Aspects of linear and nonlinear instabilities leading to transition in pipe and channel flows

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The failure of normal-mode linear stability analysis to predict a transition Reynolds number \( Re_{tr} \) in pipe flow and subcritical transition in plane Poiseuille flow (PPF) has led to the search of other scenarios to explain transition to turbulence in both flows. In this work, various results associated with linear and nonlinear mechanisms of both flows are presented. The results that combine analytical and experimental approaches indicate the strong link between the mechanisms governing the transition of both flows. It is demonstrated that the linear transient growth mechanism is based on the existence of a pair of least stable nearly parallel modes (having opposite phases and almost identical amplitude distributions). The analysis that has been applied previously to pipe flow is extended here to a fully developed channel flow predicting the shape of the optimized initial disturbance (a pair of counter-rotating vortices, CVP), time for maximum energy amplification and the dependence of the latter on \( Re \). The results agree with previous predictions based on many modes. Furthermore, the shape of the optimized initial disturbance is similar in both flows and has been visualized experimentally. The analysis reveals that in pipe flow, the transient growth is a consequence of two opposite running modes decaying with an equal decay rate whereas in PPF it is due to two stationary modes decaying with different decay rates. In the first nonlinear scenario, the breakdown of the CVPs (produced by the linear transient growth mechanism) into hairpin vortices is followed experimentally. The associated scaling laws, relating the minimal disturbance amplitude required for the initiation of hairpins and the \( Re \), are found experimentally for both PPF and pipe flow. The scaling law associated with PPF agrees well with the previous predictions of Chapman, whereas the scaling of the pipe flow is the same as the one previously obtained by Hof et al., indicating transition to a turbulent state. In the second nonlinear scenario, the base flow of pipe when it is mildly deviated from the Poiseuille profile by an axisymmetric distortion is examined. The nonlinear features reveal a \( Re_{tr} \) of approximately 2000 associated with the bifurcation between two deviation solutions.

Keywords: transient growth; streaks; hairpin vortices; optimal deviations

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1. Introduction

There has been a renewed interest in demystifying the process of transition in pipe, initiated more than a century ago by the seminal work of Reynolds (1883; e.g. Eckhardt et al. 2007). Albeit not everything is known about the precise mechanism of transition, many facts are well established. There are two parameters that govern the transition process in pipe: first, the Reynolds number (defined as $Re = 2WR/\nu$, based on the average (bulk) axial velocity $\dot{W}$, the pipe radius $R$ and the kinematic viscosity $\nu$), and second, the amplitude of the disturbance ($A$) present in the flow (at the inlet or at initial time, $t=0$). Most of the studies in transition can thus be placed under a $Re-A$ plane. The two extreme cases where results are well established (due to its mathematical ease) are when $A$ is infinitesimally small and $Re$ is increased and when $Re$ is small and $A$ is increased.

The scenario that is most studied is the one in which the disturbance amplitude is infinitesimally small and Reynolds number is increased indefinitely. Even Reynolds (1883) in his experiments saw that, when the pipe is smooth and inlet conditions are carefully controlled (i.e. the disturbance amplitudes are very small), transition could be delayed to very high $Re$ of 12 000–14 000. This has been confirmed by many later experiments including the one by Pfeniger (1961), where laminar flow has been sustained for Reynolds numbers up to $10^5$. According to linear stability analysis, a developed pipe flow is known to be stable to any kind of infinitesimal normal-mode disturbances for Reynolds numbers up to $10^6$–$10^7$ for a wide range of axial and azimuthal wavenumbers (e.g. Meseguer & Trefethen 2003). This shows that flow in pipe does not undergo transition even for extremely high $Re$ when $A$ is infinitesimally small.

Joseph & Carmi (1969) showed that any amount of initial energy of the disturbance (or any initial value of $A$) will decay monotonically for $Re < Re_\xi = 81.5$. Thus, turbulent flow cannot persist below $Re_\xi$. This corresponds to the second case in which $Re$ is small ($< Re_\xi$) and $A$ is finite.

The complexity of pipe flow comes into play when the amplitude of the disturbance is finite and $Re > Re_\xi$. In many experiments performed during the past decades, transition to turbulence has been observed at Reynolds numbers $1800 < Re < Re_{tr}$, where the upper bound value $Re_{tr}$ depends on the magnitude of the disturbance’s amplitude, naturally initiated or artificially introduced into the flow (e.g. Wygnanski & Champagne 1973; Darbyshire & Mullin 1995; Han et al. 2000). The commonly used critical Reynolds number for transition, $Re_{tr} \sim 2000$, comes from the friction factor versus $Re$ plot, where a sudden change in friction factor (corresponding to initiation of turbulence) is observed at around $Re_{tr}$ (e.g. McKeon et al. 2004). Consequently, the investigations to understand the transition scenario in pipe have rightly concentrated most of their efforts for moderate values of $Re$ (centred at approx. 2000 and ranging from a few hundreds to a few thousands).

The failure of the normal-mode linear stability analysis to predict a critical Reynolds number ($Re_{\nu}$) in the pipe flow has led to the search of other linear/nonlinear scenarios to explain energy amplification. The linear transient growth scenario is the one in which $A$ increases transiently and may reach a significant amplitude that can trigger nonlinear mechanisms before its long time decay due to viscous effects. Recourse to nonlinear studies is no less in the
literature. For example, numerical results obtained by Orszag & Patera (1983) showed the existence of non-axisymmetric secondary instabilities, where the primary state consisted of axisymmetric finite-amplitude waves.

Recently, a new parameter, apart from the already existing parameters of $Re$ and $\mathcal{A}$, has been found relevant in the pipe flow analysis. This parameter is the length of the pipe in which the flow remains turbulent ($L$, or the equivalent time) and beyond which the flow is laminar. Even though it was shown using numerical methods previously by Brosa (1989) that pipe flow turbulence consists of transients and it exhibits a chaotic nature (Faisst & Eckhardt 2004), it was only later that experimental evidence for such a scenario present in pipe flow has been found by Hof et al. (2006) and Peixinho & Mullin (2006). Their results suggest that the initiation of turbulence is probabilistic and that turbulence in pipe is not a permanent phenomenon (at least up to $Re \approx 2200$), instead it decays and has a finite lifetime (or distance $L$).

The general complications associated with the pipe flow also exist in plane Poiseuille flow (PPF). Both flows undergo subcritical transition (i.e. $Re_{tr} < Re_c$). Here, the $Re$ for PPF is defined using centreline velocity $U_{cl}$ and half channel height $h$. In PPF $Re_c = 5772$ (Orszag 1971) and $Re_{tr} \sim 1000$ (e.g. Davies & White 1928), whereas in pipe flow as mentioned above $Re_c \rightarrow \infty$ and $Re_{tr} \sim 2000$. A detailed study of this special flow has driven the field of hydrodynamic stability along with the stability of pipe flow from the onset (e.g. Rayleigh 1892).

In this work, we present various results associated with linear and nonlinear mechanisms of both plane and pipe Poiseuille flows. The results that combine analytical and experimental approaches are shown (in most cases) side by side, indicating the strong link between the mechanisms governing the transition of both flows, and in general, for the transition scenarios present in wall-bounded shear flows. In the following, we first describe briefly the experimental set-ups and then discuss linear and nonlinear transition mechanisms.
2. Experimental set-ups

This section describes experimental facilities of pipe flow and PPF, wherein water and air, respectively, are employed as working media for studying various aspects of linear and nonlinear instabilities. Techniques of hot-wire anemometry and flow visualization with dye and smoke are used to study the evolution of coherent structures. All scales are non-dimensionalized using centreline velocity \( U_{cl} \) and depending on the flow, by pipe radius \( R=0.98 \text{ cm} \), or half channel height \( h=2.5 \text{ cm} \).

\( \text{(a) Pipe flow facility} \)

The pipe flow facility that stands vertically consists of two flow circuits, one for base flow and another one for disturbance. Figure 1 describes the various components associated with both circuits. For the base flow, water is stored in an overhead tank of 30 l capacity. The flow passes through a honeycomb and then through a smooth converging nozzle before entering the circular pipe having an inner diameter \( d \) of 1.96 cm. Water exits the pipe at the bottom into a damper (to damp the fluctuations in the flow caused by the motor), and then to a temperature controller (which maintains the temperature of water up to \( \pm 0.1^\circ \text{C} \)). The circuit is closed by the motor that pumps the water back into the overhead tank. The flow rate can be controlled by increasing the r.p.m. of the motor, which translates to a \( Re \) range of 50–3500.

The disturbance flow is an open-loop facility (unlike the base flow), wherein a water soluble dye is mixed with water, held in a small overhead tank. The mixture is pumped through a small motor and after passing through a flow meter, metering valve and a solenoid valve (which can be computer controlled), it is injected into the base flow at a downstream distance of approximately \( 57d \), where the flow is close to fully developed. The disturbance is introduced into the base flow using a custom-made injection system, which has the possibility to inject four different water jets (mounted 90° apart) and having various inlet velocities. A section of the system is shown on the right-hand side of figure 1. The system is designed such that there would be optical access for visualization beginning as close as possible to the injection point (approx. 0.75\( d \)). In the present work, only one injection hole (out of four) is used, with an inner diameter of 0.8 mm and flow rates \( Q_{\text{inj}} \) ranging from 0.5 to 2.0 ml min\(^{-1}\). The parameter describing the disturbance flow is \( \nu_0 \), which is defined as \( \nu_0 = Q_{\text{inj}}/(S_{\text{inj}} U_{cl}) \), where \( S_{\text{inj}} \) is the cross-sectional area through which the disturbance is injected. Flow visualization is accomplished by using a digital camera and a strong incandescent light source. Two perpendicular views of coherent structures are obtained at the same time on the camera by fixing an inclined mirror close to the pipe. All the experiments begin by running the base flow for an initial period of time until the temperature reaches a nominal value of 21.5°C.

\( \text{(b) Plane Poiseuille flow facility} \)

The Poiseuille flow facility is the same one that was used by Svizher & Cohen (2006) and Philip et al. (2007). The set-up consists of an open circuit air facility, a centrifugal blower, a noise reduction chamber, a diffuser and a settling chamber followed by a contraction nozzle and a channel. A 3 m long channel is placed...
horizontally downstream of a contracting nozzle. The channel consists of two glass plates separated by 50 mm (2h) bars, positioned to give a channel width of 570 mm (figure 2). Two 300 mm diameter aluminium plugs are flush mounted at the lower and upper plates in the middle of the channel, 2 m downstream of the channel inlet. An optical window, 70 mm in diameter, is installed within each plug, allowing one to observe or/and illuminate the region of interest from the vertical direction. Two rectangular (80×50 mm) windows, mounted on the side bars, provide observation of the region of interest from the horizontal (spanwise) direction.

Disturbances are introduced into the laminar Poiseuille flow by continuous air injection through a streamwise slot (57×1 mm), drilled in the lower plug. This provides an almost streamwise-independent initial vertical velocity with a spanwise variation. The coordinate system has the X-, Y- and Z-axes aligned in the streamwise, wall-normal and spanwise directions, respectively. The origin is at the downstream edge of the injection slot. The normalized disturbance amplitude ($v_0$) is defined in exactly the same manner as for pipe flow. The disturbance is visualized by adding tracer particles to the secondary flow, produced by a fog machine via the condensation of oil vapour, and illuminated by a laser light sheet. Velocity measurements are accomplished using a hot-wire traversing mechanism.

3. Results

The results are divided into two subsections. The first describes the linear transient growth phenomenon in both pipe flow and PPF. It is demonstrated that this mechanism is based on the natural existence of a pair of two nearly parallel modes. These modes have opposite phases and almost identical amplitude

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1 The term ‘modes’ refers to the eigenfunctions of $OS$ and $Squire$ equations.
distributions. This part includes spatial and temporal analyses and is supported by some experimental evidence. The second subsection addresses two possible nonlinear scenarios. We begin by experimentally following the breakdown of the counter-rotating pair of vortices (produced by the linear transient growth mechanism) into a packet of hairpins. The associated scaling laws, relating the minimal disturbance amplitude required for the initiation of hairpins and the Reynolds number, are found experimentally for both flows and compared to previous theoretical and experimental results. Finally, we examine the base flow of pipe when it is mildly deviated from the Poiseuille profile by an axisymmetric distortion.

(a) Linear phenomenon of transient growth

As an alternative to normal-mode instability, transient growth has been quite successfully applied to flow systems that undergo subcritical transition. Accordingly, small disturbances may initially be significantly amplified and trigger nonlinear instabilities, before they decay due to viscous effects. It has been found that the most amplified disturbances are streamwise-independent three-dimensional structures, of two counter-rotating vortex pairs (CVPs), in both pipe (for 2π-periodic modes) and PPFs (e.g. Gustavsson 1991; Schmid & Henningson 1994; Reshotko & Tumin 2001).

Here, the procedure by Ben-Dov et al. (2003) is followed, wherein it was shown that the basic mechanism of transient growth can be explained by considering just the two least stable almost parallel modes. The pair of modes occurs naturally for stationary (spatial) and streamwise-independent (temporal) modes. We show that in PPF too, the same mechanism is of relevance.

Consider the equations for PPF having laminar solution, \( U = 1 - y^2 \), subjected to small disturbance velocities, \( u, v \) and \( w \), along \( x, y \) and \( z \) representing streamwise, wall-normal and spanwise coordinates, respectively. The coordinate \( y \) extends from \(-1\) to \(1\) and \( x \) and \( z \) spans from \(-\infty\) to \(\infty\). The associated Orr–Sommerfeld (OS) and Squire equations are

\[
\begin{align}
(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re} (D^2 - k^2)^2 \right) \tilde{v} &= 0 \quad (3.1a) \\
\left( -i\omega + i\alpha U - \frac{1}{Re} (D^2 - k^2) \right) \tilde{\eta} &= -i\beta U' \tilde{v}, \quad (3.1b)
\end{align}
\]

where \( \eta(x, y, z, t) \) is the vertical vorticity; \( \{ \tilde{v}(x, y, z, t), \tilde{\eta}(x, y, z, t) \} = \{ \tilde{v}(y), \tilde{\eta}(y) \} \times e^{i(ax+\beta y-\omega t)} \); \( D \) is derivative w.r.t. \( y \); \( \alpha \) and \( \beta \) are the streamwise and spanwise wavenumbers, respectively; \( \omega \) is the angular frequency; and \( k^2 = \alpha^2 + \beta^2 \). These equations along with the boundary conditions that \( \tilde{v} = 0, D\tilde{v} = 0 \) and \( \tilde{\eta} = 0 \) at \( y = \pm 1 \) complete the formulation of the linear problem in PPF. As the results are to be compared with those of pipe flow, the corresponding notations are as follows: \( z, r \) and \( \theta \) represent the axial, radial and azimuthal coordinates, respectively, and \( v_z, v_r \) and \( v_\theta \) the corresponding disturbance velocities, represented in normal form as \( \{ \tilde{v}_z, \tilde{v}_r, \tilde{v}_\theta \} e^{i(\alpha z+\beta r-\omega t)} \), where \( n \) is the azimuthal wavenumber.
To study the growth of disturbance the energy density integral (in PPF) is defined as

\[ E = \int_{-1}^{1} (|u|^2 + |v|^2 + |w|^2) dy. \] (3.2)

This integral can be appropriately modified for pipe flow by changing the limits to be from 0 to 1, replacing \( u, v \) and \( w \) by \( v_z, v_r \) and \( v_q \), respectively, and \( dy \) by \( r dr \).

(i) Spatial case

The spatial case of the transient growth mechanism in pipe flow is considered here. It has been observed by Reshotko & Tumin (2001) for the pipe flow analysis based on many modes that the maximum amplification of initial energy density is experienced by stationary disturbances (\( \omega=0 \)) and azimuthal wavenumber \( n=1 \). Eigenvalues (of equivalent OS equation in pipe flow) for \( n=1 \) and \( \omega=0 \) at \( Re=3000 \) are plotted in figure 3a. It is clear that the modes appear in pairs of \( \pm \alpha_r \). The least stable ones are indicated by circles and the corresponding eigenfunctions are shown in figure 3b. The axial velocity component, which is \( O(Re) \) larger than the other two, dominates the transient growth process. The cancellation of the two modes at an initial location is evident from the figure.

The work of Ben-Dov et al. (2003) is first followed for the case of pipe flow and later it shall be extended for PPF. For the analysis of transient growth in a pipe, just two least stable nearly parallel stationary modes, \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) (for \( n=1 \)), are considered:

\[ \mathbf{u}_1 = \hat{\mathbf{u}}_1(r) e^{i\theta} e^{i(\alpha_r+i\alpha_i) z}, \quad \mathbf{u}_2 = -\hat{\mathbf{u}}_1(r) A(r) e^{i\theta} e^{i(-\alpha_r+i\alpha_i) z}, \] (3.3)

where \( \alpha_r \) and \( \alpha_i \) are the real and imaginary parts of the spatial eigenvalue, \( \mathbf{u}_j = (v_{zj}, v_{rj}, v_{qj}) \) for \( j=1, 2 \). The real amplitude ratio between the two modes, \( A(r) \), can be expanded as

\[ A(r) \sim 1 + \varepsilon a_1(r) + \varepsilon^2 a_2(r) + \cdots, \] (3.4)
where \( \alpha_j (j = 1, 2, \ldots) \sim O(1) \) and \( \varepsilon \ll 1 \) and \( \varepsilon \propto 1/Re \). Substituting the above expression in equation (3.3), and then into the corresponding energy density equation (3.2) for pipe flow leads to the expression for the disturbance amplification factor, given by

\[
G(z) = \frac{E(z)}{E(0)} \approx \int_0^1 |\tilde{u}_1|^2 r \, dr \, 2e^{-2\alpha z} |1 - \cos(2\alpha r z)|. \tag{3.5}
\]

The maxima of \( G(z) \) for a particular \( z_{\text{max}}/Re \) can be obtained by \( \partial G/\partial z = 0 \). Numerical analysis for large Reynolds numbers shows that the least stable eigenvalues pair is \( \alpha = (\pm 10.82 + 28.68i)/Re \). Employing this yields \( z_{\text{max}} = Re/30 \), which is in excellent agreement with the numerical result of \( z/Re = 0.033 \), reported by Reshotko & Tumin (2001).

Figure 4a shows the eigenvalues for PPF with \( \omega = 0, \beta = 2.04 \) and \( Re = 5000 \).\(^2\) The eigenvalues are queued up on the \( \alpha_t = 0 \) line, with a separation of odd and even modes. The least stable pair of modes and their sum, \( u_1 \) and \( u_2 \) (which are both odd type) and \( u_1 + u_2/A \), respectively, are plotted in figure 4b with an amplitude ratio \( A \).\(^3\) The sum of modes shows a nearly perfect cancellation at \( x = 0 \). From numerical computations, it is found that the eigenvalues of the least stable pair of modes (the pair of odd modes circled in figure 4a) are \( \alpha_1 = 0 + 14.97i/Re \) and \( \alpha_2 = 0 + 19.03i/Re \). The growth of the disturbance due to the pair of least stable modes is given by the disturbance amplification factor \( G(x) = E(x)/E(0) \). Its value normalized with the maximum is plotted in figure 4c, with the maxima at \( x_{\text{max}}/Re = 0.059 \) for two modes (solid curve), and for an optimal disturbance using many modes, with the maxima at \( x_{\text{max}}/Re = 0.055 \) (dashed curve). Although \( G_{\text{max}} \)

\(^2\)This corresponds to the maximum energy amplification in the many mode analysis (Reddy & Henningson 1993).

\(^3\)Axial velocity components, \( u_1 \) and \( u_2 \), are not the same as \( u_1 \) and \( u_2 \) in equation (3.3), indicating vector quantities.
with many modes is about two orders of magnitude greater than that with two
modes, the location of maxima is predicted quite well with just two modes. This
shows that the role of many modes is just to make $E(0)$ much smaller (resulting
in a large $G_{\text{max}}$), but it is the two unstable modes that govern the evolution of
the disturbance.\footnote{If the next pair of unstable modes (the even one in figure 4a)
is added to the existing two modes, then $x_{\text{max}}/Re$ can be predicted closer to the many mode curve and the difference between the two
curves in figure 4c reduces.} Analytical expressions can also be derived for $G(x)$ and $x_{\text{max}}$.

Observing figure 4a, the least stable pair of eigenfunctions can be written as

$$u_1 = \tilde{u}_1(y)e^{i\beta x}e^{i(0+ia_1)x}, \quad u_2 = -\tilde{u}_1(y)A(y)e^{i\beta x}e^{i(0+ia_2)x}, \quad (3.6)$$

where, similar to equation (3.4), $A(y) \sim 1 + \varepsilon a_1(y) + \varepsilon^2 a_2(y) + \cdots$, with $\varepsilon \ll 1/Re$. The appropriate energy density functions (for $u = u_1 + u_2$) at any $x$ and $x=0$ then become

$$E(x) = (e^{-\alpha_{1x}} - e^{-\alpha_{2x}})^2 \int_{-1}^{1} |\tilde{u}_1|^2 \, dy + O(\varepsilon)$$

and

$$E(0) = \varepsilon^2 \int_{-1}^{1} |\tilde{u}_1|^2 a_1^2 \, dy + O(\varepsilon^3).$$

Subsequently, the energy amplification ratio becomes

$$G(x) = \frac{E(x)}{E(0)} \approx \frac{\int_{-1}^{1} |\tilde{u}_1|^2 \, dy}{\varepsilon^2 \int_{-1}^{1} |\tilde{u}_1|^2 a_1^2 \, dy} (e^{-\alpha_{1x}} - e^{-\alpha_{2x}})^2. \quad (3.7)$$

Thus, for high $Re$, the transient amplification ratio is proportional to $1/Re^2$, as in pipe flow. The maximum of $G(x)$ can again be found by $\partial G/\partial x = 0$, giving

$$x_{\text{max}} = \frac{\ln(\alpha_{2}/\alpha_{1})}{\alpha_{21} - \alpha_{11}}, \quad (3.8)$$

which on substituting the above found values of $\alpha_{11}$ and $\alpha_{21}$ gives $x_{\text{max}}/Re=0.059$. This can also be confirmed from figure 4c.

The spatial analysis of both pipe flow and PPF reveals that the pair of least stable modes is sufficient to predict the transient growth mechanism prevalent in these flows. There are two different physical mechanisms working in pipe flow and PPF. In pipe flow, the transient increase in $G(z)$ (equation (3.5)) is a consequence of two opposite running modes ($-\alpha_i$) decaying with an equal decay rate (same $\alpha_i$), whereas in PPF, $G(x)$ (equation (3.7)) increases initially due to two stationary modes ($\alpha_\varepsilon=0$) decaying with different decay rates (unequal $\alpha_i$).

(ii) Temporal case

For the temporal case in PPF, streamwise elongated structures ($\alpha=0$, $\beta=2.04$) are considered. Substituting $\alpha=0$ into equations (3.1a) and (3.1b) results in a homogeneous equation for $\tilde{v}$ and a forced equation for $\tilde{\eta}$, the solution of which can be found analytically (given in appendix A) and show that there exists two families of modes. The first, mode 1 (equation (A 2)), is the trivial
solution of OS equation, i.e. $\tilde{v} = 0$, and $\tilde{h}$ has both even (mode 1e) and odd (mode 1o) modes. The second solution, in turn, has two types, mode 2o (given by equations (A 4a) and (A 4b)) where $\tilde{v}$ is even and $\tilde{h}$ odd, and mode 2e (given by equations (A 7a) and (A 7b)) where $\tilde{v}$ is odd and $\tilde{h}$ even. Calculated eigenvalues are plotted on the complex $\omega_r$ plane in figure 5a. As can be noted, the eigenvalues are naturally separated into two families, mode 1 and mode 2. Both the families contain even and odd modes, and the modes come in alternating families of type 1 and on the $\omega_z = 0$ line. The dashed circles in the figure indicate the pair of modes starting with an even pair and then an odd one. From the analytical solution (see equations (A 4b), (A 7b) and (A 9)) it is evident that the axial velocities are of the order of $\Re$ larger than the other two velocity components. In figure 5b, the axial velocities associated with the first pair of least stable modes (shown by the top dashed circle) are plotted along the $y$-coordinate. The pair consists of the odd modes of both families. As was shown for the spatial case, the two modes cancel each other at the initial time giving rise to transient growth mechanism.

When the amplification factor $G(t)$ is plotted against $t/\Re$ (figure 5c, solid curve), the maximum value to $G$ can be found at $t_{\text{max}}/\Re=0.083$. The maximum value to $G$ can also be calculated analytically because as in the spatial case, here too the eigenvalues lie on $\omega_z=0$ line and an approximation similar to equation (3.6) (with $\alpha_i$ replaced by $-\omega_i$) is valid, yielding $t_{\text{max}} = \ln(\omega_{2i}/\omega_{1i})/(\omega_{1i} - \omega_{2i})$. This is close to the value of $t_{\text{max}}/\Re=0.076$, calculated by Reddy & Henningson (1993) using many modes, and recalculated by the present authors and plotted in figure 5c (dashed curve).

The pair of the two least stable modes can provide the initial structure that would lead to a maximum transient growth. This optimal initial structure for the temporal case is plotted in figure 6. Figure 6a,b shows vector plots in $r$–$\theta$ plane for pipe flow by using just two least stable modes (found analytically) and the resulting structure based on the numerical analysis of many modes. The structure is that of a CVP. For the case of PPF, the optimal initial structure is plotted in figure 6c. The similarity of CVPs obtained by transient growth

Figure 5. (a) Eigenvalues for PPF in the temporal case ($\alpha=0$, $\beta=2.04$, $\Re=5000$) (open circles, mode 1e; filled circle, mode 1o; open squares, mode 2e; filled squares, mode 2o). (b) Corresponding axial velocity components for the least stable pair of modes (dashed curve, first mode; dotted curve, second mode; solid line, pair of modes). (c) Temporal evolution of energy amplification factor with two modes (solid curve) and many modes (dashed curve).
mechanism in both pipe and PPFs shows that this mechanism has a general character and suggests that it is a common feature of wall-bounded shear flows. It is well known that such counter-rotating pairs are the dominant coherent structures in transitional and turbulent flows.

(iii) Experimental evidence of linear transient growth

The experiments in PPF and pipe flow begin with injecting the disturbance at a constant $Re$. For relatively low values of $v_0$, CVPs are observed in figure 7b,c for PPF and pipe flow, respectively. For PPF, smoke particles are introduced through the disturbance slit, and using a thin laser sheet a section in $X$–$Z$ plane (of the CVP for $Y=0.32$) is visualized. For the pipe flow, dye is injected through the disturbance hole and the CVP is visualized by illuminating it with a light source from the back. As can be seen, these structures are elongated along the axial direction (as suggested theoretically by $Z_0$ and $O(1/Re)$ for temporal and spatial cases, respectively). These pairs of vortices are in accordance with the cross-stream structures shown in figure 6.

By examining the Squire equation (3.1b) asymptotically for $Re \to \infty$ it is evident that the transient growth amplification of modes having infinitely long streamwise wavelength ($\sim Re$) is proportional to $Re$ over a time scale of $O(Re)$, i.e. $\eta \sim Re v_0 (t/Re)$, where $t$ is the time (or the proportional axial distance). When $t \sim Re$, a small $v_0$ can give an $O(1)$ growth in $\eta$. Gustavsson (1991) proved that the same scaling can be written in a different form, i.e. the distance to reach maximum vorticity ($x_{\eta_{\text{max}}}$) w.r.t. $Re$ scales as $x_{\eta_{\text{max}}} \sim Re$. For any experimental investigation of transition, the above scenario would require a sufficiently long channel of $O(Re)$. But the same effect can be achieved by maintaining $v_0$ large and thus reducing $t$ or $x$, enabling one to get the same scaling in much shorter channels or pipes. The same scaling is proved by plotting in figure 7a, on a log–log scale, $x_{\eta_{\text{max}}}$ versus $Re$. The slope of the experimental points close to 1 proves that indeed $x_{\eta_{\text{max}}} \sim Re$ (Philip et al. 2007).

The solid line in figure 7 is drawn with a fixed slope of value 1; no attempt is made to carry out any kind of curve fitting. For later reference, in figures 8 and 9 also, such solid lines of fixed slopes are drawn to facilitate the interpretation of the experimental data (given by symbols), and are not to be misunderstood as curves fit to the data.
In this section, two possible nonlinear scenarios are examined. The first one assumes that the initial disturbance is small, grows transiently and then triggers nonlinear mechanisms. The second scenario assumes the existence of a finite disturbance so that the base flow profile is mildly distorted leading to a subsequent secondary instability.

(i) Growth of hairpin vortices in channel and pipe flow

The transient growth mechanism considered above is purely linear and is assumed to be followed by a nonlinear stage, leading to transition. Accordingly, the transition consists of two stages. The first stage begins with the formation of streamwise vortices (approximately aligned with the basic laminar flow) and the subsequent formation of streamwise streaks of relatively low and high velocity in the azimuthal (spanwise) direction. This is a linear stage, governed by the transient growth mechanism. The second stage is the secondary instability of streamwise-dependent modes.

The crossover from the linear stage of transient growth to secondary instability of streaks in PPF was considered by Chapman (2002). He determined how large a streamwise streak has to be in order to make an eigenvalue of a secondary instability become unstable, and found that the centre-odd modes having a wavelength of order one are the most efficient modes to undergo secondary transition. According to his asymptotic analysis, combining the two stages, the required initial threshold amplitude ($v_0$) for transition in PPF should vary as $Re^{-3/2}$. In this transition, it is expected to see secondary odd-mode instabilities located around the centre of the channel and their wavelength should scale with the half channel height.

Experimentally, introducing a small continuous disturbance ($v_0$) through the bottom of the channel results in a linear state represented by a CVP, as already discussed previously and shown in figure 7b. Increasing the amplitude of the disturbance $v_0$ gives rise to the formation of nonlinear structures having

Figure 7. (a) A log–log plot for PPF, indicating the X-location of maximum vorticity for various $Re$. The circles represent the experimental data whereas the solid line represents the slope of 1 as predicted by algebraic growth. CVP obtained in (b) PPF at $Re=1660$ and (c) pipe flow at $Re=1050$.

(b) Nonlinear phenomena
the shape of hairpin vortices. Side (X–Y plane) and top (X–Z plane) views of the hairpins are shown in figure 8b, c, respectively. The experimental verification of the predicted scaling law $v_0 \sim Re^{-3/2}$ is shown in figure 8a (Philip et al. 2007). In this figure, $v_0$ is the critical injection rate required for the formation of hairpins. In figure 8b(i)(ii), side views (X–Y plane) of the channel are shown, where $v_0$ is just below and above its critical value, respectively.

The same experimental procedure is carried out in pipe flow leading to the development of similar coherent structures. For example, at $Re=1050$ in figure 9b, when $v_0$ is increased from figure 9b(i) to (iv), it can be noted that for very small injections where the point of observation is very close to the downstream (0.75$d$) of the injection location, streaks and CVPs are formed (figure 9b(i)(ii)), and as $v_0$ is further increased hairpins are observed (figure 9b(iii)(iv)). The scaling law of $v_0 \sim Re^{-1}$ is obtained in figure 9a by observing again the critical injection velocity (as in PPF). Here, the exponent of $-1$ is obtained in our relatively short pipe flow facility (of approx. 115$d$). The same scaling law was previously obtained by Hof et al. (2003), indicating transition to a turbulent state in their long pipe of 785 diameters.

(ii) Axisymmetric deviation and critical Reynolds number in pipe flow

Finally, we examine the stability of the base-flow profile when it is mildly deviated from the Poiseuille profile by an axisymmetric distortion. Through the examination of the nonlinear features of this problem, Ben-Dov & Cohen (2007) revealed a ‘critical Reynolds number’ of $Re_{tr} \sim 2000$ associated with the bifurcation between two deviation solutions. This result suggests a possible explanation for the well-known ‘critical’ value for ‘natural’ transition to turbulence.

A variational technique employed by Gavarini et al. (2004) is used to find the smallest axisymmetric deviations that lead to the onset of instability, for different $Re$ and axial wavenumbers ($\alpha$). The azimuthal wavenumber has been
restricted to $n=1$ since in Gavarini et al. (2004) it had been shown already that the $2\pi$-periodic disturbances are the most unstable ones. An energy constraint has been computed to yield a neutral stability. For a given deviation, the cross-sectional energy density has been defined as the energy per unit pipe length added to the flow as a result of the additional deviation:

$$
\dot{E} = \int_0^{2\pi} \int_0^1 r [(W_p + W_d)^2 - W_p^2] \, dr \, d\theta = 2\pi \int_0^1 r W_d(2W_p + W_d) \, dr,
$$

(3.9)

where $W_p = 1 - r^2$ is the pipe Poiseuille profile and $W_d$ denotes the axisymmetric deviation.

Figure 10 presents the cross-sectional energy density of the optimal deviations, yielding neutral stability ($\dot{E} = E_n$), as a function of $\alpha$ for four different Reynolds numbers. For sufficiently low Reynolds numbers ($Re<600$), only one optimal deviation exists. On increasing the Reynolds number slightly above 600 a bifurcation occurs, and two optimal deviation solutions (branches) coexist (as presented in the figure for $Re\geq1200$). Generally, as the Reynolds number is further increased the energy density of the optimal deviation required to trigger instability is decreased, and additional bifurcations are expected to take place.

Figure 10b presents the optimal deviation radial distributions of the curves in figure 10a, corresponding to the local minima at which $\alpha = \alpha_{\text{min}}$. The solid curves correspond to the minima of the lower $\alpha_{\text{min}}$ branch of solutions, whereas the dashed curves correspond to the minima of the higher $\alpha_{\text{min}}$ branch. As the Reynolds number is increased, it can be seen that for the lower $\alpha_{\text{min}}$ solutions the deviations tend to be localized next to the pipe wall, whereas for the higher $\alpha_{\text{min}}$ branch the deviations tend to be located around the centreline. In figure 10a, it can be noted that for Reynolds numbers up to values $Re\approx2000$ the global minimum energy solution is the one located near the wall, whereas for values above $Re\approx2000$ the solution located around the centreline becomes the one having the global

Figure 9. (a) Scaling law for subcritical transition (beginning of nonlinear process) in pipe flow, identified by the initiation of hairpin vortices. (b(i)–(iv)) Various coherent structures in pipe flow showing the transition from linear stage (of streaks and CVPs) to nonlinear stage (of an array of hairpins).
minimum energy, and therefore it is more likely to trigger an exponential instability. A more accurate value for this ‘critical’ Reynolds number has been found by Ben-Dov & Cohen (2007) to be $Re = 1840$. The subcritical solution (having the minimum energy below $Re = 1840$) generates unstable waves that have about twice the wavelength of the supercritical waves (having the minimum energy above $Re = 1840$), and their time scale is approximately 3.5 times longer. These two characteristics may supply an explanation for the preference of the solution having a global minimum of energy density above $Re = 1840$ to be the one leading to transition. If the deviation persists over a sufficiently long time and spatial extent, compared to the respective scales of the unstable waves, these waves can grow and initiate transition (where the length and time scales of the deviation are assumed in the analysis to be infinite). Ben-Dov & Cohen (2007) therefore proposed to associate this ‘critical’ Reynolds number with known findings, in which transition has been observed only for Reynolds numbers above approximately 1800 (see, for example, the experiments by Darbyshire & Mullin (1995) and more recently by Peixinho & Mullin (2006), and the supporting results of the direct numerical simulations by Willis & Kerswell (2007)).

The two-dimensional deviation case in Couette flow and in PPF, similar to the pipe flow axisymmetric deviations discussed above, has been addressed by Bottaro et al. (2003) and by Biau & Bottaro (2004), respectively. These two works demonstrate how a very small finite deviation destabilizes those flows at moderate Reynolds numbers ($Re = 500$ for Couette flow and $Re = 3000$ for PPF) with deviations that are approximately 1 per cent of the base flow in their magnitude. However, a critical Reynolds number has not been found for two-dimensional deviations in channel flows.

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4. Conclusions

Linear and nonlinear mechanisms that may explain subcritical transition in pipe and PPFs are analytically and experimentally addressed in an attempt to show their general character. For both flows, it is shown that the linear transient growth mechanism can be explained by simple (temporal or spatial) analysis based on the existence of a pair of least stable nearly parallel modes (having opposite phases and almost identical amplitude distributions). The analyses predict the shape of the initial disturbance, the time (or distance) at which the maximum energy amplification of the initial disturbance is achieved and the dependence of the latter on the Reynolds number. The results agree with the present experimental results in both flows and match previous results that were based on the numerical analysis of many modes. The analysis reveals that in pipe flow, the transient growth in energy ratio is a consequence of two opposite running modes decaying with an equal decay rate whereas in PPF, the energy ratio increases transiently due to two stationary modes decaying with different decay rates.

By following experimentally the formation of hairpin vortices, scaling laws relating the minimal disturbance amplitude required for their initiation and the Reynolds number are found. For PPF, this scaling is $v_0 \sim Re^{-3/2}$ in agreement with the predicted result derived by Chapman (2002) using asymptotic analysis. For pipe flow, the scaling law is $v_0 \sim Re^{-1}$, which is the same as the one experimentally obtained by Hof et al. (2003), indicating transition to a turbulent state in their long pipe. Finally, the nonlinear scenario where the pipe base flow is mildly deviated from the Poiseuille profile by an axisymmetric distortion is examined. Accordingly, a ‘critical Reynolds number’ of approximately 2000 associated with the bifurcation between two deviation solutions is revealed. This may correspond to the critical value obtained by Reynolds over more than a century ago indicating the natural transition to a turbulent state.

Appendix A. Solution of OS and Squire equations for temporal case in PPF with $\alpha=0$

For the case of $\alpha=0$, the OS and Squire equations (3.1a) and (3.1b) become

$$\left[-i\omega(D^2-\beta^2) - \frac{1}{Re}(D^2-\beta^2)^2\right] \tilde{v} = 0$$  \hspace{1cm} (A 1a)

and

$$\left[-i\omega - \frac{1}{Re}(D^2-\beta^2)\right] \tilde{\eta} = -i\beta U' \tilde{v},$$  \hspace{1cm} (A 1b)

with boundary conditions $\tilde{v} = 0$, $D\tilde{v} = 0$ and $\tilde{\eta} = 0$ at $y = \pm 1$.

These equations can be solved subject to boundary conditions to obtain two different kinds of solutions. The first one, mode 1 is given by

$$\tilde{v} = 0, \quad \tilde{\eta} = i \cos(p_1) \sin(p_1 y) - i \sin(p_1) \cos(p_1 y),$$  \hspace{1cm} (A 2)

with the eigenvalue

$$\omega = -\frac{i}{Re} (\beta^2 + p_1^2),$$  \hspace{1cm} (A 3)
where \( p_1 = n\pi/2 \) and \( n = 0, 1, 2, \ldots \). The eigenfunctions\(^6\) (A 2) are a set of orthogonal functions, with \( \tilde{\eta} \) representing odd modes (mode 1o) for \( n = 0, 2, \ldots \) and even modes (mode 1e) for \( n = 1, 3, \ldots \).

The second set of solutions consists of two types. The first type, mode 2o, has \( \tilde{\eta} \) odd and \( \tilde{v} \) even, and is as follows (also see Drazin & Reid 1981, p. 159):

\[
\tilde{v} = \frac{\cosh(\beta y)}{\cosh(\beta)} - \frac{\cos(py)}{\cos(p)} \tag{A 4a}
\]

and

\[
\tilde{\eta} = C_1 \sin(py) + B \left[ y \cosh(\beta y) - \frac{2\beta}{p^2 + \beta^2} \sinh(\beta y) \right] \\
+ K \left[ y^2 \sin(py) + \frac{1}{p} y \cos(py) \right], \tag{A 4b}
\]

with the eigenvalue

\[
\omega = -\frac{i}{Re} (\beta^2 + p^2), \tag{A 5}
\]

where \( p \) is the positive solutions of the equation \( \beta \tanh(\beta) + p \tan(p) = 0 \). In the above equation for \( \tilde{\eta} \) the various constants \( C_1, B \) and \( K \) are given by

\[
C_1 = \frac{-B}{\sin(p)} \left[ \cosh(\beta) - \frac{2\beta}{p^2 + \beta^2} \sinh(\beta) \right] - \frac{K}{\sin(p)} \left[ \sin(p) + \frac{1}{p} \cos(p) \right], \tag{A 6}
\]

\[
B = \frac{-i2\beta Re}{(p^2 + \beta^2)\cosh(\beta)} \quad \text{and} \quad K = \frac{i\beta Re}{2p \cos(p)}.
\]

The second type of solution, mode 2e, has \( \tilde{\eta} \) even and \( \tilde{v} \) odd, and is as follows:

\[
\tilde{v} = \frac{\sinh(\beta y)}{\sinh(\beta)} - \frac{\sin(qy)}{\sin(q)} \tag{A 7a}
\]

and

\[
\tilde{\eta} = C_2 \cos(qy) + M \left[ -y \sinh(\beta y) + \frac{2}{\beta} \cosh(\beta y) \right] \\
+ N \left[ -y^2 \cos(qy) + \frac{1}{q} y \sin(qy) \right], \tag{A 7b}
\]

with the eigenvalue

\[
\omega = -\frac{i}{Re} (\beta^2 + q^2), \tag{A 8}
\]

where \( q \) is the positive solutions of the equation \( \beta \coth(\beta) - q \cot(q) = 0 \). In the above equation for \( \tilde{\eta} \) the various constants \( C_2, M \) and \( N \) are given by

\[
C_2 = \frac{-M}{\cos(q)} \left[ -\sinh(\beta) + \frac{2}{\beta} \cosh(\beta) \right] - \frac{N}{\cos(q)} \left[ -\cos(q) + \frac{1}{q} \sin(q) \right],
\]

\[
M = \frac{-i2\beta Re}{(q^2 + \beta^2)\sinh(\beta)} \quad \text{and} \quad N = \frac{i\beta Re}{2q \sin(q)}.
\]

\(^6\)Note that the eigenfunction is arbitrarily multiplied by \( i \) for later convenience.
For all cases above, using the continuity equation and the definition for vertical vorticity (with $\alpha=0$), we can find the two remaining velocity components,

$$\tilde{u} = -\frac{i}{\beta} \tilde{\eta} \quad \text{and} \quad \tilde{w} = -\frac{i}{\beta} \frac{d\tilde{v}}{dy}. \quad (A.9)$$

References


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