Mind the gap: a guideline for large eddy simulation

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This paper briefly reviews some of the fundamental ideas of turbulence as they relate to large eddy simulation (LES). Of special interest is how our thinking about the so-called ‘spectral gap’ has evolved over the past decade, and what this evolution implies for LES applications.

Keywords: turbulence; large eddy simulation; boundary layer; inertial subrange

1. Introduction

The first section of this brief paper reviews some of the fundamental ideas in turbulence that are relevant to generating (or solving) it numerically. Subsequent sections deal with ideas that are particularly relevant to large eddy simulation (LES). Some of these ideas are well known (at least in the turbulence community), others are less so. A particular emphasis is placed on how our ideas have evolved over the past decade, and what the implications of this evolution are on attempts at LES.

2. The ‘turbulence’ computational problem

The primary problem with numerically computing (as well as measuring) turbulence is the enormous range of scales that must be resolved. The size of the computational domain must typically be at least an order of magnitude larger than the scales characterizing the turbulence energy (e.g. Wang & George 2002). For high Reynolds number turbulence this is characterized by the ‘pseudo’-integral length-scale, \( L_e = u^3/\varepsilon \), where \( K = 3u^2/2 \) is the kinetic energy and \( \varepsilon \) is the rate of dissipation of turbulence energy. (Note that the ‘pseudo’-integral scale is often confused with the physical integral length-scale defined from the correlation, but their ratio is Reynolds number dependent and only constant in the infinite Reynolds number limit (Gamard & George 2000).) The smallest

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dynamically significant length-scale (from dimensional analysis) is the viscous length-scale, the Kolmogorov microscale, defined by \( \eta_K = [\nu^3/\varepsilon]^{1/4} \). It follows immediately that the ratio of length-scales is \( L_\varepsilon/\eta_K = Re_L^{3/4} \), where \( Re_L = uL_\varepsilon/\nu \) is the ‘turbulence Reynolds number’ of interest (cf. Tennekes & Lumley 1972). Moreover, since the essence of turbulence is the three-dimensional amplification of vorticity, the computations must be three-dimensional. For flows of engineering importance, \( Re_L > 10^4 \), and sometimes much greater. This clearly puts the direct computation of most engineering flows well out of reach of contemporary (or even near future) numerical capabilities.

There is another problem with computational turbulence as well. Given the inherent scale complexity of turbulent flows, a compromise is necessary with respect to truncating the smallest scales. Depending on the objective of the computation different resolutions are required. For example, resolving down to a few Kolmogorov microscales will usually suffice if only second-order statistics are needed. But the general need to truncate the smallest scales comes with a price. It is commonly (and quite erroneously) assumed that the Kolmogorov microscale is the smallest scale of the turbulence. In fact, turbulence scales exist all the way down to the continuum limit, but with ever decreasing energy. If only the dissipation need be resolved, then in fact approximately 99 per cent of the dissipation is by turbulence scales larger than \( 2\pi \eta_K \), so \( \eta_K \) sets the resolution limit. The easiest way to see this is to examine decaying homogeneous isotropic turbulence for which the Reynolds averaged Navier–Stokes (RANS) energy equation reduces to

\[
\frac{dK}{dt} = -\varepsilon. \tag{2.1}
\]

If the field is decomposed into Fourier modes of wavenumber vector \( \mathbf{k} \), then both \( K \) and \( \varepsilon \) can be represented in terms of the three-dimensional spectrum function \( E(k, t) \) (\( E \) is the average over spherical shells of radius \( k = |\mathbf{k}| \) of the three-dimensional spectrum) as (Monin & Yaglom 1975)

\[
K = \int_0^\infty E(k, t) dk, \tag{2.2}
\]

\[
\varepsilon = 2\nu \int_0^\infty k^2 E(k, t) dk. \tag{2.3}
\]

If we think of \( k \) as being proportional to the inverse ‘eddy’ size, it is easy to see why considerably more resolution is required to obtain the dissipation rate, \( \varepsilon \), than to obtain the kinetic energy, \( K \). The integral of \( k^2E \) converges to within a per cent or so if \( k\eta_K = 1 \); hence the resolution criterion above.

But this is still problematical for non-stationary flows, since the dynamics of the dissipation itself (or equivalently, the enstrophy) is at least in part determined by the dissipation of dissipation, say \( \varepsilon_2 \), for which scales of at least a factor of four smaller are required. Again this is easy to see from the RANS equation for \( \varepsilon \) which can be written as

\[
\frac{d\varepsilon}{dt} = P_\varepsilon - \varepsilon_2, \tag{2.4}
\]
where $P_\varepsilon$ is the ‘production’ of dissipation (by vortex stretching and turning).

The *dissipation of dissipation* in terms of the spectral integrals is given by

$$\varepsilon_2 = 4\nu^2 \int_0^\infty k^4 E(k, t) dk.$$  \hfill (2.5)

Unfortunately, the presence of the fourth spectral moment requires in turn much higher wavenumbers (or equivalently, resolution to smaller scales) to faithfully produce $\varepsilon_2$ than even $\varepsilon$. But the *dissipation of the dissipation of the dissipation*, say $\varepsilon_3$, is required to produce $\varepsilon_2$ and it depends on

$$\varepsilon_3 = 8\nu^3 \int_0^\infty k^6 E(k, t) dk.$$  \hfill (2.6)

This process obviously can be continued indefinitely.

So there really is no satisfactory answer as to what resolution is required to generate turbulence numerically. Certainly resolving the Kolmogorov microscale is a good beginning, but as computational time goes on the effects of the unresolved scales will probably be felt, no matter the resolution. It is important to note this is not a problem for experiments, since the flow itself *always* contains the relevant scales (assuming it is allowed to develop properly). So at least the experimentally realized flow is in principle correct, even if the measurements cannot resolve all scales that are present.

### 3. The ‘gap’ and its relevance to LES

Since (at least by the argument above) even a direct numerical simulation (DNS) in effect truncates the computation below some scale, why not just truncate the computation at an arbitrary and more convenient larger scale? This is in essence the whole point of LES, and at first glance it seems a rather straightforward and logical way to attack the turbulence problem. Obviously it would simplify things considerably if the truncation could be performed in some manner so that the computation could be effectively independent of precisely the scale at which the truncation were done; i.e. scale invariant. Then any closure model would depend only on the wavenumber (or length) at which the truncations were made (e.g. Smagorinsky). Traditional turbulence theory (Kolmogorov 1941) suggests exactly where this can be done (at least in the limit of infinite Reynolds number); namely in the inertial subrange which lies somewhere between the integral scale and the Kolmogorov microscale, but sufficiently removed from either (cf. Tennekes & Lumley 1972; Meneveau & Katz 2000). The most significant feature of this region, or gap if you will, is the constancy of the spectral flux, say $\varepsilon_K$, and in fact its equality with the dissipation, $\varepsilon$.

If such a range exists, the velocity spectrum has been believed to be uniquely characterized by the so-called ‘inertial subrange’ spectrum given by

$$E(k) = \alpha_k \varepsilon^{2/3} k^{-5/3},$$

where $\alpha_k$ is a universal constant. Similar relations can be derived for the velocity structure functions in physical space as well (e.g. $\langle[u_i(x + r, t) - u_i(x, t)]^2\rangle$), so the arguments can be applied in either physical or wavenumber space. Since the underlying hypothesis is the idea of local homogeneity, it really does not matter which.
The ideas of the preceding paragraph have been widely believed, at least until recently, to apply to most turbulence. This was, of course, in the absence of careful systematic measurements over a range of Reynolds numbers. Such measurements have now been carried out by Mydlarski & Warhaft (1996) in a wind tunnel using an active grid. They were able to obtain much higher Reynolds numbers than was possible before, $50 \leq Re_L \leq 500 \ (Re_L = Re_{\lambda}^2/15)$, so it should not be too surprising that their results have forced us to modify some of our traditional ideas. Figure 1 shows their one-dimensional spectra normalized by Kolmogorov variables ($\nu$ and $\epsilon$) and multiplied by $k^{5/3}$. In this plot, a $k^{-5/3}$ range would appear as a region for which $(k_1 \eta_K)^{5/3} F_{a,a}(k_1)/\nu^{5/3} \epsilon^{1/4} = \text{const.}$, where $a = 1$ or 2 (no sum). In spite of the relatively high Reynolds numbers, as noted by the authors, there was really no such range, and at most it could be inferred that it might be reached at much higher Reynolds numbers. The inset on the plot shows the departures, $\mu$, of the observed power law region from the expected $-5/3$ behaviour, along with the finite Reynolds number theory of Gamard & George (2000). The concurrence of theory and experiment confirms that the classical spectral gap result can be expected only at Reynolds numbers greater than $Re_L > 10^4$, perhaps much greater.

Figure 1. One-dimensional wavenumber spectra pre-multiplied by $k^{5/3}$ for $Re_\lambda = 50, 100, 124, 174, 207, 275, 330$ and 473 showing absence of universal $k^{-5/3}$ range; $a = 1$, longitudinal spectra (squares), $a = 2$, lateral spectra (triangles). The inset shows the deviation, $\mu$, of one-dimensional spectra from $k^{-5/3}$ law (squares, composite model; circles, data; solid curve, model). (Reproduced from Gamard & George (2000).)
LES truncates the scales which must be resolved, usually with a closure which depends only on the computational grid and with the assumption of scale invariance. The important implication of the above for LES is that only when the ratio $L_0/\eta_K$ is more than 1000 is it reasonable to assume that there exists a Reynolds number independent range of turbulence scales over which such a closure is valid. Clearly, this is problematical since there are many flows where the turbulence Reynolds number is below this value (and as shown below, always near smooth walls).

Now it might be argued that the observations above apply only in wavenumber space and are, therefore, not relevant to volume-averaged methodologies where an eddy viscosity-like term (e.g. Smagorinsky) is added to the usual viscosity term. There are two problems with this. First, Kolmogorov-type arguments can also be applied to the so-called velocity structure functions which examine how velocity differences depend on space. And similar to the spectra presented above, they show the same type of Reynolds number dependence (Lundgren 2002). So whether working in wavenumber space or in real physical space there really is no gap, unless the ratio of scales is approximately $10^3$ or greater. Second, the idea is also wrong that simply keeping the viscous term is sufficient. This is most easily seen by observing the presence of a $k^{-1}$ range in the one-dimensional spectrum. As shown by Lin (1972; see also Gamard & George 2000), this can be accounted for by directly cutting the spectral flux itself as the wavenumber is increased into the viscous range. This would seem to imply that any closure model for the subgrid scales should explicitly include a viscous damping, over and above the direct presence of viscosity in the instantaneous volume-averaged equations, at least when applied to flows of moderate to low Reynolds numbers. Moreover, it should also be clear that a comparison between LES and DNS is probably never going to yield very useful closure evaluations, because of the low Reynolds number of the latter and the consequent absence of any range of scales for which the LES closure can be presumed valid. By contrast, applications of LES to high Reynolds number free shear flows for which there is a clear spectral gap would seem to be ideal (see Grinstein & Fureby (2002) for examples).

4. The problem for wall-bounded LES

The above considerations about the presence (or absence) of a spectral gap have direct implications for the application of LES to wall-bounded flows, at least with smooth walls. While there have been indications from earlier experiments that there is a problem, it is only very recently that we have data at Reynolds numbers sufficiently high to evaluate when the near-wall region has a spectral gap (Carlier & Stanislas 2005; Tutkun et al. in press). The Reynolds number based on momentum thickness was approximately 20 000, and the data were obtained using hot-wire rakes in the boundary layer wind tunnel at LML/Lille. Note that 20 000 is approximately the lower limit for which one can expect a true inertial layer or spectral gap in the flow, meaning that boundary layers at lower Reynolds numbers will still exhibit viscous effects in the outer (or main) part of the boundary layer, and as well show residual outer effects in the near-wall region. The data of interest here are the one-dimensional velocity spectra,
again pre-multiplied by $k_{5/3}$, as a function of distance from the wall (in viscous units). The spectra are plotted together for nine locations in figure 2. What is striking about the figures is the absence of a $k^{-5/3}$ range. Clearly, there is nothing resembling an inertial subrange exhibited by the spectra inside of at least $y^+ = y u_\tau / \nu \leq 445$. Outside this location, the spectra closely resemble the higher Reynolds number spectra of figure 1, consistent with the fact that the local Reynolds number increases with increasing distance from the wall. In particular, they exhibit a range which has a slope of nearly $-5/3$, but not quite, suggesting the residual importance of viscous effects at even the highest local Reynolds numbers.

At first glance these results seem counter-intuitive since it is well known that viscous effects play no role in the single-point RANS equations for $y^+ \geq 30$. But the spectral observations are exactly consistent with the predictions of George & Castillo (1997; see also George 2006) who postulated the existence of a mesolayer from approximately $30 < y^+ < 300–600$ in which viscosity would continue to affect directly the energetic scales of motion so there could be no spectral gap. The spectral gap can develop only when the energy-containing (and Reynolds

Figure 2. One-dimensional wavenumber spectra pre-multiplied by $k_{5/3}$ (so $k^{-5/3}$ range appears as constant) at different wall-normal positions. The solid straight lines indicate the slope that would correspond to a $k^{-1}$ range.
stress producing) scales are nearly inviscid, which can happen only outside the mesolayer, and even then only when the Reynolds number of the boundary layer is high enough ($Re > 10^4$ approx.).

The implications for an LES closure are also clear: it makes no sense (particularly in light of the preceding section) to apply an LES (or LES closure) inside of $y^+$ of a few hundred (at least without explicitly including viscosity in the model). Nor does it make sense to compare LES simulations of wall-bounded flows with the relatively low DNS of such flows, most of which are at Reynolds numbers too low to even develop an inertial range in the outer (or main) part of the flow. (Such comparisons might make sense, however, if the goal were to establish how the closure relations depend on Reynolds number in the absence of a gap.) By contrast, LES applied near rough surfaces or the atmospheric boundary layer does make sense (e.g. Porté-Agel et al. 2000), since these almost always have a well-developed inertial subrange in the velocity spectra, almost to the surface, thus giving clear evidence of a spectral gap.

5. Is there any hope for near-wall LES model?

In light of the above, there would seem to be little hope for using an LES near smooth surfaces, at least without some significant new Reynolds number dependent implementation. A new strategy is called for. Figure 3 shows two-point (actually space-time plus Taylor’s frozen field hypothesis) correlation coefficient data from the Lille/WALLTURB experiment described above (Tutkun et al. in press). The figures show how the correlation coefficient varies with distance from the wall, $y$, and downstream, $x$, for various heights of the reference probe.

Clearly, there is a strong correlation at almost all heights, and at a single time. This suggests that it might be possible to build a near-wall model that is in sync with the outer flow (i.e. follows it), perhaps quite independent of considerations such as Reynolds number and spectral gaps. Note that this is quite the opposite of the prevailing view for the last 40 years or so that it is the wall region (with its streaks and bursts) that drives the outer flow. In fact, it assumes the opposite: namely that the inner flow is driven by the outer. As pointed out by George & Castillo (1997), at high Reynolds number most of the energy production is not near the wall (even though the peak in production is there), but in the main part of the boundary layer. The same is true for the dissipation. Thus, if there were any region of a boundary layer flow where a ‘universal’ model might be found, the near-wall region would seem to be the most likely place to look. Interestingly, higher order RANS models seem to have the near-wall physics right, so they should provide a good starting point.

6. Summary and conclusions

Recent ideas in turbulence have been reviewed, especially as they might affect our understanding of attempts to implement LES. The whole idea of a closure that depends only on a filtering by the grid (or spectral truncation) depends crucially on the idea of a spectral gap which ensures scale invariance. All of the evidence from the last decade suggests strongly that such a ‘gap’ exists only at much higher Reynolds numbers than previously believed. This, of course, has no
effect on current attempts to apply LES in high Reynolds number free shear flows, or in the atmosphere. But it would seem to imply that it is futile to try to use existing LES models at low Reynolds numbers, or even to attempt to validate such models with current DNS. On the other hand, perhaps an effort to evaluate how to include viscosity in the subgrid closure might be useful. Also it means that attempts to use LES near smooth walls are doomed to failure, at least

Figure 3. Two-point correlation coefficients on streamwise–wall-normal plane at $Re_\theta$ of 19 100. The figures present the correlation between the reference probe at one wall-normal position, $y$, and the rest of the probes at the same spanwise location at different wall-normal positions, $y'$. (a) $y^+ = 7$, (b) $y^+ = 22$, (c) $y^+ = 50$, (d) $y^+ = 105$, (e) $y^+ = 220$, (f) $y^+ = 445$, (g) $y^+ = 890$, (h) $y^+ = 1805$, $y = 0.255\delta$. The contour values: (0.025 (outermost), 0.05, 0.1, 0.2, 0.4, 0.8, 1.0 (innermost)).
without a significant change in strategy: either by incorporating viscosity directly in the closure or by constructing a synthetic wall model that follows the outer flow.

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References


