Interplay of time-delayed feedback control and temporally correlated noise in excitable systems

BY S. BRANDSTETTER, M. A. DAHLEM AND E. SCHÖLL*

Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstrasse 36, D-10623 Berlin, Germany

The interplay of time-delayed feedback and temporally correlated coloured noise in a single and two coupled excitable systems is studied in the framework of the FitzHugh–Nagumo (FHN) model. By using coloured noise instead of white noise, the noise correlation time is introduced as an additional time scale. We show that in a single FHN system the major time scale of oscillations is strongly influenced by the noise correlation time, which in turn affects the maxima of coherence with respect to the delay time. In two coupled FHN systems, coloured noise input to one subsystem influences coherence resonance and stochastic synchronization of both subsystems. Application of delayed feedback control to the coloured noise-driven subsystem is shown to change coherence and time scales of noise-induced oscillations in both systems, and to enhance or suppress stochastic synchronization under certain conditions.

Keywords: time-delayed feedback control; delay; correlated noise; coupled excitable systems; excitable system

1. Introduction

Control of unstable or irregular states of nonlinear dynamic systems is one of the main problems of applied nonlinear dynamics and a central issue of current research (Schöll & Schuster 2008). A particularly simple and efficient control scheme is time-delayed feedback control, introduced by Pyragas (1992), which is also known as time-delayed autosynchronization (TDAS). The TDAS control scheme uses a time-delayed difference \( s(t) - s(t - \tau) \) of a system variable \( s \), which is coupled back into the system. It is robust and universal to apply, easy to implement experimentally, and has been used in a large variety of systems in physics, chemistry, biology, medicine and engineering (Schöll 2001, 2009; Pyragas 2006; Schimansky-Geier et al. 2007; Schöll et al. 2009b), in purely temporal dynamics as well as in spatially extended systems. Moreover, it has recently been shown to be applicable also to noise-induced oscillations and patterns (Janson et al. 2004, 2008; Pomplun et al. 2005; Stegemann et al. 2006; Flunkert & Schöll 2007; Hizanidis & Schöll 2008).

In the context of excitable systems, the effects of time-delayed feedback control on a FitzHugh–Nagumo (FHN) system driven by white noise have been studied in Balanov et al. (2004), Janson et al. (2004), Prager et al. (2007) and

*Author for correspondence (schoell@physik.tu-berlin.de).

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Pototsky & Janson (2008), and it has been shown that both the time scales and the coherence of the noise-induced oscillations can be significantly influenced in their dependence on the delay time. Moreover, the stochastic synchronization of coupled excitable systems under the influence of white noise can be deliberately controlled by local time-delayed feedback. By appropriate choice of the delay time, synchronization can be either enhanced or suppressed (Hauschildt et al. 2006; Hövel et al. in press), and this depends sensitively upon the coupling scheme (Hövel et al. 2010). In the context of neural excitable systems, such time-delayed feedback loops often occur naturally due to neuro-vascular couplings, or signal propagation delays of coupled neurons (Schneider et al. 2009; Schöll et al. 2009a). Moreover, time-delayed feedback loops might be deliberately implemented to control neural disturbances, e.g. to suppress pathological synchrony of firing neurons in diseases such as Parkinson’s or epilepsy (Schiff et al. 1994; Rosenblum & Pikovsky 2004; Popovych et al. 2005; Richardson et al. 2005; Gassel et al. 2007).

One of the most remarkable effects discovered in excitable systems under the influence of noise is the phenomenon of coherence resonance (CR; Hu et al. 1993; Longtin 1997; Pikovsky & Kurths 1997), which is characterized by the existence of an optimal noise intensity for which the noise-induced oscillations are most coherent. It is thus an indication of the constructive influence of noise in nonlinear systems (García-Ojalvo & Sancho 1999; Masoller 2002; Linder et al. 2004; Hizanidis et al. 2006; Sagüés et al. 2007). Most of the work on stochastic excitable systems assumes the stochastic input to be uncorrelated white noise. This approximation is only justified if the typical correlation times of the fluctuations are much smaller than all the characteristic time scales of the deterministic system. Moreover, in the context of neural systems, the noise is assumed to be mostly due to synaptic input from other neurons. On average a single neuron receives synaptic input from approximately 1000 other neurons, which is integrated in the soma. Since the neuron cannot distinguish between different sources, but reacts to the summed signal, correlations between these input signals transform into temporal correlations. Hence it is of great importance to understand how the temporal correlations of an input signal to a neuron—or a generic excitable system—influence its response. One simple method to model these temporal correlations is to use exponentially correlated coloured noise. The influence of coloured noise on a single excitable system—FHN system or Oregonator model or integrate-and-fire neuron—was studied in Casado (1997), Beato et al. (2005, 2008), Beato (2006) and Schwalger & Schimansky-Geier (2008), and CR with respect to the correlation time of the noise was observed. Also, delayed feedback control of noise-induced pulses in the Oregonator model driven by coloured noise was investigated (Balanov et al. 2006).

In this paper, we study the combination of time-delayed feedback and temporally correlated coloured noise in excitable systems. Apart from the fact that coloured noise is more realistic than white noise, the combination of time-delayed feedback and temporally correlated noise is interesting from a theoretical point of view: both the delay time of the control and the correlation time of the coloured noise influence the coherence of oscillations; hence the interplay of the delay time and the noise correlation time will be the central issue.
To get a thorough understanding of the effects of coloured noise, we first review an FHN system driven by coloured noise without control (§2). Then we apply time-delayed feedback to a single FHN system (§3) and investigate the influence of the control on varying the noise correlation time; in particular, we will study the control of CR.

In §4, which is concerned with two coupled excitable systems, the focus will lie on the synchronization of the two subsystems. We will systematically investigate the influence of the noise correlation time on the dynamics of the coupled systems, and its interplay with other parameters of the system. Furthermore, we will study the influence of time-delayed feedback on two coupled excitable systems driven by coloured noise, and, in particular, compare the results with the findings for coupled excitable systems driven by white noise.

2. Model system

In this work, we use the FHN system as a generic model for excitable systems. It is a generalization of the van der Pol oscillator (van der Pol 1926) and is also known as the Bonhoeffer–van der Pol oscillator (Bonhoeffer 1948, 1953). The model was first suggested by FitzHugh (1961), and independently by Nagumo et al. (1962). As a two-variable simplification of the four-variable Hodgkin–Huxley model, which describes the propagation of action potentials along the giant squid axon (Hodgkin & Huxley 1952), it describes the response of an excitable nerve membrane to external current stimulation:

\[
\varepsilon \dot{x} = x - \frac{x^3}{3} - y
\]

and

\[
\dot{y} = x + a + \eta(t).
\]

The original interpretation of the FHN model equation (2.1) is based on a single point-like neuron: the variable \( x \) models fast changes of the electrical potential across the membrane (occurring as spikes in the time series), and \( y \) is the recovery variable related to the gating mechanism of the membrane channels (FitzHugh 1961). The noise term \( \eta(t) \) is a stimulating current representing external input to the cell, and \( \varepsilon \), usually chosen to be much smaller than unity, represents the time-scale ratio of the two variables. The fast variable \( x \) is called the activator variable, whereas the slow variable \( y \) is referred to as the inhibitor variable. The excitability parameter \( a \) determines whether the system is excitable \((a > 1)\) or exhibits self-sustained periodic oscillations \((a < 1)\). In the following we choose \( a = 1.05 \) in the excitable regime close to the threshold.

(a) The excitable regime

Figure 1 shows a schematic phase portrait of a single deterministic system \((\eta = 0)\) in the excitable regime \((a > 1)\) with the cubic activator nullcline and the vertical inhibitor nullcline (solid black lines). The system has a single fixed point,
which is stable for $a > 1$ and located on the left branch of the cubic nullcline. At $a = 1$ a supercritical Hopf bifurcation of a limit cycle occurs, and the fixed point becomes unstable and is shifted to the middle branch of the nullcline for $a < 1$. The excitable behaviour of the system is crucially determined by the cubic nonlinearity of the activator equation and the separation of time scales between the two variables: when the system is perturbed by an external stimulus, which can be regarded as a setting of the initial condition, the system undergoes a large excursion in phase space. Starting from its initial condition, the system performs, owing to the strong time-scale separation $\varepsilon \ll 1$, a fast transition to the stable right branch of the activator nullcline. After that, it travels slowly upwards approximately along this nullcline, until the phase points jumps back to the left branch, and returns, along the left branch of the nullcline, slowly downwards to the fixed point. Without further external stimulation the system remains in the fixed point.

The threshold-like behaviour of the FHN system is associated with the canard-like trajectory (dash-dotted line in figure 1), which is the trajectory passing through the local maximum of the cubic nullcline, and which is often referred to as the threshold of the FHN system. The region around the canard-like trajectory is extremely sensitive to initial conditions: for initial conditions only slightly below the canard-like trajectory the system will perform a large excursion in phase space, whereas for initial conditions only slightly above the canard-like trajectory the excursion will be small. In principle, the transition from small to large relaxation oscillations is continuous; in fact, however, phase space excursions of intermediate amplitude are very rare. Correspondingly, small sub-threshold stimulations (dashed line in figure 1) will result in only small relaxation oscillations, while super-threshold stimulations (solid grey line in figure 1) induce a full excursion in phase space, corresponding to a characteristic spike in the time evolution of the $x$-variable.
When the system is perturbed by external noise, the system will be driven out of the fixed point persistently, resulting in noise-induced oscillations. In the following we will refer to the excursions in phase space as either oscillations or spiking and use the two terms interchangeably.

Note that the FHN system only models neurons of Type II excitability, which is associated with a Hopf bifurcation and a non-zero frequency at the onset of self-sustained oscillations. Type I excitability is associated with a saddle-node-infinite-period (SNIPER) bifurcation and zero frequency at the onset of self-sustained oscillations, which corresponds to the characteristic frequency dependence of the SNIPER bifurcation. This type of global bifurcation is not encountered in the FHN model, but is met in other two-variable models (Hu et al. 1993; Hizanidis et al. 2008), for example the Hindmarsh–Rose model.

(b) The oscillatory regime

At \( a = 1 \) a supercritical Hopf bifurcation of a limit cycle occurs, and the fixed point becomes unstable and is shifted to the middle branch of the nullcline for \( a < 1 \). Hence the deterministic system exhibits sustained oscillations even without external stimuli, which motivates the term oscillatory for this parameter regime. This type of bifurcation is associated with characteristic dependences of the frequency \( f(a) = O(1) \) and amplitude \( r(a) = O(\sqrt{|a - 1|}) \) of oscillations upon the bifurcation parameter \( a \). However, for the FHN system these relations are valid only for a very small range of the bifurcation parameter \( a \). The square-root-shaped increase of the amplitude can be observed only for \( a \) values extremely close to the bifurcation point; for \( a \) only slightly larger the amplitude increases very rapidly. This phenomenon is known as canard explosion or canard transition (Benoit et al. 1981; Broens & Bar-Eli 1991; Peng et al. 1991). The effect is influenced by the time-scale separation \( \varepsilon \) of the two variables: the smaller \( \varepsilon \), the smaller the bifurcation parameter range for which the square-root-shaped increase is observable. Owing to the very rapid increase of the amplitude, limit cycles of intermediate amplitude, referred to as canard cycles, are very rare. In fact, for \( \varepsilon = 0.01 \) limit cycles of small or intermediate amplitude are hardly ever observed; instead the amplitude immediately rises to the maximal amplitude, corresponding to large excursions in phase space.

The very rapid increase of the amplitude of oscillations is also reflected in the frequency or period of the oscillations. The non-zero frequency or finite period of oscillations associated with the supercritical Hopf bifurcation is a local property of the unstable focus. In contrast, the large-amplitude oscillations are induced by the global dynamics of the system. Hence their period is not determined by the local properties of the fixed point, but by the global structure of the phase space: the period is dominated by the slow motion along the two stable branches of the activator nullcline. The dependence of the period of oscillations on the bifurcation parameter \( a \) can be estimated by analytical approximations, very similar to the approximations for the firing time in Schöll et al. (2009b).

Owing to the small value of \( \varepsilon \ll 1 \) there is a strong time-scale separation between the fast activator and the slow inhibitor variable, and the limit cycle can be approximated as shown in figure 2a. Starting from point \( A \), corresponding to the local minimum of the activator nullcline, the system performs a rapid horizontal transition to point \( B \) on the right branch of the \( x \) nullcline, followed by
a slow motion along this nullcline to its local maximum $C$. Then the system jumps back to point $D$ on the left branch and then slowly moves downwards along the left branch until it reaches the starting point $A$, and a new cycle begins. The four points of this excursion are approximately given by $A = (-1, -\frac{2}{3})$, $B = (2, -\frac{2}{3})$, $C = (1, \frac{2}{3})$ and $D = (-2, \frac{2}{3})$. With regard to the period of the oscillations, we can neglect the fast transitions between the left and right branches of the $x$ nullcline. The slow motion along the left and right branches of the $x$ nullcline can be approximated by $y = x - (x^3/3)$ and hence $\dot{y} = \dot{x}(1 - x^2) = x + a$, which gives

$$
\dot{x} = \frac{x + a}{1 - x^2}.
$$

This equation can be integrated analytically

$$
\int_{\pm 2}^{x} \frac{dx'}{1 - x'^2} = (a^2 - 1) \ln \frac{a + 2}{a + x} - a(\pm 2 - x) + 2 - \frac{x^2}{2} = t.
$$

The duration of the motion on the right branch $T_r$ is given by

$$
T_r = \int_{2}^{1} dx \frac{x + a}{1 - x^2} = (a^2 - 1) \ln \frac{a + 2}{a + 1} - a + \frac{3}{2},
$$

while for the left branch we obtain

$$
T_l = \int_{-2}^{-1} dx \frac{x + a}{1 - x^2} = (a^2 - 1) \ln \frac{a - 2}{a - 1} + a + \frac{3}{2}.
$$

Hence, the period of oscillations $T = T_r + T_l$ can be estimated as

$$
T = (a^2 - 1) \ln \frac{a^2 - 4}{a^2 - 1} + 3.
$$
In figure 2b we compare the analytical result with numerical simulations of a deterministic oscillatory FHN system. For $\varepsilon = 0.0001$ the analytical approximation (2.6) (solid black line) is in good agreement with the numerical results (black circles). For $\varepsilon = 0.01$, however, the numerical simulation yields a considerably larger period for all values of $a$ (grey circles). The reason for this discrepancy is revealed by the corresponding phase portrait for $a = 0$, shown in figure 2a. The system does not perform a direct transition to the left or right branch of the activator nullcline, but ‘overshoots’ the approximated straight line and follows a curved trajectory in the vicinity of the four points $A$, $B$, $C$ and $D$, which elongates the oscillation period noticeably. To account for this delay we simply assume that the extra time depends in a similar way on the parameter $a$ and rescale our formula with $(1 + e)$, where $e$ is a constant parameter, determined by fitting the rescaled analytical solution

$$T = \left( (a^2 - 1) \ln \frac{a^2 - 4}{a^2 - 1} + 3 \right)(1 + e)$$

(2.7)

to the numerical simulations. For $e = 0.185$ the numerical solutions for $\varepsilon = 0.01$ (grey circles in figure 2b) and the analytical approximation (2.7) (solid grey line in figure 2b) are in good agreement. As seen from figure 2b the oscillation period decreases monotonically for $0 < a < 1$ and is a minimum for $a = 0$. Moreover, the minimum period is about a factor of 2 smaller than the maximum period for $a \to 1$. We will see in §3 that an FHN system driven by coloured noise can be driven occasionally rather far into the oscillatory regime; thus the oscillation period can vary substantially during these phases.

(c) Two coupled systems

In the following, we consider two FHN systems coupled via symmetric diffusive coupling (coupling constant $C$) and driven by two independent noise sources, as illustrated by the schematic structure in figure 3. The first subsystem is driven by coloured noise $\eta(t)$ and the second one by white noise $\xi(t)$. Additionally, we apply a linear time-delayed feedback control term to the first subsystem with feedback gain $K$ and delay time $\tau$. The dynamical equations are given by

$$\varepsilon_1 \dot{x}_1 = x_1 - \frac{x_1^3}{3} - y_1 + C(x_2 - x_1),$$
$$\dot{y}_1 = x_1 + a + \eta(t) + K[y_1(t - \tau) - y_1(t)],$$
$$\varepsilon_2 \dot{x}_2 = x_2 - \frac{x_2^3}{3} - y_2 + C(x_1 - x_2)$$

and

$$\dot{y}_2 = x_2 + a + D_2 \xi(t),$$

(2.8)

where $(x_1, y_1)$ and $(x_2, y_2)$ represent the two subsystems.

We consider two non-identical excitable systems with time-scale separations $\varepsilon_1 = 0.005$ and $\varepsilon_2 = 0.1$. This difference of parameters is associated with different sensitivities of the system to input in the activator variable: with
the given symmetric coupling strength $C$, the first subsystem shows much greater sensitivity or excitability to spikes from the second subsystem than the second subsystem does to spikes from the first subsystem. In other words, nearly every spike from the second system will result in a spike of the first one, but hardly any spike from the first subsystem will be reflected in the second one.

The interaction between the two subsystems is modelled as diffusive, i.e. the coupling term vanishes if the $x$ variables are equal. The coupling is realized as symmetric activator–activator coupling, i.e. the difference between the two activator variables is coupled into the activator variables, with the coupling strength $C$ taken to be equal in both subsystems for simplicity. Note that with the given form we assume the coupling to be instantaneous, i.e. we neglect any delay in the connection and assume infinite signal propagation speed. Delay-coupled FHN systems and their control have been studied in Dahlem et al. (2009), Rüdiger & Schimansky-Geier (2009) and Schöll et al. (2009a).

The first system is driven by Gaussian temporally correlated coloured noise $\eta(t)$ with zero mean and autocorrelation function $\Psi(s) = \langle \eta(t)\eta(t+s) \rangle = \sigma^2 \exp(-|s|/\tau_c)$, where $\sigma^2$ is the variance and $\tau_c$ the correlation time of the coloured noise. The term ‘coloured’ noise means that, in contrast to white noise, the frequency spectrum is not flat, but is a Lorentzian with maximum at zero and half-width at half-maximum $1/\tau_c$. However, since the peak of the spectrum is at zero, there is no preferred frequency that could be referred to as the ‘colour’ of the noise.

Exponentially correlated coloured noise with this autocorrelation function can be generated by an Ornstein–Uhlenbeck process, defined by

$$\tau_c \dot{\eta} = -\eta + \sqrt{2\sigma^2 \tau_c} \xi(t),$$

where $\xi(t)$ is Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$. The noise correlation time $\tau_c$ and the variance $\sigma^2$ are parameters that can be varied independently. Coloured noise generated by an Ornstein–Uhlenbeck process with this parametrization is referred to as power-limited coloured noise, since the total power of the noise (the integral over the spectral density of the process) is conserved upon varying the noise correlation time (Jung et al. 2005). For the numerical generation of exponentially correlated coloured noise we use the integral algorithm proposed by Fox et al. (1988), corresponding to a direct numerical generation of the conditional probability distribution of the Ornstein–Uhlenbeck process.
The second system is driven by Gaussian white noise $\xi(t)$ with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$; $D_2$ is the noise intensity of the white noise.

The parameters of the time-delayed feedback scheme in equations (2.8) are the feedback gain $K$ and the delay time $\tau$.

(d) Characterization of noise-induced oscillations

(i) Time scales

The Fourier power spectral density reveals the frequency content of random oscillations and is therefore the most universal tool to characterize the time scales involved. In the following we will use the terms power spectral density and spectrum interchangeably. Since the oscillations of the FHN system have a distinct spiky shape, one can define the interspike interval (ISI), denoted as $T_{\text{ISI}}$, as the time interval between two subsequent spikes (corresponding to excursions in the phase space). When considering long time series of the system, we can calculate the relative frequency of ISIs, referred to as the distribution of ISIs in the following. The mean ISI $\langle T_{\text{ISI}} \rangle$ is another convenient quantity to characterize the time scales of the system.

(ii) Coherence

In order to measure the coherence of oscillations by a single scalar quantity, we use the correlation time defined as

$$t_{\text{cor}} = \frac{1}{\sigma^2} \int_0^\infty |\Psi(s)| \, ds,$$

where $\Psi(s) = \langle [x(t - s) - \langle x \rangle][x(t) - \langle x \rangle] \rangle$ is the autocorrelation function of signal $x(t)$ and $\sigma^2 = \Psi(0)$ its variance. The larger the $t_{\text{cor}}$ is, the more coherent $x(t)$ is. To distinguish between the correlation time $t_{\text{cor}}$ as a measure of coherence of an arbitrary signal, and the correlation time $\tau_c$ as a parameter of the coloured noise, we will in the following refer to the latter as noise correlation time.

Additionally we use a second quantity to measure the coherence of the oscillations and introduce the normalized standard deviation of ISIs $R_T$, defined by

$$R_T = \frac{\sqrt{\langle T_{\text{ISI}}^2 \rangle - \langle T_{\text{ISI}} \rangle^2}}{\langle T_{\text{ISI}} \rangle}.$$

For highly coherent oscillations the standard deviation of the ISIs is small, while for incoherent oscillations the standard deviation is large. This quantity is also referred to as the noise-to-signal ratio in the context of stochastic resonance (Pikovsky & Kurths 1997), coefficient of variation (Schwalger & Schimansky-Geier 2008) or normalized fluctuations (Beato et al. 2008).

Together with the correlation time $t_{\text{cor}}$, we now have two measures of coherence at hand. We will see that both quantities show consistent behaviour for an FHN system driven by coloured noise with small noise correlation time. However, the two quantities can exhibit contradictory behaviour, e.g. for an FHN system driven by coloured noise with large noise correlation time. The correlation time not only measures the coherence of oscillations, but also reaches high values for coherent phases without spiking. In contrast, the normalized standard deviation of ISIs by
definition only quantifies the coherence of oscillations; however, this quantity is only suitable for unimodal ISI distributions: for example, a bimodal distribution with two very pronounced peaks with considerable distance will yield a high value of the standard deviation, although the system shows very coherent oscillations with two distinct frequencies. We will see that in an FHN system subject to time-delayed feedback the ISI distribution becomes bimodal; hence the normalized standard deviation of ISIs is not applicable in that case.

(iii) Synchronization

Generally, synchronization means adjustment of time scales of oscillations in interacting subsystems (Rosenblum et al. 2001a; Balanov et al. 2009). Originally defined as the adjustment of the frequencies of self-sustained oscillators due to their weak interaction, the concept has been generalized to interacting noisy oscillators. In our case the coupling between the two subsystems can lead to synchronization. Here we consider two forms of stochastic synchronization: firstly, frequency synchronization or frequency locking, meaning a fixed ratio $n : m$ of the frequencies of the two subsystems, where $n$ and $m$ are integers; secondly, phase synchronization or phase locking, which is associated with a constant phase difference between the two subsystems.

In general, the two uncoupled systems will have different time scales, independent of each other and not rationally related. However, the coupling between the two subsystems can shift their time scales, so that their ratio is closer to a rational number $n : m$, where $n$ and $m$ are integers. As introduced above, the mean ISI $\langle T_i \rangle$, $i = 1, 2$, characterizes the time scales involved. Hence the ratio of mean ISIs $R_{\text{ISI}} = \langle T_1 \rangle / \langle T_2 \rangle$ provides a suitable measure for frequency synchronization. The closer this ratio is to the chosen rational number $n : m$, the more synchronized the systems are. In the following we will focus on 1:1 frequency synchronization and thus regard a system with $R_{\text{ISI}}$ close to unity as highly frequency synchronized, while $R_{\text{ISI}}$ far from unity characterizes systems with significantly different time scales and thus corresponds to weak frequency synchronization.

The mean ISIs $\langle T_i \rangle$, $i = 1, 2$, are calculated by averaging over the whole time series of the respective subsystem. Therefore frequency synchronization as defined above is associated with the time-averaged behaviour of the coupled oscillators and does not imply synchronous behaviour of the time series in the sense that both subsystems spike at the same time. Consider for example two identical uncoupled FHN systems driven by white noise of equal intensity: the ratio of mean ISIs equals unity; however, the spiking of one subsystem occurs completely independent of the other subsystem. Although both subsystems feature the same basic time scales, they are obviously completely independent and not in synchrony. To measure the instantaneous coordination of the interacting systems we therefore introduce the concept of phase synchronization or phase locking. First we need to define a phase for each oscillator; owing to the spiky nature of the oscillations this can be accomplished by using the spike times. The phase of each system is defined as

$$
\phi_j(t) = 2\pi \frac{t - t_{i-1}}{t_i - t_{i-1}} + 2\pi (i - 1), \quad j = 1, 2,
$$

(2.10)
where \( t_i \) is the time at which we observe a spike in the respective time series. The time of the spike may be defined as a local maximum or the start of the spike. Subtraction of the phases leads to the phase difference calculated as

\[
\Delta \phi_{n,m}(t) = \phi_1(t) - \frac{m}{n} \phi_2(t),
\]

where \( n \) and \( m \) are integers for which synchronization is suspected. In deterministic systems phase synchronization would be reflected in a locked phase, meaning that the phase difference is constant for all times. In stochastic systems such synchronization is not possible due to the noise. The phase difference is noisy and phase slips occur in the form of abrupt 2\( \pi \) jumps of the phase difference. However, the more synchronized the systems are, the longer the phase remains constant (apart from fluctuations due to noise). Thus, if synchronization takes place, the phase difference demonstrates plateaux occasionally interrupted by 2\( \pi \) jumps. To quantify the phase synchronization we choose the synchronization index, which is introduced in Rosenblum et al. (2001b). The synchronization index for \( n = m = 1 \) is defined as

\[
\gamma_{1,1} = \sqrt{\langle \cos \Delta \phi_{1,1}(t) \rangle^2 + \langle \sin \Delta \phi_{1,1}(t) \rangle^2},
\]

and so \( \gamma_{1,1} \) can vary between 0 and 1, with 0 representing no phase synchronization and 1 corresponding to perfect 1:1 phase synchronization.

### 3. Single FitzHugh–Nagumo system

(a) Coherence resonance with coloured noise

We shall first consider a single FHN system driven by coloured noise without time-delayed feedback control, corresponding to equation (2.1). Under white noise conditions this system exhibits CR (Hu et al. 1993; Pikovsky & Kurths 1997), i.e. the coherence of noise-induced oscillations becomes optimum (\( t_{\text{cor}} \) maximum, \( R_T \) minimum) for a certain finite value of the noise intensity. Here we consider CR under coloured noise on varying the noise correlation time.

A systematic overview of the effects upon the coherence of the resulting spike trains \( x(t) \) is shown in figure 4a: the correlation time \( t_{\text{cor}} \) and the normalized standard deviation of ISIs \( R_T \) are calculated for varying noise correlation time \( \tau_c \). Both the correlation time and the normalized standard deviation of ISIs are strongly affected by the noise correlation time \( \tau_c \) and show non-monotonic behaviour. The normalized standard deviation of ISIs is at a global minimum at \( \tau_c \approx 0.05 \), indicating an optimal value \( \tau_{\text{opt}} \) of the noise correlation time. The system shows CR with respect to the noise correlation time with a coherence maximum at \( \tau_c = 0.05 \), thus confirming the results in Casado (1997), Beato (2006) and Beato et al. (2008).

The correlation time \( t_{\text{cor}} \) of the \( x \)-variable, which has not been studied so far, shows consistent behaviour for \( \tau_c < 1 \): it reaches a local maximum at \( \tau_c = \tau_{\text{opt}} \), thus reconfirming the coherence maximum obtained from the normalized standard deviation of ISIs. However, for \( \tau_c > 1 \) the two coherence measures
exhibit contradictory behaviour: while the normalized standard deviation of ISIs monotonically grows from its minimum onward, thus indicating a monotonic decrease of the coherence, the correlation time has a turning point at $\tau_c \approx 1$ and also grows monotonically for $\tau_c > 1$, which corresponds to a monotonic increase of the coherence. We shall investigate the origin of this diverging behaviour later on by looking at the autocorrelation function and the distribution of ISIs.

Yet, beforehand, let us explain the origin of the CR with respect to the correlation time by having a close look at the threshold dynamics of the FHN system: the additive noise term in the inhibitor variable can be viewed as a shifting of the inhibitor nullcline to the right or left, rendering the system effectively oscillatory when the Hopf point is reached. This means that each noise parameter acts as a kind of dynamically changing bifurcation parameter, switching the system between the excitable and the oscillatory regimes.

To clarify this, we first focus on the effect of a single perturbation. Initially, we prepare the system in its fixed point and apply at time $t$ a perturbation of duration $\Delta$ and constant amplitude $A$. This perturbation will shift the $y$-nullcline to $x = -a + A$ at time $t$; after the time interval $\Delta$ the nullcline will ‘jump’ back to its original state. Though simplified, this picture will help us to qualitatively understand the basic dynamics.

Firstly, we use it to explain the increase of coherence when going from very small values of the correlation time to the coherence maximum: for white noise or coloured noise with small correlation times, the perturbations are rather short, so that they trigger one spike at most, before the system returns to the excitable regime. Whether or not a full spike is triggered depends on both the duration and the amplitude of the perturbation: when the $y$-nullcline is shifted quickly beyond the threshold, the system will follow with some delay. If the perturbation is too short, the system will not have reached the threshold and will simply return to the fixed point in a relaxation motion. Therefore the mean duration of the perturbations is critical. On the other hand, the amplitude determines how fast the system reacts and moves beyond the threshold. Hence spikes can also be induced by very short but large perturbations.
For white noise the mean duration of the perturbations is very short, but with increasing noise intensity—and therefore increasing mean amplitude—the mean number of sufficiently large spikes increases. Accordingly, the activation time decreases rapidly with the noise amplitude according to the Kramers formula \( \langle t_a \rangle \sim \exp(\text{const} \times D^{-2}) \), leading to a most coherent spike train at the coherence maximum (Pikovsky & Kurths 1997).

For coloured noise, on the other hand, we keep the intensity and accordingly the mean amplitude constant. The mean duration of the disturbances, however, increases with increasing noise correlation time. Therefore the mean number of spike-triggering perturbations also grows, leading to a decreasing activation time.

From that point of view the origin of the increase of coherence is similar for the white noise and the coloured noise cases. Though different features of the noise—the mean amplitude and the mean duration of the perturbations, respectively—are varied, both cases have a common feature: the number of spike-triggering perturbations grows, leading to a decreasing activation time.

The decline of coherence for \( \tau_c > \tau_{\text{opt}} \) can be understood in a similar manner. The mean duration of the disturbances still grows, but so does the duration of unsuccessful perturbations, during which the system does not spike. The duration of these intervals is irregular, and their variance grows with \( \tau_c \), increasing both the excursion time and the activation time and their variance. The spikes are separated by increasing irregular time intervals, during which the nullcline is shifted to the left of the unperturbed fixed point.

For large \( \tau_c \) a completely new effect comes into play: the nullcline is shifted beyond the Hopf point long enough to trigger more than one spike; accordingly the system is effectively oscillatory for the duration of the perturbation and fires regularly, which leads to clusters of spikes. These bursts are spaced by intervals of about the same duration, in which the fluctuations drive the system above the Hopf point, so that no spikes are triggered. The regularity of these subsequent active and inactive phases is low. The period of oscillations during these bursts can vary noticeably and is related to the noise variable \( \eta(t) \): when \( \eta(t) \) undershoots the threshold only by a small amount, the period is rather large; the period is minimum when \( \eta(t) \) is far below the threshold. In §2a it was shown that in the oscillatory regime the period of oscillations depends on the bifurcation parameter \( a \). In this case the noise acts as a slowly varying bifurcation parameter and thus determines the period of oscillations according to equation (2.7). Figure 4a shows that the correlation time of these bursting spike trains strongly increases with increasing \( \tau_c \). The coherence of the oscillations in the active phases increases with increasing \( \tau_c \). The influence of the inactive phases is even more important; the strong increase of the correlation time can be shown to be mostly due to the very coherent inactive phases: the slowly exponentially decaying autocorrelation function of the coloured noise is reflected in the autocorrelation function of the \( x \)-variable, so that the integral over this function is large.

Figure 4b shows the correlation time \( t_{\text{cor}} \) as a function of the noise intensity \( \sigma^2 \) and the noise correlation time \( \tau_c \). The most striking feature in figure 4b is the linear structure of the \((\sigma^2, \tau_c)\)-plane for noise correlation times \( \tau_c < 0.1 \). The black line through the coherence maximum in the log–log plot serves as a guide to the eye, indicating a power-law relation \( \sigma^2 \times \tau_c = \text{const} \). Thus in this regime the correlation time (and also the normalized standard deviation of ISIs) is a
function of $D = \sigma^2 \tau_c$: $t_{\text{cor}} = f(D)$ and $R_T = f(D)$. From that point of view the coherence depends only on the variance of the white noise source driving the Ornstein–Uhlenbeck process.

This can also be shown analytically by a simple approximation. Since we use an Ornstein–Uhlenbeck process to generate the coloured noise $\eta(t)$, we can rewrite the two-variable FHN system driven by coloured noise as a three-variable system

$$\varepsilon \dot{x} = x - \frac{x^3}{3} - y, \quad (3.1)$$

$$\dot{y} = x + a + \eta \quad (3.2)$$

and

$$\tau_c \dot{\eta} = -\eta + \sqrt{2 \sigma^2 \tau_c} \xi(t) \quad (3.3)$$

driven by white noise $\xi(t)$ and featuring three different time scales $\varepsilon$, 1 and $\tau_c$.

For $\tau_c < 0.1$ the time scale of $\eta(t)$ is at least one order of magnitude smaller than the time scale of the $y$-variable; hence, we can eliminate the fast variable $\eta$ adiabatically (Gardiner 2002). Compared with the slow motion along the activator nullcline the change of the noise variable $\eta$ is rather fast. We assume $\eta$ to relax rapidly and approximate the dynamics by setting the left-hand side of equation (3.3) equal to zero, so that

$$\eta(t) = \sqrt{2 \sigma^2 \tau_c} \xi(t), \quad (3.4)$$

and thus we arrive at a simple FHN system driven by white noise

$$\varepsilon \dot{x} = x - \frac{x^3}{3} - y \quad (3.5)$$

and

$$\dot{y} = x + a + \sqrt{2 \sigma^2 \tau_c} \xi(t) \quad (3.6)$$

with noise intensity

$$D_{\text{wn}} = \sqrt{2D} = \sqrt{2 \sigma^2 \tau_c}. \quad (3.7)$$

Since the time scale of the $x$-variable $\varepsilon = 0.01$ is even smaller, one should approximate its dynamics as well. This reduces the dynamics to a motion along the left and right branches of the cubic activator nullcline. In fact, Lindner et al. (2004) used this approximation to derive an analytical approximation for the CR curve for $R_T$ using Fokker–Planck equations (Lindner & Schimansky-Geier 1999; Lindner et al. 2004). They also presented an approximation for the position of the coherence maximum

$$D_{\text{max}} \approx 2 \left( 2 - \sqrt{3} \right) \Delta U_1, \quad (3.8)$$

where

$$U_1(y) = \frac{(y - a)^2}{2} \quad (3.9)$$

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Figure 5. (a) Power spectral density $S(T)$ versus the inverse frequency $T = 1/f$ and the noise correlation time $\tau_c$. (b) Distribution of ISIs $f_{ISI}$ versus $T_{ISI}$ and $\tau_c$ for an FHN system driven by coloured noise, corresponding to figure 4a. Parameters are $a = 1.05$, $\epsilon = 0.01$ and $\sigma^2 = 0.04$.

is the effective potential corresponding to the motion along the left branch along the $x$-nullcline, and

$$\Delta U_l = U_l(y_-) - U_l(y^*),$$  \hspace{1cm} (3.10)

where $y_-$ is the threshold value, where the motion jumps to the right branch of the activator nullcline, and $y^*$ is the value at the fixed point. For our system this yields $D_{\text{max}} \approx 0.0023$. Hence, we can approximate the position of the coherence maximum for CR with respect to the noise correlation time and to the noise intensity by the relation

$$D_{\text{max}} = \sigma^2 \tau_c \approx 0.0023. \hspace{1cm} (3.11)$$

As figure 4b shows, equation (3.11), plotted as the black line, fits very well the coherence maximum.

We emphasize that we do not take the white noise limit of the coloured noise, since the time scale of the coloured noise is not small compared to all system time scales, and, furthermore, would lead to white noise of zero intensity in the noise scaling we use. We rather approximate only the dynamics and show that for $\tau_c < 0.1$ we can replace the coloured noise with white noise of variance (or intensity) $2\sigma^2 \tau_c$, which is equivalent with respect to the effects on the coherence of the spike trains. Although the mean amplitude and duration of the noise perturbations can be controlled separately by $\sigma^2$ and $\tau_c$, respectively, for $\tau_c < 0.1$ both parameters have similar effects upon the number of spike-triggering perturbations.

Figure 5a shows the Fourier power spectral density $S(T)$ as a function of the period $T = 1/f$ for varying $\tau_c$. The peak corresponds to the main period of the oscillations and therefore to the time scale of the oscillations. One can see the time scale moving towards lower periods $T$ with increasing $\tau_c$, but for $\tau_c > 5$ the main period remains constant.

The distribution of ISIs is shown in figure 5b. Below a certain threshold the ISI density is zero, corresponding to the absence of spikes. For increasing $\tau_c$ the peak of the distribution decreases until $\tau_c$ equals approximately 2. For $\tau_c > 2$ the peak position remains constant at $T_{ISI} = 2.7$. The peak at $T_{ISI} = 2.7$ becomes sharper with increasing $\tau_c$, reaching its maximum at $\tau_c = 100$, which seems to contradict
the increase of the normalized standard deviation of ISIs $R_T$ in the CR curve in figure 4a. The reason for the large variance is the appearance of large ISIs, which are in fact inter-burst intervals, since the system shows bursting behaviour for $\tau_c = 100$. The mean ISI on the other hand remains nearly constant, since the large ISIs are very rare. Thus the normalized standard deviation of ISIs grows with increasing $\tau_c$.

The apparently contradictory behaviour of the two coherence measures $\tau_c$ and $R_T$ for large correlation times can therefore be explained by their different reactions to the long inactive phases. The normalized standard deviation of ISIs, which by definition only measures the coherence of oscillations, neglects the inactive phases, or rather treats them as strong disturbances to the coherence. On the other hand, the correlation time measures the coherence of both phases, and is very sensitive to coherent inactive phases.

(b) Control of coherence resonance by delay time and noise correlation time

Now we apply time-delayed feedback control with delay time $\tau$ and control gain $K$ to the inhibitor in equation (2.1). In figure 6a the correlation times $t_{\text{cor}}$ is shown for varying delay times $\tau$ for two different values of $\tau_c$ (solid lines); $\tau = 0$ corresponds to the uncontrolled system. For both values of $\tau_c$ the correlation time $t_{\text{cor}}$ oscillates with varying delay time $\tau$. While oscillating, $t_{\text{cor}}$ increases until it saturates for $\tau \approx 20$. Interestingly, for $\tau_c = 0.05$ (grey line) the correlation time reaches a global minimum at $\tau \approx 2.5$, even below the correlation time of the uncontrolled system. This means that one can not only enhance but also diminish the coherence by choosing $\tau$ appropriately. For $\tau_c = 100$ (black line) the modulation is weaker but qualitatively similar.

After investigating the effects of feedback control on the coherence we now proceed to study the time scales introduced by the control. The power spectrum reveals the whole frequency content of the system; the positions of its maxima
correspond to the dominant time scales of the system. It is changed in a characteristic way by the time-delayed feedback control. Generally, the variation of $\tau$ leads to a variation of the number, position, height and width of spectral peaks. The feedback control shifts the existing peak nearly linearly and introduces new peaks, which are also shifted almost linearly (entrainment of time scales by delayed feedback; Balanov et al. 2004; Janson et al. 2004; Prager et al. 2007).

We calculate the power spectrum of the activator variable with varying time delay $\tau$ and extract the periods of the dominant spectral peaks $T_{\text{max}}$ and plot them versus $\tau$ (figure 6a) for $\tau_c = 0.05$ (full circles) and $\tau_c = 100$ (open circles). The evolution of $T_{\text{max}}$ exhibits a piecewise linear dependence upon $\tau$: starting at $T_{\text{max}} = T_0$ for $\tau = 0$ it monotonically increases until at $\tau \approx 2.5$ the first newly introduced peak becomes dominant and $T_{\text{max}}$ abruptly drops to the lower period of the next peak branch. From then it again increases with $\tau$, although with a slightly smaller slope. These quick transitions to the successive lower branches occur roughly every $T_0$ time units, and each subsequent entrainment happens at a lower slope. Thus the graph of $T_{\text{max}}$ as a function of $\tau$ shows a piecewise linear behaviour, with declining slopes for each subsequent line segment.

Figure 6a shows that the positions of the maxima of coherence (solid lines) coincide with the $\tau$-values where the period of the dominant spectral peak $T_{\text{max}}$ becomes equal to $T_0$. Thus time-delayed feedback control induces various time scales dependent on the delay time $\tau$; these oscillations are most coherent when the dominant period of the induced oscillations is equal to the period of the uncontrolled system. The delayed feedback control can be regarded as being in resonance with the major time scale of the noise-driven system, when the delay time is chosen appropriately.

We have seen that the coherence of the time series oscillates with varying delay time $\tau$. The period of these oscillations depends on the main period of the uncontrolled system, which in turn is influenced by the noise correlation time $\tau_c$.

On the other hand the uncontrolled system exhibits CR with respect to the noise correlation time $\tau_c$. Therefore the coherence of the time series is heavily influenced by the temporal correlations of the driving noise. How this CR can be controlled by time-delayed feedback, or, generally speaking, the interplay of these two types of coherence modulations, is the subject of this section.

In order to investigate the interplay of the delay time $\tau$ with the noise correlation time $\tau_c$ systematically, we vary independently both $\tau$ and $\tau_c$ and calculate the correlation time $t_{\text{cor}}$ of the $x$-variable. Figure 6b summarizes how the correlation time $t_{\text{cor}}$ is modulated as a function of the delay time $\tau$ and the noise correlation time $\tau_c$.

A cut through this figure for constant $\tau = 0$ corresponds to the CR curve with respect to $\tau_c$ (figure 4a). The plots of the correlation time for varying delay time $\tau$ (figure 6a; grey and black lines) are represented by cuts for constant $\tau_c = 0.05$ and $\tau_c = 100$, respectively. One can clearly see the modulations of $t_{\text{cor}}$ for varying $\tau$ at $\tau_c = 0.05$ and 100. The modulations in the intermediate regime ($\tau_c \approx 1$) are hardly recognizable in this representation since they are of relatively small amplitude compared to the two other cases. We find that the positions of the maxima and minima of the CR with respect to $\tau_c$ are hardly changed by the feedback control. However, the frequency of the modulations for varying delay time $\tau$ is considerably affected by the noise correlation time $\tau_c$. The white lines in figure 6b mark the positions of the coherence maxima with respect to $\tau$ for

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different $\tau_c$; the distance between two subsequent maxima, which we have related to the dominant period of the uncontrolled system, is clearly decreasing with increasing $\tau_c$.

Thus the interplay of time-delayed feedback and temporal correlations of the coloured noise can be characterized as follows. Firstly, the coloured noise and the feedback control do not influence each other directly, since the feedback signal is calculated from the $y$-variable and does not depend directly on the driving noise, but rather on the frequency of oscillations present in the uncontrolled system. Moreover, the coloured noise does not introduce a separate preferred frequency of oscillations, but shifts the existing spectral peak of the system. Nevertheless, this shift of the main spectral peak is reflected in the frequency of the modulations due to the time-delayed feedback; in that sense, one can observe an indirect interaction of the feedback control and the coloured noise. On the other hand, the CR with respect to the noise correlation time is only marginally affected; the positions of the maximum and minimum hardly vary for varying delay times $\tau$ of the feedback control.

4. Coupled FitzHugh–Nagumo systems

Next, we will study the coupled system equations (2.8), where only subsystem 1 is driven by coloured noise. In order to investigate the influence of coloured noise on the coupled system, we will keep the white noise intensity in the second subsystem constant and vary the parameters of the noise input to the first subsystem. Unless stated otherwise, we will fix the noise intensity of the second subsystem to $D_2 = 0.09$, corresponding to rather large time scales of oscillations of the single uncoupled system. We shall see that, by variation of the noise correlation time in the first subsystem only, we can significantly influence the time scales and coherence of the oscillations of both subsystems.

(a) Variation of noise correlation time

First we consider the coupled system without time-delayed feedback ($K = 0$). To investigate the effects of temporal noise correlations in the coupled systems, we first fix the noise intensity at $\sigma^2 = 0.04$ and vary the noise correlation time $\tau_c$. To characterize the time scales involved we calculate the average ISI $\langle T_i \rangle$ of both subsystems as well as their ratio $R_{ISI}$ (figure 7a(i), b(i), c(i), d(i)) for varying $\tau_c$. Moreover, the influence on the coherence of the local dynamics, quantified by the correlation time $t_{\text{cor}}'$ from $x_1$ and $x_2$, and of the global behaviour ($t_{\text{cor}}$ estimated from the sum $x_\Sigma = x_1 + x_2$) are shown in figure 7a(ii), b(ii), c(ii), d(ii). Additionally we calculate the synchronization index $\gamma_{1.1}$ to measure the effects on the phase synchronization (grey line in figure 7a(ii), b(ii), c(ii), d(ii)). To account for the influence of varying coupling strength $C$ we repeat this procedure for various values of $C$ and compare the results (figure 7a–d).

Considering the effects of both varying noise correlation time and different coupling strength $C$ we will sequentially step through figure 7(a)–(d) and describe the characteristic features. In each panel the noise correlation time is varied in logarithmic scaling from $\tau_c = 0.001$, where the first subsystem spikes very rarely without input from the second subsystem, to $\tau_c = 100$, where the single subsystem exhibits bursting behaviour.
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Figure 7. Time scales and coherence of noise-induced oscillations in two coupled FHN systems, driven by coloured noise (first subsystem) and white noise (second subsystem) according to equations (2.8), versus correlation time $\tau_c$ at $\sigma^2 = 0.04$: (a(i), b(i), c(i), d(i)) average ISI $\langle T_1 \rangle$ (solid line), $\langle T_2 \rangle$ (dashed line), and their ratio (circles); (a(ii), b(ii), c(ii), d(ii)) correlation time $t_{cor}$ obtained from $x_1$ (solid line), $x_2$ (dashed line), and $x_{\Sigma}$ (circles). Synchronization index $\gamma_{1,1}$ (grey line). See text for details ($a = 1.05$, $\varepsilon_1 = 0.005$, $\varepsilon_2 = 0.1$, $D_2 = 0.09$, $\sigma^2 = 0.04$). Coupling strength: (a) $C = 0.04$, (b) $C = 0.1$, (c) $C = 0.2$ and (d) $C = 0.5$.

The system shows the same behaviour at $\tau_c = 0.001$ for all considered values of $C$: for $\tau_c = 0.001$ the noise input to the first subsystem induces hardly any spikes by itself. However, due to the coupling to the second subsystem, the first subsystem exhibits forced oscillations whose properties are defined by those of the second subsystem. Thus the subsystems are perfectly frequency and phase synchronized, which is reflected in the ratio of average ISIs $R_{ISI} \approx 1$ and the synchronization index $\gamma_{1,1} \approx 1$.

(i) Very weak coupling ($C = 0.04$).

With $\tau_c$ increasing from zero the time scales of the two subsystems separate, until at $\tau_c \approx 0.1$ their difference is maximum and upon further increasing $\tau_c$ they approach each other again (see figure 7a(i), b(i), c(i), d(i)). The average ISI of the first subsystem is governed by the coloured noise input, while the time scales of the second subsystem are only weakly affected. Correspondingly the ratio of
average ISIs drops from 1 to approximately 0.5 at $\tau_c \approx 0.1$ and afterwards again increases slightly. Since the time scales of the second subsystem are only weakly changed, the evolution of the ratio of average ISIs is dominated by the change of time scales in the first subsystem.

The effects on the coherence of the system are also dominated by the effects on the first subsystem (see figure 7a(ii), b(ii), c(ii), d(ii)): the graph of correlation time $t_{cor}$ of the first subsystem shows the characteristic shape of the CR curves observed in §3, featuring a local maximum at $\tau_c \approx 0.04$, a local minimum at $\tau_c \approx 1$ and a strong increase for large $\tau_c$ due to the bursting behaviour. The correlation time of the second subsystem is only weakly influenced with a small maximum at $\tau_c \approx 0.005$ and a generally increasing trend. Thus the correlation time of the sum is dominated by the changes to the first subsystem, showing a local maximum at $\tau_c \approx 0.04$ and a strong increase for large $\tau_c$. The synchronization index shows a strong decrease from 1 to a minimum of approximately 0.1 at $\tau_c \approx 0.1$, followed by a slight increase for further increasing $\tau_c$. The very low value of the ratio of average ISIs and the synchronization index reflect the weak coupling strength: when the first subsystem starts to develop an independent dynamics with increasing $\tau_c$, the second system only rarely responds with a spike to the spikes of the first subsystem. Accordingly the spiking frequency of the second subsystem is only lowered weakly (cf. figure 7a(i), b(i), c(i), d(i)).

(ii) Intermediate coupling ($C = 0.1$)

The influence of the first subsystem on the second subsystem is increased, leading to similar effects. The average ISIs of the second subsystem are equal to the first subsystem up to $\tau_c = 0.01$ and show a similar trend upon further increasing $\tau_c$. The ratio of average ISIs is monotonically decreasing, reaching its minimum of $R_{ISI} \approx 0.7$ at $\tau_c = 100$. For $\tau_c < 0.01$ the system shows almost complete frequency synchronization. An interesting effect can be observed for the correlation time of the second subsystem: the position of the maximum is shifted to the maximum of the first subsystem. Moreover, it is much more pronounced than for $C = 0.04$ and even more than for $C = 0.2$. Owing to the shift of the maximum all three graphs of $t_{cor}$ exhibit a maximum at the same position, leading to a very large correlation time of the sum. In fact, it is even larger than for higher values of the coupling strength (cf. figure 7c,d). However, although the time series of the $x$ variables of both subsystems and their sum are most coherent at $\tau_c \approx 0.04$, this is not reflected in the ratio of average ISIs nor in the synchronization index, which do not show any significant change in behaviour at this point. The synchronization index monotonically decreases with increasing $\tau_c$, exhibiting similar behaviour as the ratio of average ISIs.

(iii) Strong coupling ($C = 0.2$)

For coupling strength $C = 0.2$ the subsystems show synchronized behaviour for a wide range of $\tau_c$. The time scales of the two subsystems separate only for $\tau_c \gtrsim 0.5$; correspondingly the ratio of average ISIs equals 1 up to $\tau_c \approx 0.5$, thus indicating perfect frequency synchronization, and drops to approximately 0.9 for higher $\tau_c$, which corresponds to strong frequency synchronization. The synchronization index exhibits similar behaviour, showing perfect phase synchronization for $\tau_c < 0.1$, and decreasing afterwards to approximately 0.85 for...
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\[ \tau_c = 0.85. \] Interestingly, the position of the maxima of all three graphs of \( t_{\text{cor}} \) is shifted to \( \tau_c \approx 0.1 \), showing that the coupling can influence the CR not only in the ‘resonating’ second subsystem, but also in the first subsystem, to which the CR-inducing noise source is applied. Furthermore the maxima are less pronounced than for \( C = 0.1 \).

(iv) Very strong coupling (\( C = 0.5 \))

In this case the two subsystems show complete frequency and phase synchronization, indicated by the ratio of average ISIs and the synchronization index, which are both equal to unity for the whole range of \( \tau_c \). Correspondingly the time scales of the two subsystems are equal for all values of \( \tau_c \). The positions of the maxima of the correlation times are shifted further to \( \tau_c \approx 0.2 \), and the maxima are only weakly pronounced compared with smaller coupling strengths. Although the two subsystems are completely frequency and phase synchronized and thus spike at the same times, the correlation times of the two subsystems are equal only for \( \tau_c < 10 \). For \( \tau_c > 10 \) the correlation time of the first subsystem increases strongly with increasing \( \tau_c \), while the correlation time of the second subsystem is only moderately increased. This can be explained using the results from §3. The regime of large \( \tau_c \) is characterized by bursts of spikes separated by long inactive phases without any spiking. We observed that the correlation time increases strongly for large \( \tau_c \) mainly due to the highly coherent inactive phases. The correlation time of the first subsystem demonstrates this mechanism. In the second subsystem a different situation occurs. Owing to the strong coupling the systems spike synchronously; thus each spike occurring in the first subsystem during a burst is transferred to the second one. During the inactive phases, however, the time series of the second subsystem is strongly incoherent as a result of fluctuations of the inherent white noise. Thus the correlation time of the second subsystem measures only the increasing coherence of oscillations during the bursts, which leads to the moderate increase of the correlation time for large \( \tau_c \).

To summarize this subsection, we can characterize the effect of variation of the noise correlation time as follows. Temporal correlations of the driving noise in one system only can significantly influence the time scales and coherence of both subsystems, provided that the coupling strength is sufficiently large. In particular, we observed CR with respect to the correlation time not only in the coloured noise-driven subsystem, but also in the second one driven by a white noise source. For all considered values of the coupling strength \( C \) the ratio of average ISIs and the synchronization index showed similar dependence on the noise correlation time \( t_{\text{cor}} \). Thus, phase synchronization occurs together with frequency synchronization and vice versa.

(b) Variation of the noise intensity and the coupling strength

In §4a we have varied the noise correlation time \( \tau_c \), the noise intensity \( \sigma^2 \) and the coupling strength \( C \) independently and have thus shown the dependences of frequency and phase synchronization on all three parameters. In this subsection, however, we take another approach by continuously varying the coupling strength and the noise intensity independently. To account for the influence of the noise correlation time we vary \( \sigma^2 \) and \( C \) independently for various fixed values of \( \tau_c \) and calculate the ratio of average ISIs \( R_{\text{ISI}} \) and the synchronization index \( \gamma_{1,1} \).
Figure 8. Two coupled FHN systems driven by coloured noise in the first subsystem and white noise in the second one: (a(i), b(i)) ratio of average ISIs \( \langle T_1 \rangle / \langle T_2 \rangle \) from the two systems; and (a(ii), b(ii)) synchronization index \( \gamma_{1,1} \) versus noise intensity \( \sigma^2 \) and coupling strength \( C \). (a) \( \tau_c = 0.03 \) and (b) \( \tau_c = 100 \).

This representation has the advantage that we can compare these \((\sigma^2, C)\)-planes directly with the corresponding \((D_1, C)\)-planes for white noise-driven coupled FHN systems. Furthermore, we will use this picture when we apply time-delayed feedback for various noise correlation times.

In figure 8 we show the frequency and phase synchronization for (a) \( \tau_c = 0.03 \) and (b) \( \tau_c = 100 \). Note that we have varied the \( \sigma^2 \) range appropriately for each case. For \( \tau_c = 0.03 \) one can clearly see the 1:1 synchronization tongue (light area) for both frequency and phase synchronization. For high noise intensity \( \sigma^2 \) the frequencies of the two subsystems are considerably different; therefore a large coupling strength is necessary to synchronize the two subsystems. For small noise intensity \( \sigma^2 \) the spiking in the first subsystem is induced by signals from the second subsystem; thus synchronization also occurs for low coupling strength \( C \). Also for \( \tau_c = 100 \) the \((\sigma^2, C)\)-plane exhibits a tongue-like structure of 1:1 phase and frequency synchronization. The plots show similar structure as for \( \tau_c = 0.03 \), except for minor changes: for \( \tau_c = 100 \) and constant coupling strength \( C = 0.1 \) the ratio of average ISIs \( R_{\text{ISI}} \) decreases more slowly with increasing \( \sigma^2 \) than for smaller \( \tau_c \) (see figure 8a,b).

To emphasize the relationship with the case of white noise, we have directly compared the results for \( \tau_c = 0.01 \) with the results for white noise: the ratio of average ISIs and the synchronization index as functions of noise intensity \( D_1 \) and coupling strength \( C \) of two coupled FHN systems driven by white noise are very similar to the result for coloured noise with \( \tau_c = 0.01 \) and rescaled noise intensity \( D_{\text{wn}} = \sqrt{2\sigma^2 \tau_c} \) according to equation (3.7).

Generally, for all considered values of \( \tau_c \), frequency synchronization is accompanied by phase synchronization and vice versa. The noise correlation time \( \tau_c \) has a strong influence on the synchronization of the two subsystems; however,
if one accounts for the interplay of the noise correlation time and intensity, and incorporates the additional influence of $\sigma^2$ by appropriately adjusting the range of the noise intensity, the $(\sigma^2, C)$-planes show similar structure.

In the following section we will apply time-delayed feedback control to the coupled FHN system. Thereby we will use these $(\sigma^2, C)$-planes to identify parameters for which the systems are weakly, moderately or strongly synchronized in the uncontrolled case, and investigate the influence of control on the synchronization.

(c) Control of a moderately synchronized system

In this subsection we investigate whether the feedback applied to only one of the interacting subsystems, in other words locally applied feedback, is able to manipulate the global properties of coupled oscillators. In analogy to Hauschildt et al. (2006), where the control of white noise-driven coupled excitable systems was studied, we apply time-delayed feedback to the first subsystem alone, while the second subsystem is only indirectly influenced by the control via the coupling between the two subsystems. We will investigate the effects on global time scales, on coherence, and especially on synchronization. In particular, we will focus on the influence of the temporal noise correlations quantified by the noise correlation time $\tau_c$, and compare our results with the case of white noise, where no temporal correlations are present in the driving noise source. Our system is given by equations (2.8).

In §3 we have observed that the noise correlation time $\tau_c$ significantly influences the synchronization of the two subsystems. Thus we cannot vary $\tau_c$ without simultaneously changing the synchronization of the uncontrolled system. To investigate the influence of time-delayed feedback on a moderately, weakly and strongly synchronized system, we therefore take the following steps. For small noise correlation time $\tau_c = 0.03$ as well as for large noise correlation time $\tau_c = 100$ we identify pairs of parameter values $(\sigma^2, C)$ for which the two systems show (i) weak, (ii) moderate, and (iii) strong synchronization. We use the respective $(\sigma^2, C)$ planes (figure 8a for $\tau_c = 0.03$ and figure 8b for $\tau_c = 100$) to select pairs of parameters at which the two systems are (i) far away from, (ii) closer to, and (iii) almost inside the 1:1 synchronization region. In this section we study in detail the effects of time-delayed feedback on a moderately synchronized system. For weakly and strongly synchronized systems we have found similar results (not shown).

For large noise correlation time $\tau_c = 100$ we select the parameter values $\sigma^2 = 0.2$ and $C = 0.2$ (cf. figure 8b), and for small correlation time $\tau_c = 0.03$ parameter values $\sigma^2 = 7$ and $C = 0.2$ (cf. figure 8a). Our aim is to find out if the feedback can enhance or suppress the synchronization of the subsystems, and make their global dynamics more or less coherent. In particular, we are interested in whether perfect 1:1 synchronization can be induced by the local feedback control, and whether the synchronization can be destroyed.

Figure 9 shows the ratio of average ISIs $R_{\text{ISI}}$ (figure 9a(i), b(i), c(i)) and the synchronization index $\gamma_{1,1}$ (figure 9a(ii), b(ii), c(ii)), for wide ranges of both the feedback delay time $\tau$ and control strength $K$. Figure 9(a) corresponds to white noise-driven coupled oscillators (reproducing fig. 9 from Hauschildt et al. 2006), figure 9(b,c) correspond to coloured noise-driven coupled oscillators with noise.
correlation times $\tau_c = 0.03$ and 100, respectively. The light areas are associated with 1:1 synchronization; the borders of the plane at $K = 0$ and $\tau = 0$ represent the uncontrolled state of the system without feedback.

As seen from figure 9 the case of white noise and the case of small $\tau_c = 0.03$ are very similar, which is in accordance with the results from §3, where we showed that coloured noise with $\tau_c < 0.1$ has similar effects as white noise of appropriate intensity $D_{\text{wn}} = \sqrt{2\sigma^2\tau_c}$ (see equation (3.7)). For $\tau = 0.03$ and $\sigma^2 = 7$ this yields $D_{\text{wn}} \approx 0.65$, which is in good accordance with the noise intensity $D = 0.6$ for white noise. In both cases the locally applied feedback is able to increase the 1:1 synchronization considerably with suitable feedback parameters. On the other hand, for $\tau \approx 2.5$ (black area) 1:1 synchronization is suppressed.

For $\tau_c = 100$ the effects are qualitatively similar to some extent: the delayed feedback is able to increase the synchronization considerably, and for $\tau \approx 2.5$ the synchronization can be suppressed. However, some crucial differences can be observed. Firstly, the synchronization area for $\tau < 2$ has a different shape; for $\tau_c = 100$ the system can reach perfect 1:1 frequency synchronization for $\tau \approx 1.9$, while for $\tau_c = 0.03$ the maximum of synchronization occurs at different $\tau \approx 0.8$ and does not reach perfect synchronization. Furthermore, for $\tau_c = 100$ the synchronization can only be suppressed for $K < 1$, and for $K > 1$ further modulations of the ratio of average ISIs occur, displaying a second local maximum of synchronization for $\tau \approx 4$ and $K > 1.5$.

Next, we keep $K = 1.5$ constant and vary $\tau$, corresponding to a cut through figure 9c at $K = 1.5$, for which pronounced modulations of the frequency are encountered. Average ISIs, their ratio, correlation times and synchronization index are shown in figure 10a. An increase of $\tau$ from zero leads to an increase of both average ISIs; however, $\langle T_1 \rangle$ grows faster than $\langle T_2 \rangle$, thus their ratio increases.

Figure 9. Effect of delayed feedback on frequency and phase synchronization between the two subsystems for moderately synchronized subsystems: (a(i), b(i), c(i)) ratio of average ISIs $\langle T_1 \rangle / \langle T_2 \rangle$ from the two systems; (a(ii), b(ii), c(ii)) synchronization index $\gamma_{1,1}$ versus control strength $K$ and delay time $\tau$. (a) First subsystem driven by white noise; parameters $D_1 = 0.6$, $C = 0.2$. (b,c) First subsystem driven by coloured noise; parameters (b) $\tau_c = 0.03$, $\sigma^2 = 7$, $C = 0.2$ and (c) $\tau_c = 100$, $\sigma^2 = 0.2$, $C = 0.2$. 

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Figure 10. (a) Time scales, coherence and synchronization index of noise-induced oscillations in two coupled FHN systems, driven by coloured noise (first subsystem) and white noise (second subsystem) according to equations (2.8), versus delay time $\tau$ of the local feedback at $K = 1.5$, $\tau_c = 100$, $\sigma^2 = 0.2$ and $C = 0.2$, corresponding to moderately synchronized systems. (b) Corresponding plot for two moderately synchronized coupled FHN systems driven by white noise with noise intensity $D = 0.6$. (a(i), b(i)) Average ISI $\langle T_1 \rangle$ (solid line), $\langle T_2 \rangle$ (dashed line), and their ratio (circles). (a(ii), b(ii)) Correlation time $t_{\text{cor}}$ obtained from $x_1$ (solid line), $x_2$ (dashed line), and $x_\Sigma$ (circles). Synchronization index $\gamma_{1,1}$ (grey line).

almost up to 1 at $\tau \approx 1.8$, in contrast to the case of white noise (figure 10b). After a decrease of both average ISIs for $\tau \geq 1.5$, they increase again for $\tau \geq 2.5$. However, $\langle T_2 \rangle$ shows a clear maximum at $\tau \approx 3$, while $\langle T_1 \rangle$ only increases slightly; hence, their ratio shows a pronounced minimum. For further increasing $\tau$, they both oscillate, but with different frequencies, resulting in strong modulations of their ratio $R_{\text{ISI}}$. The synchronization index $\gamma_{1,1}$ generally exhibits very similar behaviour as the ratio of ISIs. Hence both frequency and phase synchronization show strong sustained modulations with increasing $\tau$, in contrast to white noise (figure 10b), where the modulations are less pronounced and, moreover, have a larger period. The correlation time of the first subsystem is very large and only slightly affected by the control, which is characteristic for large noise intensities and large noise correlation time. Interestingly the correlation time of the second subsystem shows two clear maxima at $\tau \approx 2.5$ and 7. The correlation time of the sum is almost constant due to the high constant value of the correlation time of the first subsystem.

(d) Control of synchronization by delay time and noise correlation time

While we have kept the noise correlation time $\tau_c$ constant in the previous sections, we now take a different approach: for constant noise intensity $\sigma^2$, coupling strength $C$ and control strength $K$, we vary $\tau_c$ and the delay time $\tau$ independently and investigate the interplay of the two parameters with respect to frequency and phase synchronization. As shown above the noise correlation time $\tau_c$ influences the synchronization of the uncontrolled system. We choose two pairs of parameters: (i) $C = 0.2$ and $\sigma^2 = 1$, for which the synchronization...
Figure 11. Effect of delayed feedback on frequency and phase synchronization between the two subsystems at $\sigma^2 = 1$ and $C = 0.2$. (a(i), b(i), c(i)) Ratio of average ISIs $\langle T_1 \rangle / \langle T_2 \rangle$ from the two systems. (a(ii), b(ii), c(ii)) Synchronization index $\gamma_{1,1}$ versus noise correlation time $\tau_c$ of the driving noise in the first subsystem and delay time $\tau$. Control strength (a) $K = 0.5$, (b) $K = 1.0$ and (c) $K = 1.5$.

of the uncontrolled system is varied from strong synchronization to weak synchronization with increasing $\tau_c$, and (ii) $C = 0.2$ and $\sigma^2 = 0.04$, for which the subsystems are strongly to moderately synchronized.

(i) **Strongly–weakly synchronized systems**

Figure 11 shows the ratio of average ISIs $R_{\text{ISI}}$ (figure 11a(i), b(i), c(i)) and the synchronization index $\gamma_{1,1}$ (figure 11a(ii), b(ii), c(ii)) for $\sigma^2 = 1$, $C = 0.2$ and three different values of the control strength (a) $K = 0.5$, (b) $K = 1$ and (c) $K = 1.5$. The delay time $\tau$ is varied from 0 to 10, and the noise correlation time $\tau_c$ from 0.001 to 100. The left border of the plots at $\tau = 0$ corresponds to the state of the uncontrolled system.

As seen from figure 11 the synchronization of the uncontrolled system is changed from perfect 1:1 synchronization to weak synchronization at $\tau_c = 100$.

The influence of the delayed feedback control can be characterized as follows. Most prominent is the wave-like modulated structure of the $(\tau, \tau_c)$-planes for $\tau \in [0, 4]$, which is observable for all three considered values of the control strength. Except for very small values of $\tau_c$, the synchronization is generally increased for $\tau \in [0, 2]$ and decreased for $\tau \in [2, 24]$. The monotonic decrease of the synchronization for the uncontrolled system is not changed by the feedback control, but rather shifted to a higher level for $\tau \in [0, 2]$ and to a lower level for $\tau \in [2, 24]$, which leads to the wave-like modulated features of the plot. In figure 11c one can see that the maximum of the phase synchronization, quantified by the ratio of average ISIs $R_{\text{ISI}}$, is shifted with increasing $\tau_c$ from $\tau \approx 1$ for $\tau_c = 0.1$ to $\tau \approx 1.5$ for $\tau_c = 100$. Both the ratio of average ISIs and the synchronization index show similar behaviour, although the effects are less pronounced for the synchronization index in this presentation. For $\tau > 4$ both quantities and thus
the frequency and phase synchronization remain approximately constant; the sustained modulations of the synchronization for $\tau_c = 100$, which we have observed in the previous section, are not resolvable in this presentation. The effects are generally stronger for increasing $K$, but qualitatively similar.

(ii) Strongly–moderately synchronized systems

Figure 12 shows the ratio of average ISIs $R_{\text{ISI}}$ (figure 12a(i), b(i), c(i)) and the synchronization index $\gamma_{1,1}$ (figure 12a(ii), b(ii), c(ii)) for $\sigma^2 = 0.04$, $C = 0.2$ and three different values of the control strength (a) $K = 0.5$, (b) $K = 1.0$ and (c) $K = 1.5$. The delay time $\tau$ is varied from 0 to 10, and the noise correlation time $\tau_c$ from 0.001 to 100. The left border of the plots at $\tau = 0$ corresponds to the state of the uncontrolled system.

Figure 12 shows that the synchronization of the uncontrolled system is changed from perfect 1:1 synchronization to moderate synchronization at $\tau_c = 100$.

The influence of the delay time can be characterized as follows. For $K = 0.5$ (figure 12a) the feedback control is able to both increase and decrease synchronization for $\tau_c \gtrsim 1$. For $\tau \in [0.5, 2]$ the system reaches perfect or almost perfect 1:1 synchronization independently of the noise correlation time $\tau_c$. For $\tau \approx 2.5$ the synchronization is lowered, most pronounced for $\tau_c \approx 10$. A second local maximum occurs for $\tau \approx 5$; for further increased $\tau$ both the ratio of average ISIs and the synchronization index remain more or less constant, but on a lower level than in the original state for $\tau_c$ around 10 and on about the same or higher level for $\tau_c = 100$. For $\tau_c \lesssim 0.1$ the system is perfectly 1:1 synchronized in the uncontrolled state, and the delayed feedback control is not able to destroy the synchronization. Note that the coherence maximum with respect to the correlation time for the uncontrolled systems is at $\tau_c \approx 0.03$, thus the perfect synchronization is not simply due to very infrequent spiking of the first subsystem.

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For $K = 1$ (figure 12b) and $K = 1.5$ (figure 12c), the effects are similar. Again, for $\tau \in [0.5, 2]$ the system reaches perfect or almost perfect 1:1 synchronization independent of the noise correlation time $\tau_c$. However, the minimum for $\tau \approx 2.5$ is less pronounced and for further increasing $\tau$ the system shows about the same synchronization as in the uncontrolled state.

To sum up, for $\tau \in [0.5, 2]$ the delayed feedback control is able to induce perfect 1:1 synchronization independently of the noise correlation time; for further increasing $\tau$ the synchronization can be lowered, but only for a certain range of the noise correlation time $\tau_c$ and appropriate choice of the control strength $K$.

5. Conclusions

The influence of time-delayed feedback on an FHN system driven by coloured noise was investigated by considering the effects on time scales and coherence of noise-induced oscillations. The application of delayed feedback renders the distribution of ISIs bimodal or multimodal, thus disqualifying the normalized standard deviation of ISIs as a suitable measure of coherence. Therefore, we solely used the correlation time to quantify the regularity of spike trains. We observe linear entrainment of the spectral peaks by the feedback control, as is known for white noise-driven FHN systems; for large noise correlation times this phenomenon can also be observed for the distribution of ISIs. Also the coherence of oscillations can be influenced considerably. Both effects are known for white noise-driven FHN systems; in fact, for small noise correlation time the effects are equivalent to those for white noise, in accordance with the findings for the uncontrolled system. The distribution of ISIs features characteristic gaps, which correspond to the delay of spikes due to the control. The interplay of the delay time and the noise correlation time is associated with the major time scales of the uncontrolled system: the modulations of the coherence by the feedback control depend on the dominant spectral peak of the uncontrolled system, the position of which is shifted for varying noise correlation time.

The synchronization of two coupled FHN systems was investigated by the consideration of the ratio of ISIs and the synchronization index. Coloured noise was applied to the first subsystem, while the second one was driven by white noise. It was shown that the noise correlation time can influence both frequency and phase synchronization of the two oscillators considerably. Moreover, temporal correlations present in the noise input of the first system only are able to significantly influence the time scales and coherence of both systems. In particular we observed CR with respect to the correlation time also in the second subsystem, which is only indirectly influenced by the coloured noise.

The effects of delayed feedback control on two coupled FHN systems were investigated by applying the control to the first subsystem only. Nevertheless, the control was shown to be able to change the time scales and coherence of oscillations considerably in both subsystems, and also to influence the global characteristics of the two coupled neurons. The influence of the synchronization between the two oscillators was studied for a moderately, a weakly and a strongly synchronized system. It was shown that both frequency and phase synchronization could be increased and decreased. Again the effects for small noise correlation times were generally equivalent to those for white noise, while
for large noise correlation times noticeable changes occurred: the synchronization shows strong sustained modulations with increasing delay time, different from the case of white noise.

Moreover, the interplay of the delay time and the noise correlation time was studied. We varied both parameters independently and investigated the effects on the synchronization. The synchronization is monotonically decreasing with increasing noise correlation time; the feedback control shifts this monotonic decrease of the synchronization to a higher or lower level, depending on the delay time, which gives the resulting plots a wave-like modulated appearance.

In conclusion, the main results of our investigation of the effects of time-delayed feedback control on excitable systems driven by coloured noise can be summarized as follows. Both the time scales and coherence of oscillations can be influenced significantly. Moreover, the synchronization of two coupled systems can be enhanced or suppressed. For small noise correlation times the effects of feedback control are generally equivalent to those for white noise of appropriate intensity. Noticeable changes are found for large noise correlation times, where the system shows bursting behaviour. The interplay of the delay time and the noise correlation time is determined by the shift of the major time scale of the uncontrolled system with varying noise correlation time.

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References


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