The stability of a steady-state front (kink) subject to a time-delayed feedback control (TDFC) is examined in detail. TDFC is based on the use of the difference between system variables at the current moment of time and their values at some time in the past. We first show that there exists a bifurcation to a moving front. We then investigate the limit of large delays but weak feedback and obtain a global bifurcation diagram for the propagation speed. Finally, we examine the case of a two-dimensional front with radial symmetry and determine the critical radius above which propagation is possible.

Keywords: kink; front; reaction–diffusion; delayed feedback; bifurcation

1. Introduction

A localized structure in a driven non-equilibrium system consists of one or more regions in one state surrounded by a region in a qualitatively different state. Such patterns may be stationary or oscillatory, static or moving, and they are important in a wide variety of fields; see Akhmediev & Ankiewicz (2008) and Tlidi et al. (2007) for recent reviews. In optics, they are called cavity solitons (CSs), and they are potentially interesting for the all-optical control of light. To this end, a delayed feedback is sometimes used to control these CSs (Tanguy et al. 2008). Slowly moving spots and fronts subject to delay appear in various contexts. In population biology, the time needed for successive generations to reproduce and migrate affects the spread of a population (Fort & Méndez 2002; Ortega-Cejas et al. 2004). In another setting, propagating depolarization waves appearing for both migraine and stroke have been described as reaction–diffusion waves. Here, too, a delayed feedback is present, as experimental data suggest (Schneider et al. 2009; Dahlem et al. in press). This and other practical problems (Erneux 2009) have motivated a surge of research activity around spatially localized structures with delay.

Several analytical studies have recently been proposed for localized patterns controlled by a delayed feedback and described mathematically by the reaction–diffusion equation

\[ u_t = \Delta u + f(u) + \gamma g(u, u(t - \tau)) \tag{1.1} \]

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One contribution of 13 to a Theme Issue ‘Delayed complex systems’.
where \( f(u) \) is a cubic-like nonlinear function and \( g(u, u(t - \tau)) \) is a linear function of \( u(t - \tau) \) and possibly \( u \). Rubinstein et al. (2007) showed that a localized solution of the one-dimensional Ginzburg–Landau equation can be stabilized by a delayed feedback control (\( f = u + |u|^2 u \) and \( g = -\max_x |u(t - \tau)| \)). Coombes & Laing (2009) constructed a stable front in the special case of a delayed threshold nonlinearity. They showed that the speed of propagation decreases with the delay (\( f = -u + H(a - u(t - \tau)) \), \( 0 < a < 1/2 \), and \( \gamma = 0 \), where \( H(u) \) is the Heaviside step function). On the other hand, Boubendir et al. (submitted) demonstrated that stable one-dimensional and two-dimensional propagating fronts may undergo a Hopf bifurcation (\( f = \frac{1}{2} (u - u^3) \) and \( g = \langle u(t - \tau) \rangle \), where \( \langle u \rangle \) denotes the spatial average).

Of particular physical interest are time-delayed feedback controls (TDFCs) (Pyragas 1992; Schöll & Schuster 2008). TDFC is based on the use of the difference between system variables at the current moment of time and their values at some time in the past. They have the property that steady states of the unperturbed problem are solutions of the delayed feedback problem. The simplest TDFC for a single variable \( u \) is provided by the control force

\[
g = \gamma (u(t - \tau) - u(t)),
\]

where \( \gamma \) and \( \tau \) denote the rate and the delay of the feedback, respectively. A recent combined analytical and numerical study of the two-dimensional Swift–Hohenberg equation suggests that steady localized spots can become mobile as soon as \( \gamma \tau < -1 \) (Tlidi et al. 2009).

The main objective of this paper is to demonstrate that this transition at \( \gamma \tau = -1 \) is indeed a bifurcation. To this end, we consider a steady-state front subject to a linear TDFC described by equation (1.1), where \( f = \frac{1}{2} (u - u^3) \) and \( g \) is as given by equation (1.2). Section 2 examines the stability of the one-dimensional steady-state front, determines its bifurcation point and summarizes the main result of a local analysis. We then investigate in §3 the case of large delays but weak feedback, which allows a global bifurcation diagram for the propagation speed as a function of the feedback rate. In §4, we analyse the case of a two-dimensional front with radial symmetry and determine the critical radius above which propagation is possible. We discuss our main result in §5.

2. Kink bifurcation

In one dimension, equation (1.1) takes the form

\[
u_t = u_{xx} + f(u) + \gamma [u(t - \tau, x) - u(t, x)]
\]

and admits a localized front (or kink) given by \( u = \tanh(x/2) \). We wonder whether a bifurcation to a moving front is possible. To answer this question, we look for a front solution of equation (2.1) of the form \( u = u(z, \varepsilon) \), where

\[
z \equiv x - \varepsilon s(t, \varepsilon)
\]

and \( s(t, \varepsilon) = O(1) \); see figure 1. The small parameter \( \varepsilon \) measures the speed of the moving front, and it needs to be related to the feedback parameters \( \gamma \) and \( \tau \).
Substituting $u = u(z, \varepsilon)$ into equation (2.1) leads to the equation

$$-\varepsilon s'u_z = u_{zz} + f(u) + \gamma [u(z + \varepsilon(s(t) - s(t - \tau)))] - u,$$

where $s' \equiv ds/dt$. Together with the boundary conditions $u(-\infty) = -1$ and $u(\infty) = 1$, we solve equation (2.3) by seeking a perturbation solution for $u$ and $s$ in power series of $\varepsilon$. From the first two problems, we sequentially find that $u = u_0 \equiv \tan(z/2)$ and that $s = s_1(t)$ needs to satisfy the solvability condition

$$s' + \gamma(s_1(t) - s_1(t - \tau)) = 0. \quad (2.4)$$

The steady state $s_1 = cst$ is stable if $\text{Re}(\lambda) < 0$, where $\lambda$ satisfies the characteristic equation

$$\lambda + \gamma[1 - \exp(-\lambda \tau)] = 0. \quad (2.5)$$

A necessary condition for stability is obtained by first determining the real roots. Keeping $\tau$ fixed, we determine $\lambda = \lambda(\gamma)$ from equation (2.5) in the implicit form $\gamma \tau = \lambda \tau[\exp(-\lambda \tau) - 1]^{-1}$ (figure 2). We then find that $\lambda$ is unique and negative if

$$-1 < \gamma \tau < 0. \quad (2.6)$$

Equation (2.5) admits no purely imaginary roots, and it can be shown that all the complex roots have a negative real part (the roots of $\lambda = -a - b \exp(-\lambda \tau)$ are studied, for example, in Michiels & Niculescu (2007, p. 60)). The critical point $\gamma = \gamma_c \equiv -\tau^{-1}$ is a bifurcation point that corresponds to a double-zero eigenvalue of the characteristic equation (2.5). A local analysis near this critical point with $s_1 = c_1 t$ is possible and leads to the following bifurcation equation for the speed $\varepsilon c_1$:

$$\frac{\tau}{30}(\varepsilon c_1)^3 + (\gamma - \gamma_c)(\varepsilon c_1) = 0, \quad (2.7)$$

Figure 1. Slowly propagating front induced by a delayed feedback.
where $\gamma - \gamma_c = O(\varepsilon^2)$. From this equation, we find that a non-trivial solution is given by

$$
\varepsilon c_1 = \pm \sqrt{-30 \frac{\gamma - \gamma_c}{\tau}} \quad (\gamma \leq \gamma_c).
$$

(2.8)

The bifurcation is supercritical because, if $\gamma < \gamma_c$, the two solutions overlap the unstable zero solution. The small parameter $\varepsilon$ can now be defined as

$$
\varepsilon \equiv \sqrt{\gamma_c - \gamma},
$$

(2.9)

implying a constant value for the coefficient $c_1 = \pm \sqrt{30/\tau}$. We omit all details of the bifurcation analysis because we verify it later by considering a different asymptotic limit of equation (2.1).

In summary, we have shown that a slowly propagating front is bifurcating from the kink at $\gamma = \gamma_c$. The bifurcation is supercritical and the speed of the front is proportional to the deviation $\sqrt{\gamma_c - \gamma}$. For a fixed deviation $\gamma_c - \gamma$, the speed is a decreasing function of the delay $\tau$.

3. Large delay and weak feedback

We next wish to determine an expression for the propagation speed that is not limited to the vicinity of the bifurcation point. To this end, we consider the large delay limit but keep $\gamma \tau = O(1)$. Specifically, we redefine our small parameter $\varepsilon$ as

$$
\varepsilon \equiv \tau^{-1}
$$

(3.1)

and introduce the new time $\tilde{t}$ and new feedback amplitude $\gamma_1 = O(1)$ as

$$
\tilde{t} \equiv \varepsilon t \quad \text{and} \quad \gamma_1 \equiv \varepsilon^{-1} \gamma.
$$

(3.2)
In terms of the new variables (3.2), equation (2.1) becomes

$$\varepsilon u_t = u_{xx} + f(u) + \varepsilon \gamma_1 [u(t-1, x) - u(t, x)],$$

(3.3)

where we have renamed $\overline{t}$ as $t$. Introducing

$$z \equiv x - s(t, \varepsilon)$$

(3.4)

into equation (3.3) leads to the following equation for $u(z, t)$:

$$\varepsilon(-s'u_z + u_t) = u_{zz} + f(u) + \varepsilon \gamma_1 [u(z - (s(t-1) - s), t-1) - u],$$

(3.5)

where $s' \equiv ds/dt$. We again seek a perturbation solution for $u$ and $s$ in power series of $\varepsilon$. The leading problem admits the solution $u_0 \equiv \tanh(z/2)$. But the solvability condition for the first correction of the solution now requires that

$$-s'_0 \int_{-\infty}^{\infty} u_0^2 \, dz - \gamma_1 \int_{-\infty}^{\infty} [u_0(z - (s_0(t-1) - s_0)) - u_0] u'_0 \, dz = 0.$$ 

(3.6)

Note that $\int_{-\infty}^{\infty} u_0 u'_0 \, dz = 0$. The coefficient of $\gamma_1$ then reduces to the integral $I \equiv \int_{-\infty}^{\infty} u_0 (z + z_0) u'_0 \, dz$ where

$$z_0 \equiv s_0(t) - s_0(t-1).$$

(3.7)

We introduce the expression for $u_0$ into $I$ and solve it by partial fractions. From equation (3.6), we then obtain the following equation for $s_0:

$$s'_0 = -3\gamma_1 F(z_0),$$

(3.8)

where $F(y)$ is defined by

$$F(y) \equiv -\frac{2\exp(-y)}{[\exp(-y) - 1]^2} y + \frac{1 + \exp(-y)}{1 - \exp(-y)}.$$ 

(3.9)

Note that $F(-y) = -F(y)$, $F(y) = y/3 - y^3/90 + \cdots$ as $y \to 0$ and $F(y) > 0$ if $y > 0$. These properties will be useful in our analysis of equation (3.8). We may determine a constant-speed solution by substituting $s_0 = ct$ into equation (3.9). Since $z_0 = s_0(t) - s_0(t-1) = c$, we obtain

$$c + 3\gamma_1 F(c) = 0.$$ 

(3.10)

The two solution branches are shown in figure 3. Near $\gamma_1 = -1$, the limit $c \to 0$ of equation (3.10) is

$$c(1 + \gamma_1) + \frac{c^3}{30} + \cdots = 0$$ 

(3.11)

and matches equation (2.7) (inserting $c = \tau \varepsilon c_1$ and $\gamma_1 = \gamma \tau$ into equation (3.11) and reorganizing leads to equation (2.7)). The numerical solution of equation (3.8) is shown in figure 4. After an exponential increase, the position of the front follows its constant-speed regime as predicted by the bifurcation diagram in figure 3.

In summary, the large delay and low feedback limit lead to a richer bifurcation equation for the propagation speed. Away from the bifurcation point, the speed becomes proportional to the feedback rate.
Figure 3. Propagation speed. Near the bifurcation point $\gamma_1 = -1$, $c$ is a parabolic function of $|\gamma_1 + 1|$. For large negative $\gamma_1$, $c \simeq \pm 3\gamma_1$ is a linear function of $\gamma_1$. At $\gamma_1 = -1.5$ (broken line), the positive solution is $c = c_\infty \simeq 3.98$.

Figure 4. Evolution of the position of the front for $\gamma_1 = -1.5$. The solution has been obtained numerically from the delay–differential equations (3.7) and (3.8). The initial conditions are $s_0(t-1) = 0$ for $-1 \leq t < 0$ and $s(0) = 0.01$. The broken line is the constant-speed solution predicted by the bifurcation analysis.

4. Quasi-two-dimensional kinks

In two dimensions, planar curved fronts move with a normal velocity proportional to their curvature. For a circular interface and in the absence of feedback, the curvature is the reciprocal of the radius $r = s(t)$, and its evolution satisfies $s' = -s^{-1}$ whose solution is given by $s = \sqrt{s(0)^2 - 2t}$, i.e. circles shrink to a point
at finite time. In this section, we again consider the limit of large delays and weak feedback, but, because of the effect of curvature, we need to change the interface coordinate. We neglect angular variations for simplicity and introduce \( \varepsilon, \tilde{t} \) and \( \gamma_1 \) defined by equations (3.1) and (3.2). Equation (1.1) in two dimensions then takes the form

\[
\varepsilon u_t = u_{rr} + r^{-1} u_r + f(u) + \varepsilon \gamma_1 [u(t - 1, r) - u(t, r)],
\]

where we have renamed \( \tilde{t} \) as \( t \). In two dimensions, there exist front solutions (called quasi-two-dimensional kinks) that are locally kinks along the normal direction to the interface. The interface can be approximated by the set of points \( r = S(\varepsilon t)/\varepsilon \) for which \( u \) vanishes. Introducing the interface variable \( z \equiv r - \frac{S(\varepsilon t, \varepsilon)}{\varepsilon} \) into equation (4.1), we find

\[
\varepsilon (u_t - S' u_z) = u_{zz} + \frac{\varepsilon}{S + \varepsilon z} u_z + f(u)
\]

\[
+ \varepsilon \gamma_1 \left[ u\left(z - \frac{S(\varepsilon t - \varepsilon) - S}{\varepsilon}, t - 1\right) - u\right],
\]

where \( S' \equiv \frac{dS}{d(\varepsilon t)} \). We solve this equation by seeking a perturbation solution for \( u \) and \( S \) in power series of \( \varepsilon \). The leading-order solution is \( u_0 = \tanh(z/2) \), and the solvability condition for the first correction of the solution requires that

\[
\left(S_0' + \frac{1}{S_0}\right) \int_{-\infty}^{\infty} u_0^2 z \, dz + \gamma_1 \int_{-\infty}^{\infty} [u_0(z + z_0) - u_0] u_0 \, dz = 0,
\]

where

\[
z_0 = \varepsilon^{-1} (S_0 - S_0(\varepsilon t - \varepsilon)).
\]

The integrals in equation (4.4) are identical to the integrals solved in the previous section, and equation (4.4) simplifies as

\[
S_0' = -\frac{1}{S_0} - 3\gamma_1 F(z_0),
\]

where \( F(y) \) is defined by equation (3.9). If \( \varepsilon t = O(1) \), we may expand \( S_0(\varepsilon t - \varepsilon) \) for small \( \varepsilon \), and \( z_0 \), defined by equation (4.5), equals \( S_0' \) as \( \varepsilon \to 0 \). On the other hand, if \( \varepsilon t = O(\varepsilon) \), \( S_0(\varepsilon t - \varepsilon) \) cannot be expanded for small \( \varepsilon \). But the condition

\[1\]The two-dimensional interface variable \( z \equiv r - \varepsilon^{-1} S(\varepsilon t, \varepsilon) \) differs from the one-dimensional interface variable \( z \equiv x - s(t, \varepsilon) \) by the fact that we introduce \( \varepsilon^{-1} S(\varepsilon t, \varepsilon) \) instead of \( s(t, \varepsilon) \). For sufficiently large values of \( S \), the effect of curvature becomes weaker, and we expect that the two-dimensional front propagates as a one-dimensional front. Mathematically, this means that \( S = c_\infty \varepsilon t \) as \( t \to \infty \), and the two-dimensional interface variable becomes \( z = r - c_\infty t \) as \( t \to \infty \).
Figure 5. Solution of equation (4.7) for $\gamma_1 = -1.5$. (a) Trajectory in phase plane. If $S_0'(0) > c_c \simeq 2$ ($S_0'(0) < c_c$), the solution follows the upper branch (lower branch) of the C-shaped trajectory. The broken line indicates the long-time speed $S_0' = c_\infty$, which is identical to $c_\infty$ computed for the one-dimensional problem. (b) Solution starting at $S_0'(0) = 1.5$ and $S_0'$ is continuously decreasing. (c) Solution starting at $S_0'(0) = 2.5$ and $S_0'$ is approaching $c_\infty$ as $\varepsilon t \to \infty$.

that $z_0$ is $O(1)$ as $\varepsilon \to 0$ then requires that $S_0 - S_0(0) = O(\varepsilon)$, meaning an initial layer. The asymptotic analysis thus reduces to an ‘outer solution’ for $S_0$ valid as a long-time solution ($t = O(\varepsilon^{-1})$) and an ‘inner solution’ for $S_0$ that describes its evolution near $S_0(0)$ ($t = O(1)$).

For the long-time solution, $z_0 = S_0'$, and equation (4.6) takes the form

$$S_0' = -\frac{1}{S_0} - 3\gamma_1 F(S_0').$$

Equation (4.7) is an equation relating $S_0'$ and $S_0$ and describes a trajectory in the phase plane $(S_0, S_0')$ (the C-shaped curve in figure 5a). Figure 5b,c shows the two possible time evolutions for $S_0 = S_0(\varepsilon t)$ for $S_0'(0) < c_c$ and $S_0'(0) > c_c$, respectively. They have been obtained from solving equation (4.7) numerically. Note that these solutions are physically valid only if they can be connected to the initial layer solution.

To examine the initial layer solution, we introduce

$$S_0 = S_0(0) + \varepsilon S_1(t)$$

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into equation (4.5) and obtain $z_0 \equiv S_1(t) - S_1(t - 1)$. Equation (4.6) then leads to the following equation for $S_1$: \[ S_1' = -\frac{1}{S_0(0)} - 3\gamma_1 F(S_1 - S_1(t - 1)), \] (4.9) where $S_1' \equiv dS_1/dt$. This equation is a delay–differential equation that must be solved using appropriate initial conditions. But it admits solutions of the form $S = ct$, where $c$ satisfies \[ c = -\frac{1}{S_0(0)} - 3\gamma_1 F(c). \] (4.10) Equation (4.10) is equivalent to equation (4.7), with $c$ and $S_0(0)$ replacing $S'_0$ and $S_0$, respectively. Provided that $S_0(0) > S_c$, there exist two solutions. Analysing their linear stability, we obtain the following linearized problem for the small perturbation $v \equiv S_1 - ct$: \[ v' = -3\gamma_1 F'(c)(v - v(t - 1)). \] (4.11) Equation (4.11) has the same form as equation (2.4). Analysing the real roots, we find that stability requires the condition \[ 1 + 3\gamma_1 F'(c) > 0, \] (4.12) and this condition is verified for the upper branch in the phase plane $(S_0, S'_0)$ in figure 5a. This can be demonstrated by determining $dS_0(0)/dc$.
from equation (4.10). We find

$$\frac{d(S_0(0))}{dc} = S_0(0)^2[1 + 3\gamma_1 F'(c)].$$

(4.13)

The condition (4.12) thus implies that \(d(S_0(0))/dc > 0\), which corresponds to the upper branch in figure 6 (\(S'_0 = c\) increases with \(S_0(0)\)).

5. Discussion

In this paper, we considered the simplest case of a localized pattern subject to a delayed feedback, namely a one-dimensional steady-state front of a scalar reaction–diffusion equation with a Pyragas-type control (Pyragas 1992). The control has the advantage that it does not modify the steady states of the unperturbed system (as the kink) but only affects their stability properties. Because the leading solution is known analytically, asymptotic studies can be developed in detail. Our analysis is much in the spirit of the recent work by Boubendir et al. (submitted) who considered a global delayed feedback. It differs by the bifurcation analysis and by the large delay limit that allows a richer equation for the propagation speed. We found that the steady-state front may undergo a bifurcation to a propagating front. For two-dimensional fronts with radial symmetry, this bifurcation is still possible, but a critical radius needs to be surpassed.

As for all delay–differential equation problems, asymptotic results are not routine applications of singular perturbation techniques and they need to be carefully checked numerically. In both one and two dimensions, we did not find a Hopf bifurcation to sustained oscillations. In future work, we plan to investigate our reaction–diffusion equation numerically and address this question.

The work by the three authors was supported by the Fonds National de la Recherche Scientifique (FRS-FNRS, Belgium).

References


Bifurcation to fronts due to delay


