, due to nonlinear focusing when the initial condition exceeds a certain threshold (Ablowitz & Segur 1979; Papanicolaou et al. 1994). Given that the ground state is on the borderline between existence for all time and collapse, one would expect small-amplitude lumps to be prone to nonlinear focusing; the numerical simulations of Kim & Akylas (2006) though did not reveal any sign of a related instability.

As a first step towards a comprehensive study of the stability properties of gravity–capillary lumps of the wave packet type, rather than tackling the full water-wave equations, we shall discuss a simpler model problem. Specifically, we consider the fifth-order Kadomtsev–Petviashvili (KP) equation, an extension to two spatial dimensions of the fifth-order KdV equation, which describes weakly nonlinear long water waves for Bond number close to one-third. While ignoring viscous effects cannot be justified under these flow conditions, the fifth-order KP equation admits lumps of the wave packet type analogous to those found in the full water-wave problem for \( B < 1/3 \), and provides a useful model for theoretical purposes.

A linear stability analysis of depression lumps (the primary solution branch) of the fifth-order KP equation confirms that they are unstable in the small-amplitude limit. An exchange of stability takes place at finite wave steepness though, above which lumps become stable. Moreover, based on unsteady numerical simulations, a small-amplitude wave packet, under conditions for which it would have experienced wave collapse according to the BRDS equations, in fact evolves into an oscillatory state near a finite-amplitude lump that is stable. This clarifies the meaning of the wave collapse predicted by the approximate envelope equations and suggests an explanation for the emergence of stably propagating finite-amplitude lumps in the simulations of Kim & Akylas (2006).

Even though the present study deals with a model equation only, it is probable that locally confined gravity–capillary wave packets in the full water-wave problem also would have a tendency to form steep lumps as a result of nonlinear focusing. This would seem consistent with laboratory observations of wind-driven gravity–capillary waves having profiles similar to those of steep depression lumps (Zhang 1995).

2. Preliminaries

As remarked earlier, in order for lumps to be possible in the water-wave problem, it is necessary that the linear phase speed features a minimum, and this requires both gravity and surface tension to be present. Using the water depth \( h \) as length scale and \( (h/g)^{1/2} \) as time scale, \( g \) being the gravitational acceleration, the character of gravity–capillary lumps hinges on the value of the Bond number \( B = \sigma / (\rho gh^2) \), where \( \sigma \) denotes the coefficient of surface tension and \( \rho \) the fluid density. This is clear from the dispersion relation, \( \omega^2 = k(1 + Bk^2) \tanh k \), \( k \) being the wavenumber and \( \omega \) the wave frequency, according to which the minimum of the phase speed \( \omega/k \) is realized at \( k = 0 \) when \( B > 1/3 \), whereas for
$B<1/3$, it occurs at a non-zero wavenumber, $k=k_{m}$. Accordingly, in the former case, lumps are two-dimensional generalizations of KdV solitary waves that also bifurcate at $k=0$, while, in the latter case, lumps are fully localized counterparts of the plane solitary wave packets which bifurcate at $k=k_{m}$ as well.

As our interest centres on lumps of the wave packet type, we shall take $B<1/3$, and, for convenience, it will be further assumed that $B$ is close to one-third so that $k_{m}/1.0$. The neighbourhood of the phase-speed minimum may then be captured via a long-wave approximation ($k/1$) to the dispersion relation,

$$
\omega = k \left\{ 1 + \frac{1}{2} \left( B - \frac{1}{3} \right) k^{2} + \frac{1}{90} k^{4} + \cdots \right\}.
$$

(2.1)

Combined with a quadratic nonlinear term of the KdV type, (2.1) leads to the fifth-order KdV equation, which has served as the starting point in several prior theoretical investigations of plane solitary waves of the wave packet type (see Grimshaw 2007 for a review). In the same vein, we shall make use of the fifth-order KP equation, an extension to two spatial dimensions of the fifth-order KdV equation, in order to discuss lumps of the wave packet type. To this end, assuming nearly unidirectional wave propagation along $x$, the leading-order effects of transverse ($z$) variations are taken into account in (2.1) by writing $k=(l^{2}+m^{2})^{1/2} \approx l+(1/2)m^{2}/l$, $l$ being the longitudinal and $m<<l$ the transverse wavenumber components.

The fifth-order KP equation then follows from this weakly two-dimensional approximation to the linear dispersion relation, combined with a KdV-type quadratic nonlinear term. A systematic derivation was presented in Paumond (2005). Here, we shall work with the fifth-order KP equation (for $B<1/3$) in the normalized form

$$
\{ \eta_{t} + 3(\eta^{2})_{x} + 2\eta_{xxx} + \eta_{xxxxx} \} + \eta_{zz} = 0,
$$

(2.2)

where $\eta(x, z, t)$ stands for a variable associated with the wave disturbance, such as the free-surface elevation, in a frame moving with the linear-long-wave speed. The linear dispersion relation of the model equation (2.2) for unidirectional propagation along $x$,

$$
\omega = -2k^{3} + k^{5},
$$

(2.3)

is of the same form as (2.1) when allowance is made for the change in reference frame.

In preparation for discussing the dynamics of lumps, we first consider small-amplitude modulated wave packets of (2.2) and obtain the corresponding evolution equations. This derivation follows along the lines of the classical weakly nonlinear stability theory (Stuart 1960). Briefly, introducing the amplitude parameter $0<\epsilon<1$, the appropriate expansion takes the form

$$
\eta = \frac{1}{2} \epsilon \{ A(X, Z, T)e^{i\theta} + c.c. \} + \epsilon^{2} A_{0}(X, Z, T)
$$

$$
+ \epsilon^{2} \{ A_{2}(X, Z, T)e^{2i\theta} + c.c. \} + \cdots,
$$

(2.4)

where $\theta=kx-\omega t$, $k$ and $\omega$ being the carrier wavenumber and frequency, respectively, which satisfy the dispersion relation (2.3), and $(X, Z, T)=\epsilon(x, z, t)$ are the ‘slow’ envelope variables.

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Upon substituting (2.4) into (2.2) and collecting zeroth-harmonic terms, the induced mean flow $A_0$ satisfies, to leading order,

$$A_{0XT} + A_{0ZZ} = -\frac{3}{2} (|A|^2)_{XX}. \quad (2.5)$$

Similarly, collecting second-harmonic and primary-harmonic terms, the envelope $A(X, Z, T)$ of the primary harmonic satisfies

$$i(A_T + c_g A_X) + \epsilon \left( \frac{1}{2} c_g' A_{XX} + \frac{1}{k} A_{ZZ} \right) + \frac{3}{2} \epsilon \frac{A^2 A^*}{k(5k^2 - 2)} - 6\epsilon k A A_0 = 0, \quad (2.6)$$

where $c_g = d\omega/dk$, $c_g' = d^2\omega/dk^2$ and $*$ denotes complex conjugate.

As expected, to leading order, the wave packet propagates with the group velocity $c_g$. Adopting a reference frame moving with $c_g$, $X' = X - c_g T$, and defining $T' = \epsilon T$, it follows from (2.5) and (2.6) (after dropping the primes) that $A$ and $A_0$ satisfy

$$-c_g A_{0XX} + A_{0ZZ} = -\frac{3}{2} (|A|^2)_{XX}, \quad (2.7a)$$

$$iA_T + \frac{1}{2} c_g' A_{XX} + \frac{1}{k} A_{ZZ} + \frac{3}{2} \frac{A^2 A^*}{k(5k^2 - 2)} - 6\epsilon k A A_0 = 0. \quad (2.7b)$$

The coupled-equation system (2.7a) and (2.7b) is of the same form as the BRDS equations that govern the evolution of the wave envelope and the induced mean flow of a modulated wave packet in the water-wave problem.

As explained by Ablowitz & Segur (1979), the signs of the various coefficients in these equations are critical in determining the character of the solution. Here, of particular interest is the case of both (2.7a) and (2.7b) being elliptic; this requires $c_g < 0$ and $c_g' > 0$, which is realized if $\sqrt{3/5} < k < \sqrt{6/5}$. For $k$ in this range, moreover, the coefficient of the cubic term in (2.7b) is positive, implying that the self-interaction of the envelope has a focusing effect, and the sign of the second nonlinear term in (2.7b) is such that the interaction of the envelope with the mean flow enhances this nonlinear self-focusing. As a result, when $\sqrt{3/5} < k < \sqrt{6/5}$, the BRDS equations (2.7a) and (2.7b) predict the formation of a singularity in finite time, or the so-called wave collapse, of localized initial conditions above a certain threshold amplitude (Ablowitz & Segur 1979; Papanicolaou et al. 1994). In the water-wave problem, this situation arises for gravity–capillary wave packets in sufficiently deep water, suggesting that a small-amplitude wave pulse under these flow conditions would evolve to a peaked nonlinear disturbance.

It is well known (see Papanicolaou et al. 1994; Ablowitz et al. 2005) that the threshold for wave collapse is provided by a special locally confined solution of (2.7a) and (2.7b), the so-called ground state, $A = A(X, Z) \exp(iT)$. Physically, this solution, when combined with the carrier signal, corresponds to a fully localized wave packet with the envelope of permanent form moving with the group speed $c_g$ and crests travelling at the linear phase speed, $\omega/k$, slightly modified by nonlinear effects. In general, as these two speeds are different, the disturbance, as a whole, does not represent a lump. However, at the specific wavenumber $k = k_m = 1$ and

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frequency $\omega = \omega_m = -1$ corresponding to the minimum of the linear phase speed, the group speed $c_{g|m}$ is equal to $\omega_m/k_m = -1$; so the minimum phase speed becomes the bifurcation point of lumps of the wave packet type. It is important to note that $k_m = 1$ is within the range $\sqrt{3/5} < k < \sqrt{6/5}$ where the BRDS equations (2.7a) and (2.7b) are of the elliptic–elliptic focusing type.

Returning to (2.4), near their bifurcation points, these lumps can be approximated as localized wave packets with envelope and crests moving at the same speed, $\ddot{\eta} = \epsilon \dot{A}(x, t) \cos(x - ct) + \cdots$; where $\xi = \epsilon(x - ct)$, $c = -1 - \epsilon^2$, $\dot{A}$ being the ground-state solution of the BRDS equations for $k = k_m = -1$; from (2.7a) and (2.7b), it follows that $A$ satisfies the coupled system

$$\ddot{A}_{0\xi} + \ddot{A}_{0ZZ} = -\frac{3}{2}(|\dot{A}|^2)_{\xi\xi}, \quad (2.8a)$$

$$-\ddot{A} + 4\dddot{A}_{\xi\xi} + \dddot{A}_{ZZ} + \frac{1}{2}\dddot{A}^3 - 6\dot{A}_0\dddot{A} = 0. \quad (2.8b)$$

Note that two symmetric solution branches bifurcate at $\epsilon = 0$, one corresponding to elevation and the other to depression lumps, depending on whether the maximum of $A$ at the origin $\xi = Z = 0$ coincides with a crest or a trough, respectively. Small-amplitude lumps near the minimum phase speed of the full gravity–capillary problem for $B < 1/3$ were constructed in a similar way by Kim & Akylas (2005). In the special case where transverse variations are absent ($\partial/\partial Z = 0$), (2.8a) and (2.8b) reduce to the steady nonlinear Schrödinger (NLS) equation that admits the well-known envelope soliton solution with a ‘sech’ profile; one thus recovers the elevation and depression plane solitary waves that also bifurcate at $c = -1$.

To trace lump-solution branches, $\bar{\eta}(\chi, z; c)$ where $\chi = x - ct$, in the finite-amplitude range away from their bifurcation point, we solve directly the steady version of (2.2),

$$(-c\dddot{\eta} + 3\dddot{\eta}^2 + 2\dddot{\eta}_{xx} + \dddot{\eta}_{xxxx})_{xx} + \dddot{\eta}_{zz} = 0, \quad (2.9)$$

via a Fourier iteration method, analogous to the one used in Musslimani & Yang (2004), using $1024 \times 128$ modes along the horizontal and transverse directions, in a computational domain $120\pi \times 60\pi$. Figure 1 shows the peak amplitude $\eta_0 = \bar{\eta}(\chi = 0, z = 0; c)$ of depression lumps (also known as the primary solution branch) as $c$ is decreased below $-1$. Also, for comparison, we have plotted the peak amplitude of plane solitary waves of depression. Although, for the same speed, lumps have roughly twice the peak amplitude of solitary waves, the two solution branches behave in a similar manner qualitatively as $c$ is varied; also lump profiles along the centreline $z=0$ are similar to those of plane solitary waves. The computations of Parau et al. (2005, 2007) revealed an analogous behaviour for gravity–capillary lumps and solitary waves on water of finite or infinite depth.

On the other hand, considering that the ground state of the BRDS equations is prone to collapse while NLS envelope solitons are stable, one would expect the dynamics of lumps to be quite different from that of plane solitary waves. This issue is taken up below by first examining the stability of lumps to infinitesimal perturbations.
3. Exchange of stability

We shall focus on the primary (depression) lump-solution branch, \( \eta(\chi, z; c) \), parametrized by the wave speed \( c \), of the fifth-order KP equation (2.2). Upon superposing a perturbation \( \eta'(\chi, z, t) \) and linearizing about the basic state, \( \eta' \) satisfies

\[
\{ \eta'_t - c \eta'_\chi + 6(\eta \eta'_t)_\chi + 2 \eta_{\chi \chi \chi} + \eta_{\chi \chi \chi \chi \chi} \} + \eta_{zz} = 0. \tag{3.1}
\]

Before considering the associated eigenvalue stability problem, in a preliminary stability analysis, we integrated the linearized perturbation equation (3.1) numerically by a spectral method that parallels the one used in Kim & Akylas (2006), for various values of \( c \) below the bifurcation point \( (c = -1) \), taking as initial condition \( \eta'(\chi, z, t = 0) = \hat{\eta}(\chi, z; c) \). The spatial resolution was the same as that used earlier for the steady equations (2.8a) and (2.8b), and the time step \( \Delta t = 5 \times 10^{-3} \). To monitor the evolution of the disturbance, we compute

\[
E'(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta'^2 \, d\chi \, dz. \tag{3.2}
\]

We first considered values of \( c \) slightly below \(-1\), for instance \( c = -1.02 \), for which lump profiles resemble small-amplitude modulated wave packets. Following an initial transient period, \( E'(t) \) eventually grows linearly with time, indicating exponential growth of \( E'(t) \), and this trend also was confirmed by monitoring the growth of the peak amplitude of the disturbance with time. On the other hand, for values of \( c \) farther from the bifurcation point, for example \( c = -1.64 \), \( E'(t) \) exhibits quite different behaviour: as suggested by a log–log plot of \( E'(t) \) against \( t \), eventually, \( E'(t) \propto t^2 \), so the disturbance now grows linearly with time. This is not a sign of instability, however, as it is easy to show that the perturbation equation (3.1) admits the linearly growing disturbance \( \eta' = t \tilde{\eta}_\chi - \partial \tilde{\eta} / \partial c \) as a solution, which may be interpreted as a shift in the speed parameter of the base state \( \tilde{\eta}(\chi, z; c) \). Consistent with this reasoning, the numerical results confirm that \( \eta' \propto t \tilde{\eta}_\chi \) at large time.

Figure 1. Peak amplitudes of lumps (solid line) and plane solitary waves (dashed line) of depression as the wave speed \( c \) is decreased below the bifurcation point \( c = -1 \).

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The overall picture suggested by these computations is that $h_0$ grows exponentially, implying instability, when $c < K_{1.053}$. Figure 2 shows the computed growth rate $\lambda$ for $c$ in this range; as $c$ is decreased below the bifurcation point, $\lambda$ first increases until $c \approx -1.03$ where a maximum is reached, and then it decreases sharply. On the other hand, for $c > K_{1.09}$, there is no evidence of instability as $h_0$ eventually grows linearly with time, $h_0 \propto t$. Finally, for $c$ in the range $K_{1.09} > c > K_{1.053}$, our numerical results are not entirely conclusive, and we suspect that the computation has to be carried out to very long times ($t \gg 200$) in order to see a definite trend in this transitional regime between unstable and stable behaviours.

Turning to a formal stability analysis, upon substituting $\eta' = \hat{\eta}(\chi, z) \exp(\lambda t)$ in (3.1), $\hat{\eta}$ is governed by

$$
(\lambda \hat{\eta} - c \hat{\eta}_x + 6(\eta \hat{\eta})_x + 2\hat{\eta}_{xxx} + \hat{\eta}_{xxxx})_x + \hat{\eta}_{zz} = 0, \quad (3.3a)
$$

$$
\hat{\eta} \rightarrow 0 \quad (\chi^2 + z^2 \rightarrow \infty), \quad (3.3b)
$$

assuming that the perturbations are locally confined. This defines an eigenvalue problem, the eigenvalues occurring in quartets ($\lambda$, $-\lambda$, $\lambda^*$, $-\lambda^*$); instability thus is associated with eigenvalues that are not purely imaginary.

In the light of the numerical results presented above, one would expect a pair of real eigenvalues when $-1.053 < c < -1$, as disturbances were found to grow exponentially for $c$ in this range. Combined with the fact that no instability was detected when $c < -1.09$, this suggests that, for a certain $c$ between $-1.053$ and $-1.09$, an exchange of stability takes place, the pair of real eigenvalues colliding at the origin $\lambda = 0$ as this critical value of $c$ is approached from above.

To further explore this hypothesis, a bifurcation analysis near the possible onset of instability ($|\lambda| \ll 1$) was carried out, adapting to the case of lumps a procedure suggested in Pelinovsky & Grimshaw (1997) for the stability of plane solitary waves. Specifically, introducing the expansion

$$
\hat{\eta} = \hat{\eta}^{(0)} + \lambda \hat{\eta}^{(1)} + \lambda^2 \hat{\eta}^{(2)} + \cdots, \quad (3.4)
$$
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\[ \dot{\eta}^{(0)} \] is posed as a linear combination of \( \eta_\chi \) and \( \eta_z \), \( \dot{\eta}^{(0)} = \eta_\chi + C\eta_z \), which are solutions of (3.3a) and (3.3b) for \( \lambda = 0 \) by virtue of the invariance of \( \eta \) to translations in \( \chi \) and \( z \).

At \( O(\lambda) \), \( \dot{\eta}^{(1)} \) is governed by the forced equation \( L\dot{\eta}^{(1)} = -\eta_{\chi\chi} - C\eta_{\chi z} \), \( L \) denoting the linear operator in (3.3a) for \( \lambda = 0 \), and is subject to condition (3.3b) at infinity. The solution is expressed as

\[ \dot{\eta}^{(1)} = -\frac{\partial \eta}{\partial \chi} - C\frac{\partial \eta}{\partial c_2} \bigg|_0, \]  

(3.5)

where \( \eta(\chi, z; c, c_2) \) satisfies

\[ \{-c\eta_\chi - c_2\eta_z + 3(\eta^2)_\chi + 2\eta_{\chi\chi} + \eta_{\chi\chi\chi} \}_\chi + \eta_{zz} = 0, \]  

(3.6)

and \( \partial \eta / \partial c_2 \bigg|_0 \) denotes \( \partial \eta / \partial c_2 \) evaluated for \( c_2 = 0 \). (As explained in Kim & Akylas (2006), physically \( \eta \) corresponds to an oblique lump solution and can be mapped to a lump propagating along the \( x \)-direction via rotation of axes.)

Proceeding to \( O(\lambda^2) \), it follows from (3.3a), making use of (3.5), that \( \dot{\eta}^{(2)} \) satisfies

\[ L\dot{\eta}^{(2)} = \left( \frac{\partial \eta}{\partial \chi} + C\frac{\partial \eta}{\partial c_2} \bigg|_0 \right) \chi, \]  

(3.7)

subject to condition (3.3b). Appealing to the usual solvability argument, for this inhomogeneous problem to have a solution, it is necessary that the r.h.s. in (3.7) be orthogonal to the well-behaved homogeneous solution of the corresponding adjoint problem. Here, the operator adjoint to \( L \) is

\[ L^A = -c\ddot{\eta}_\chi + 6\eta\ddot{\eta}_\chi + 2\dot{\eta}_\chi^4 + \dot{\eta}_\chi^6 + \dot{\eta}_z^2, \]  

(3.8)

and a well-behaved solution of \( L^A\eta = 0 \) is readily shown to be

\[ \eta^A = \int_{-\infty}^{\chi} \eta \, d\chi'. \]  

(3.9)

(Note that \( \eta^A \to 0 \) as \( \chi \to \infty \), as required by the well-known constraint of the KP equation; see Akylas (1994).) Imposing this solvability condition on the r.h.s. of (3.7), making use of the fact that \( \partial \eta / \partial c_2 \bigg|_0 \) is odd in \( \chi \) and \( z \), it follows that

\[ \frac{d\tilde{E}}{dc} = 0, \]  

(3.10)

where \( \tilde{E}(c) = \int_{-\infty}^{\chi} \int_{-\infty}^{\infty} \eta^2 \, d\chi \, dz \). The constant \( C \) remains undetermined at this stage.

The above heuristic analysis, assuming that \( \dot{\eta}^{(0)} \) is a linear combination of \( \dot{\eta}_\chi \) and \( \dot{\eta}_z \), suggests (3.10) as a necessary condition at the critical value of \( c \) where exchange of stability takes place. Figure 3 shows \( \tilde{E}(c) \) for the depression-lump solution branch computed earlier, along with a plot of \( \tilde{E}(c) \) (per unit \( z \)) for plane solitary waves of depression. It is interesting that \( \tilde{E} \) corresponding to depression lumps features a minimum, so (3.10) is met, at \( c = -1.0576 \), consistent with our numerical results (figure 2) that indicate that exchange of stability occurs for a value of \( c \) in the range \( -1.09 < c < -1.053 \). On the other hand, as shown in figure 3, \( \tilde{E} \) grows monotonically for plane solitary waves of depression, as they are stable to longitudinal perturbations (Calvo et al. 2000).

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4. Transient evolution

The linear stability analysis presented above indicates that, close to their bifurcation, lumps are unstable to infinitesimal perturbations. Also when viewed as wave packets, these small-amplitude lumps are on the verge of wave collapse according to the BRDS equations. Here we follow the transient evolution of linear instability in the nonlinear regime, in an attempt to understand the meaning of this singularity.

For this purpose, we solve the fifth-order KP equation (2.2) numerically as an initial-value problem, employing a spectral technique analogous to the one in Kim & Akylas (2006). The spatial resolution used is the same as before and $\Delta t = 1.0 \times 10^{-3}$. It can be readily shown that, for locally confined disturbances,

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta^2(x, z, t) \, dx \, dz$$

is conserved, and this property was also used to check the accuracy of the computations.

The first set of initial conditions considered is of the form

$$\eta = (1 + \delta)\tilde{\eta}(x, z; c) \quad (t = 0), \quad (4.2)$$

where $\tilde{\eta}$ denotes the depression-lump solution branch computed earlier and $\delta$ is a parameter that controls the amplitude and sign of the perturbation imposed. For $c$ in the linearly stable range found in §3, for instance $c = -1.64$, we confirmed that the lumps propagate stably under perturbations of either sign.

However, close to the bifurcation point $c = -1$, where lumps are linearly unstable and resemble small-amplitude wave packets, the BRDS equations predict that the initial condition (4.2) would eventually either collapse (for $\delta > 0$)
or disperse out (for \( \delta < 0 \)). Figure 4a–d shows results from computations using the initial condition (4.2) for \( c = -1.0225 \), which is in the linearly unstable range (figure 2). For a positive perturbation (\( \delta = 0.01 \)), the peak amplitude of the disturbance initially grows, as expected from the results of the linear stability analysis; but rather than experiencing wave collapse as the BRDS equations would imply, the disturbance evolves into a finite-amplitude oscillatory state (figure 4a). Representative wave profiles, corresponding to the initial condition and to the times at which the (magnitude of the) peak amplitude reaches its first maximum and minimum, are displayed in figure 4b–d. The disturbance profile at the maximum of the peak amplitude (figure 4c) is close to the stable finite-amplitude lump that corresponds to \( c \approx -1.17 \) in figure 1.

On the other hand, for the same initial condition as above, but with negative perturbation (\( \delta = -0.05 \)), the peak amplitude decreases in magnitude monotonically (figure 4a), and the disturbance decays into dispersive waves. This is consistent with the BRDS equations, which predict that initial conditions below the ground state (\( \delta < 0 \)) eventually disperse out.

We next consider as initial condition a wave packet with envelope given by the ground state of the BRDS equations, but with carrier wavenumber other than \( k = k_m = 1 \) corresponding to the minimum phase speed, so the initial disturbance as a whole is far from a lump. Specifically, we chose \( k = 1.05 \), which is still in the range \( 3^{1/5} < k < 6^{1/5} \) where the BRDS equations (2.6) are of the elliptic–elliptic focusing type, and \( \tilde{\eta} \) in (4.2) was replaced by the first three harmonics in expansion (2.3), using the corresponding ground state \( A(X, Z) \) as the envelope \( A \) and \( \epsilon = 0.15 \).

The results of the evolution of this initial condition are summarized in figure 4e–h. For a positive perturbation (\( \delta = 0.05 \)), after an initial transient period, the peak amplitude of the disturbance again exhibits an oscillatory behaviour (figure 4e). Moreover, as illustrated in figure 4f–h, rather than the wave collapse predicted by the BRDS equations when \( \delta > 0 \), the disturbance oscillates essentially between two depression lumps, one of relatively small amplitude (\( \eta_0 \approx -0.24 \)) that is unstable and the other of higher amplitude (\( \eta_0 \approx -0.41 \)) that is stable. Here, the results are displayed in a frame moving with the group velocity \( c_g = -0.5375 \) of the initial packet; thus the main disturbance is seen to propagate to the left (figure 4f–h), as lumps propagate with speed less than \(-1\). By contrast, the same initial wave packet but with a negative perturbation (\( \delta = -0.1 \)) results in a decaying disturbance (figure 4e) that eventually disperses out, as also predicted by the BRDS equations for \( \delta < 0 \).

These results have brought out the prominent role of finite-amplitude lumps in the nonlinear focusing of localized wave packets. Extension of these ideas to the full water-wave problem, including the effects of damping and forcing, are under current investigation.

5. Concluding remarks

The last 50 years have been a period of significant advances in our understanding of nonlinear phenomena in fluid mechanics. The problem of water waves has been central to this remarkable progress, providing the original motivation for many key ideas that later proved useful in various other contexts in fluid mechanics as
Figure 4. (Caption opposite.)
well as different fields. In particular, the phenomenon of wave collapse considered here, although originally studied in the context of gravity–capillary wave packets, turns out to be important also in plasma physics and nonlinear optics.

This progress has been achieved by combining theoretical efforts, in which Prof. J. T. Stuart played a leading role, with detailed computational studies and carefully designed experiments. We envisage this synergy continuing, and the study of nonlinear phenomena will probably remain in the forefront of fluid mechanics research in the twenty-first century, inspired by environmental, biological and energy-related applications.

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