Particle dynamics inside shocks in Hamilton–Jacobi equations

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The characteristic curves of a Hamilton–Jacobi equation can be seen as action-minimizing trajectories of fluid particles. For non-smooth ‘viscosity’ solutions, which give rise to discontinuous velocity fields, this description is usually pursued only up to the moment when trajectories hit a shock and cease to minimize the Lagrangian action. In this paper we show that, for any convex Hamiltonian, there exists a uniquely defined canonical global non-smooth coalescing flow that extends particle trajectories and determines the dynamics inside shocks. We also provide a variational description of the corresponding effective velocity field inside shocks, and discuss the relation to the ‘dissipative anomaly’ in the limit of vanishing viscosity.

Keywords: Hamiltonian formulations; shock wave interactions and shock effects; control theory; convex sets and geometric inequalities; singularity theory

1. Introduction

The Hamilton–Jacobi equation,

$$\frac{\partial \phi}{\partial t}(t, x) + H(t, x, \nabla \phi(t, x)) = 0,$$

plays an important role in a large variety of mathematical and physical problems. Apart from analytical mechanics, it appears in the description of a whole range of extended dissipative systems featuring non-equilibrium turbulent processes, from the microscale of condensed matter and statistical physics through the mesoscale setting of a free-boundary fluid to macroscale cosmological evolution; the interested reader is referred to the non-exhaustive collection of references in Bec & Khanin (2007). The central issue in a study of nonlinear evolution for equation (1.1) is to understand, from both the mathematical and physical points of view, the behaviour of the system after the inevitable formation of singularities.

A theory of weak solutions for a general Hamilton–Jacobi equation, employing regularization by infinitesimal diffusion, has existed since the 1970s (Kruzhkov 1975; Lions 1982; Crandall et al. 1992). In the one-dimensional setting, this

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One contribution of 13 to a Theme Issue ‘Turbulent mixing and beyond’. 
theory is essentially equivalent to the earlier theory of hyperbolic conservation laws in fluid mechanics, developed in the 1950s (Hopf 1950; Lax 1954; Oleinik 1954). In more than one dimension, however, the two theories are no longer parallel.

The theory of weak solutions for the Hamilton–Jacobi equation is closely related to the calculus of variations, and from this point of view one can say that the introduction of diffusion is motivated essentially by stochastic control arguments (Fleming & Soner 2005). In the present paper we adopt a related but somewhat complementary viewpoint, in which the Hamilton–Jacobi equation is considered as a fluid dynamics model, and construct the flow of ‘fluid particles’ inside the shock singularities of a weak solution.

A useful example to be borne in mind when thinking about equation (1.1)—and arguably the most widely known variant thereof—is the Riemann, or inviscid Burgers, equation,

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = 0 \quad \text{and} \quad \nabla \times u = 0. \quad \text{(1.2)}$$

It is obtained for the Hamiltonian $H(t, x, p) = |p|^2/2$, by setting $u(t, x) = \nabla \phi(t, x)$. Note that in equation (1.2) it is essential that the velocity field $u$ is curl-free, so this model is indeed equivalent to the Hamilton–Jacobi equation (1.1). The Riemann equation may in turn be considered as a limit of vanishing viscosity of the Burgers equation,

$$\frac{\partial u^\mu}{\partial t} + u^\kappa \cdot \nabla u^\mu = \mu \nabla^2 u^\mu \quad \text{and} \quad \nabla \times u^\mu = 0. \quad \text{(1.3)}$$

Hence, solutions of equation (1.2) can be defined as limits of smooth solutions to equation (1.3) as the positive parameter $\mu$ goes to zero.

The Burgers equation is in fact very special: it can be exactly mapped by the Cole–Hopf transformation into the linear heat equation and therefore explicitly integrated (Hopf 1950). Nonetheless, the qualitative behaviour of solutions to a parabolic regularization of equation (1.1) for a general convex Hamiltonian,

$$\frac{\partial \phi^\mu}{\partial t} + H(t, x, \nabla \phi^\mu) = \mu \nabla^2 \phi^\mu, \quad \text{(1.4)}$$

in the limit of vanishing viscosity, is similar to that of the Burgers equation. It turns out that, as $\mu \to 0$, there exists a limit $\phi(t, x) = \lim \phi^\mu(t, x)$, which is called the viscosity solution. Remarkably, the viscosity solution can be described by a purely variational construction that does not use the viscous regularization at all. Below, we briefly recall the main ideas of this variational approach.

Assume that the Hamiltonian function $H(t, x, p)$ is smooth and strictly convex in the momentum variable $p$, i.e. is such that for all $(t, x)$ the graph of $H(t, x, p)$ as a function of $p$ lies above any tangent plane and contains no straight segments. This implies that the velocity $v = \nabla_p H(t, x, p)$ is a one-to-one function of $p$. In addition, the Lagrangian function

$$L(t, x, v) = \max_p [p \cdot v - H(t, x, p)]$$

under the above hypotheses is smooth and strictly convex in $v$, although it may not be finite everywhere: e.g. the relativistic Hamiltonian $H(t, x, p) = \sqrt{1 + |p|^2}$ corresponds to the Lagrangian $L(t, x, v)$, which is defined for $|v| \leq 1$ as
Therefore, the bound (1.6) is achieved for trajectories satisfying Hamilton’s mechanical action corresponding to the trajectory where at the last step the Young inequality is used. This implies a bound for the \( V \) holds for a minimizer \( \phi \) of \( V \) and of \( v \). (Usually the term ‘Legendre transform’ refers to the relation between the conjugate functions \( H \) and \( L \); here we follow the usage adopted by Fathi in his works on weak Kolmogorov–Arnold–Moser (KAM) theory (Fathi in press), which is more convenient in the present context.)

Note that if \( H(t, x, p) = |p|^2/2 \), then \( L(t, x, v) = |v|^2/2 \) and the Legendre transform reduces to the identity \( v \equiv p \), blurring the distinction between velocities and momenta. This is another very special feature of the Burgers equation.

Now assume that \( \phi(t, x) \) is a strong solution of the inviscid equation (1.1), i.e. a smooth function that satisfies the equation in the classical sense. For an arbitrary trajectory \( \gamma(t) \), the full time derivative of \( \phi \) along \( \gamma \) is given by

\[
\frac{d\phi(t, \gamma)}{dt} = \frac{\partial \phi}{\partial t} + \nabla \phi \cdot \dot{\gamma} = \nabla \phi \cdot \dot{\gamma} - H(t, \gamma, \nabla \phi) \leq L(t, \gamma, \dot{\gamma}),
\]

(1.5)

where at the last step the Young inequality is used. This implies a bound for the mechanical action corresponding to the trajectory \( \gamma \):

\[
\phi(t_2, \gamma(t_2)) \leq \phi(t_1, \gamma(t_1)) + \int_{t_1}^{t_2} L(s, \gamma(s), \dot{\gamma}(s)) \, ds.
\]

(1.6)

Equality in equation (1.5) is achieved only if \( \dot{\gamma} \) is the Legendre transform of \( \nabla \phi \) at \( (t, \gamma) \):

\[
\dot{\gamma}(t) = \nabla_p H(t, \gamma, \nabla \phi(t, \gamma)).
\]

(1.7)

This represents one of Hamilton’s canonical equations, with momentum given for the trajectory \( \gamma \) by \( p_\gamma(t) := \nabla \phi(t, \gamma(t)) \). The other canonical equation, \( \dot{p} = -\nabla_x H \), follows from equations (1.1) and (1.7) because

\[
\dot{p}_\gamma(t) = \frac{\partial \nabla \phi}{\partial t} + (\nabla \otimes \nabla \phi) \cdot \dot{\gamma} = -\nabla_x H(t, \gamma, \nabla \phi) - \nabla_p H \cdot (\nabla \otimes \nabla \phi) + (\nabla \otimes \nabla \phi) \cdot \dot{\gamma}.
\]

Therefore, the bound (1.6) is achieved for trajectories satisfying Hamilton’s canonical equations. This is a manifestation of the variational principle of least action: Hamiltonian trajectories \( (\gamma(t), p_\gamma(t)) \) are (locally) action-minimizing. In particular, if the initial condition

\[
\phi(t = 0, y) = \phi_0(y)
\]

(1.8)

is a fixed smooth function, the identity

\[
\phi(t, x) = \phi_0(\gamma(0)) + \int_0^t L(s, \gamma(s), \dot{\gamma}(s)) \, ds
\]

holds for a minimizer \( \gamma \) such that \( \gamma(t) = x \).

*Phil. Trans. R. Soc. A* (2010)
The least-action principle can be used to construct the viscosity solution corresponding to the initial data (1.8):

\[ \phi(t, x) = \min_{\gamma: \gamma(t) = x} \left( \phi_0(\gamma(0)) + \int_0^t L(s, \gamma(s), \dot{\gamma}(s)) \, ds \right). \]  

(1.9)

This is the celebrated Lax–Oleinik formula (e.g. E et al. 2000; Fathi in press), which reduces the partial differential equation problem (1.1) and (1.8) to the variational problem (1.9), where minimization is extended to all smooth curves \( \gamma \) such that \( \gamma(t) = x \).

The function \( \phi \) defined by equation (1.9) is smooth at those points \( (t, x) \) where the minimizing trajectory is unique. In this case, the minimizer can be embedded in a smooth family of minimizing trajectories whose endpoints at time 0 and \( t \) are continuously distributed about \( \gamma(0) \) and \( \gamma(t) = x \). A piece of initial data \( \phi_0 \) gets continuously deformed according to equation (1.5) along this bundle of trajectories into a piece of smooth solution \( \phi \) to equation (1.1) defined in a neighbourhood of \( x \) at time \( t \). Of course, the Hamilton–Jacobi equation is satisfied in a strong sense at all points where \( \phi \) is differentiable.

However, the crucial feature of equation (1.9) is that generally there will be points \( (t, x) \) with several minimizers \( \gamma_i \), which start at different locations \( \gamma_i(0) \), but bring the same value of action to \( x = \gamma_i(t) \). Just as above, each of these Hamiltonian trajectories will be responsible for a separate smooth ‘piece’ of solution. Thus, for locations \( x' \) close to \( x \), the function \( \phi \) will be represented as a pointwise minimum of smooth pieces \( \phi_i \):

\[ \phi(t, x') = \min_i \phi_i(t, x'). \]

As all \( \gamma_i \) have the same terminal value of action, all these pieces intersect at \( (t, x) \): \( \phi_1(t, x) = \phi_2(t, x) = \cdots = \phi(t, x) \). Thus, the neighbourhood of \( x \) at time \( t \) is partitioned into domains where \( \phi \) coincides with each of the smooth functions \( \phi_i \), and strongly satisfies the Hamilton–Jacobi equation (1.1). These domains are separated by surfaces of various dimensions where two, or possibly three or more, pieces \( \phi_i \) intersect and hence \( \phi \) is not differentiable. Such surfaces are called shocks. Note that a function \( \phi \) defined by the Lax–Oleinik formula is continuous everywhere, including the shocks; it is its gradient that suffers a discontinuity. In general, there are infinitely many continuous functions that match the initial condition (1.8) and are differentiable and satisfy the Hamilton–Jacobi equation (1.1) apart from some shock surfaces. A standard one-dimensional example of such non-uniqueness is provided by \( \phi_\alpha(t, x) = \min(\alpha|x| - \alpha^2 t/2, 0) \), which for any \( \alpha > 0 \) satisfies the initial condition \( \phi_\alpha(0, x) = 0 \) and the equation \( \partial \phi_\alpha / \partial t + |\partial \phi_\alpha / \partial x|^2 / 2 = 0 \) apart from the shock rays \( x = \pm \alpha t/2, \, x = 0 \). What distinguishes the function \( \phi \) defined by equation (1.9) from all other ‘weak solutions’, and confers on it important physical meaning, is that \( \phi \) appears in the limit of vanishing viscosity for the regularized equation (1.4) with the initial condition (1.8) (e.g. Lions 1982).

For a smooth Hamiltonian it can be proved that, once shocks are created, they never disappear, although they can merge with one another. Another important physical feature of viscosity solutions is that minimizers can only merge with shocks but can never leave them: all minimizers coming to some
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(t, x) in equation (1.9) originate at t = 0. It is easy to see that this is not so in the above example of φ_x, where minimizers emerge from x = 0 at all times t > 0.

Moreover, in a solution φ given by equation (1.9), a minimizer that has come to a shock can no longer continue as a minimizing trajectory. Indeed, wherever it comes, there will be other trajectories originated at t = 0 that will bring smaller values of action to the same location. Hence, for the purpose of the least-action description (equation (1.9)), Hamiltonian trajectories are not considered as soon as they are absorbed by shocks. The set of trajectories that survive as minimizers until time t > 0 is decreasing with t, but at all times it is sufficiently large to cover the whole space of final positions.

Let us now adopt an alternative viewpoint and consider the Hamilton–Jacobi equation as a fluid dynamics model, assuming that Hamiltonian trajectories (1.7) are described by material ‘particles’ transported by the velocity field u(t, x), which is the Legendre transform of the momentum field p(t, x) = ∇φ(t, x). From this new perspective it is no longer natural to accept that particles annihilate once they reach a shock. Can something therefore be said about the dynamics of those particles that got into the shock, notwithstanding the fact that their trajectories cease to minimize the action? The problem here comes from the discontinuous nature of the velocity field u, which makes it impossible to construct classical solutions to the transport equation ˙γ(t) = u(t, γ).

In one dimension the answer to the question above is readily available. Shocks at each fixed t are isolated points in x space and, as soon as a trajectory merges with one of them, it continues to move with the shock at all later times. This description is related to Dafermos’ theory of generalized characteristics (Dafermos 2005), which, in fact, can be extended to a much more general situation of non-convex Hamiltonians and systems of conservation laws. However, in several space dimensions, shocks become extended surfaces and, already for equation (1.1) with a strictly convex Hamiltonian, the dynamics of trajectories inside shocks is by no means trivial. The main goal of the present paper is to describe a natural and canonical construction of such dynamics.

First results in this direction were obtained by Bogaevsky (2004, 2006) for the Burgers equation (1.3). Bogaevsky suggested to consider the transport problem for a smooth velocity field u^μ:

\[ ˙γ^μ(t) = u^μ(t, γ^μ) \quad \text{and} \quad γ^μ(0) = y. \]

Since u^μ for μ > 0 is a smooth vector field, this equation defines a family of particle trajectories forming a smooth flow. The next step is to take the limit of this flow as μ ↓ 0. Bogaevsky proved that this limit exists as a non-differentiable continuous flow, for which the forward derivative ˙γ(t + 0) = lim_{τ↓0}[(γ(t + τ) − γ(t))/τ] is defined everywhere. If γ(t) is located outside shocks, this derivative coincides with u(t, γ(t)). Otherwise, there are several limit values of velocity u_i, and Bogaevsky discovered an interesting explicit representation for ˙γ(t + 0): it coincides with the centre of the smallest ball that contains all u_i. It should be remarked that the uniqueness of a limit flow in the case of a quadratic Hamiltonian was earlier observed by Cannarsa and Sinestrari in the context of the propagation of singularities for the eikonal equation and differential inclusions (Cannarsa & Sinestrari 2004, lemma 5.6.2).
The original approach in Bogaevsky (2004, 2006) is based on the specific properties of the Burgers equation and cannot be applied in the case of general convex Hamiltonians. In particular, the method uses the identity of velocities and momenta, which does not hold in the general setting. At the same time the common wisdom says that all Hamilton–Jacobi equations with convex Hamiltonians must have similar properties.

In this work we consider the above strategy, consisting of the parabolic regularization of the Hamilton–Jacobi equation and investigation of the vanishing viscosity limit for the corresponding regularized flow in the case of general convex Hamiltonians. We show that such a limit exists, and derive an explicit representation for the forward velocity of the limit flow that extends the above result for the Burgers equation. Yet, the mechanism powering these results in the general case is completely different. It is based on the fundamental uniqueness of a possible limit behaviour of $\gamma^\mu$, which we discuss in detail below.

The paper is organized as follows. In §2 we study the local structure of a viscosity solution near a singularity. We also introduce the notion of admissible velocity at a singularity and show that it can be determined uniquely. Moreover, the unique admissible velocity provides a solution to a certain convex minimization problem, which generalizes Bogaevsky’s construction of the centre of the smallest ball. In §3 we demonstrate that, for any non-smooth viscosity solution, there exists a unique continuous non-smooth flow of trajectories tangent to the field of admissible velocities. Section 4 contains concluding remarks.

2. Local structure of viscosity solutions and admissible momenta

Let $\phi$ be a viscosity solution to the Hamilton–Jacobi equation (1.1) with initial data (1.8). If there is a single minimizer coming to $(t, x)$, then $\phi$ is differentiable at this point and

$$
\phi(t + \tau, x + \xi) = \phi(t, x) + \frac{\partial \phi}{\partial t} \tau + \nabla \phi \cdot \xi + \cdots
$$

where dots $\cdots$ stand for higher-order terms. If $\phi$ is not differentiable at $(t, x)$, this means that there are several minimizers $\gamma_i$, such that $\gamma_i(t) = x$, each bringing to $(t, x)$ a different piece $\phi_i$ of solution. Then the Lax–Oleinik formula implies that

$$
\phi(t + \tau, x + \xi) = \min_i \phi_i(t + \tau, x + \xi)
= \phi(t, x) + \min_i (-H_i \tau + p_i \cdot \xi) + \cdots,
$$

where $p_i := \nabla \phi_i(t, x)$ and $H_i := H(t, x, p_i)$.

In the latter case none of the expressions $-H_i \tau + p_i \cdot \xi$ provides an adequate linear approximation to the difference $\phi(t + \tau, x + \xi) - \phi(t, x)$, but they all majorize this difference up to a remainder that is linear or higher order depending on $\tau$ and $\xi$. Evidently, so too does the linear form $-H \tau + p \cdot \xi$ for any convex combination

$$
p = \sum_i \lambda_i p_i \quad \text{and} \quad H = \sum_i \lambda_i H_i,
$$

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with \( \lambda_i \geq 0, \sum \lambda_i = 1 \). In convex analysis these convex combinations are called supergradients of \( \phi \) at \( (t, x) \) and the whole collection of them, which is a convex polytope with vertices \((-H_i, p_i)\), is called the superdifferential of \( \phi \) (Rockafellar 1970; Cannarsa & Sinestrari 2004). We use Rockafellar’s notation \( \partial \phi(t, x) \) for the superdifferential (Rockafellar 1970).

To avoid a possible misunderstanding it should be noted that, although the uniqueness of the minimizer coming to \( (t, x) \) implies differentiability of \( \phi \) at \( t \) and earlier times, it does not imply its differentiability at any \( t + \tau > t \). The following example shows how this may happen. The function defined for

\[
\phi(t + \tau, x + \xi) = \phi(t, x) - \frac{4}{3} \tau^3 - 2|\xi|\tau - \frac{4}{3}(|\xi| + \tau^2)^{3/2}
\]

is a viscosity solution of the equation \( \partial \phi/\partial t + |\partial \phi/\partial x|^2/2 = 0 \) that satisfies the smooth initial condition

\[
\phi(t, x + \xi) - \phi(t, x) = -\frac{4}{3}|\xi|^{3/2}.
\]

For \( \tau > 0 \) a shock appears at \( \xi = 0 \), but differentiability at \( \tau = 0 \) is recovered because \( \partial \phi(t + \tau, x) = (-8\tau^2) \times [-4\tau, 4\tau] \) shrinks to (0, 0) as \( \tau \downarrow 0 \). Such points \((t, x)\) are called preshocks (Bec & Khanin 2007).

The particular case of preshocks is an instance of a general fact: replacing gradients with superdifferentials allows one to recover continuous differentiability, but in a weaker sense. Namely, suppose that \((t_n, x_n)\) converges to \((t, x)\) and the sequence \((-H_n, p_n) \in \partial \phi(t_n, x_n)\) has a limit point \((-H, p)\). By definition of superdifferential, one has

\[
\phi(t_n + \tau, x_n + \xi) - \phi(t_n, x_n) \leq -H_n \tau + p_n \cdot \xi + \cdots. \tag{2.1}
\]

Passing here to the limit and using the continuity of \( \phi \), we see that \((-H, p) \in \partial \phi(t, x)\). Therefore, the superdifferential \( \partial \phi(t, x) \) contains all the limit points of the superdifferentials \( \partial \phi(t_n, x_n) \) as \((t_n, x_n)\) converges to \((t, x)\).

Suppose that \((t, x)\) is a point of shock where \( k \) smooth branches \( \phi_i \) meet. It follows from the above discussion that, for a particle moving from a shock point \((t, x)\), all possible values of the velocity \( v \) must be such that the corresponding momentum \( p(v) \) belongs to the convex hull of the available momenta \( p_i = \nabla \phi_i(t, x) \), \( 1 \leq i \leq k \). However, one can say even more. For small positive \( \tau \), not all the branches \( \phi_i \) will contribute to the solution \( \phi \) at a point \((t + \tau, x + v\tau)\), but only those of them that correspond to a minimum in \( \min_i(-H_i + p_i \cdot v) \). Denote the corresponding set of indices by

\[
I(v) := \{1 \leq j \leq k : -H_j + p_j \cdot v = \min_i(-H_i + p_i \cdot v)\}. \tag{2.2}
\]

The set \( I(v) \) has the following physical meaning: if a particle moves away from a shock with a given velocity \( v \), then only \( \phi_j \) and \( p_j \) for \( j \in I(v) \) are relevant. Geometrically, one can say that the convex hull of \( p_j, j \in I(v) \), is the \( p \)-projection of the face of the superdifferential \( \partial \phi(t, x) \) that looks towards an infinitesimal observer who has just left \((t, x)\) with velocity \( v \).

This implies that any possible velocity \( v \) must satisfy the following condition.

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Admissibility condition. A velocity $v^*$ is admissible if and only if the corresponding momentum belongs to the convex hull of the momenta $p_j, j \in I(v^*)$. Namely, one has

$$p^*(v^*) = \sum_{j \in I(v^*)} \lambda_j p_j, \quad \lambda_j \geq 0 \quad \text{and} \quad \sum_{j \in I(v^*)} \lambda_j = 1.$$  \hfill (2.3)

To make the admissibility argument rigorous, it is necessary to have some control over the remainder term in equation (2.1). A natural function class that contains viscosity solutions of Hamilton–Jacobi equations and in which such control is possible is formed by semiconcave functions (Cannarsa & Sinestrari 2004). We refer the reader interested in careful proofs of this and the other convex analytical results used in this paper to the monographs of Rockafellar (1970) and Cannarsa & Sinestrari (2004).

Remarkably, the admissibility condition determines the velocity $v^*$ uniquely.

Lemma 2.1 (Uniqueness). Let $\phi$ be a viscosity solution to the Cauchy problem (1.1) and (1.8). Then at any $(t, x)$ there exists a unique admissible velocity $v^*$. Moreover, this admissible velocity $v^*$ is the unique global minimum point for the function

$$\hat{L}(v) := L(t, x, v) - \min_i (-H_i + p_i \cdot v).$$  \hfill (2.4)

Proof. Recall that $L(t, x, v)$ is a strictly convex function of $v$ because of the assumptions formulated in §1. Rewriting

$$L_i(v) := L(t, x, v) + H_i - p_i \cdot v \quad \text{and} \quad \hat{L}(v) = \max_i L_i(v),$$  \hfill (2.5)

we see that $\hat{L}(v)$ is a pointwise maximum of a finite number of strictly convex functions and therefore is strictly convex itself. Furthermore, because the Hamiltonian $H(t, x, p)$ is assumed to be finite for all $p$, its conjugate Lagrangian $L(t, x, v)$ grows faster than any linear function as $|v|$ increases, and all its level sets are bounded. Thus, $\hat{L}(v)$ attains its minimum at a unique value of velocity $v^*$.

Let us show that this minimum point $v^*$ satisfies the admissibility condition. Indeed, one has

$$\nabla_v L_i(v^*) = \nabla_v L(t, x, v^*) - p_i = p^* - p_i.$$  

Suppose that $p^*$ does not belong to the convex hull of $p_j, j \in I(v^*)$. Then there exists a vector $h$ such that $(p^* - p_j) \cdot h < 0$ for all $j \in I(v^*)$. It follows that $L_j(v^* + \epsilon h) < L_j(v^*)$ for all $j \in I(v^*)$ if $\epsilon > 0$ is sufficiently small. Hence, $\hat{L}(v^* + \epsilon h) < \hat{L}(v^*)$ for sufficiently small $\epsilon$, which contradicts our assumption that $v^*$ is a minimum point. This contradiction proves that $v^*$ is admissible.

To prove uniqueness, we show that if $v$ is admissible then it is a unique global minimum point for the function $\hat{L}$. Using the strict convexity of $\hat{L}$, we obtain

$$L_j(v + h) = L(t, x, v + h) + H_j - p_j \cdot (v + h)$$

$$> L_j(v) + \nabla_v L(t, x, v) \cdot h - p_j \cdot h = L_j(v) + (\hat{p} - p_j) \cdot h,$$
where \( \hat{p} \) is the Legendre transform of \( \hat{v} \). Since \( \hat{v} \) is admissible, \( \hat{p} = \sum_j \lambda_j p_j \), where all \( \lambda_j \geq 0 \) and \( \sum_j \lambda_j = 1 \). Hence,

\[
\sum_j \lambda_j (\hat{p} - p_j) \cdot h = [(\sum_j \lambda_j) \hat{p} - \sum_j \lambda_j p_j] \cdot h = [\hat{p} - \hat{p}] \cdot h = 0.
\]

It follows that \( (\hat{p} - p_j) \cdot h \geq 0 \) for at least one \( j \in I(\hat{v}) \). Thus, \( \hat{L}(\hat{v} + h) > \hat{L}(\hat{v}) \), which implies that \( \hat{v} \) is a unique global minimum point for \( \hat{L} \). This observation concludes the proof.

The admissibility property, first formulated above in a somewhat unmanageable combinatorial form (2.3), turns out to be the optimality condition in a convex minimization problem given by equation (2.4), i.e. a much simpler object. In particular, if \( \phi \) is differentiable at \((t, x)\), then

\[
\hat{L}(v) = L(t, x, v) + H(t, x, \nabla \phi) - \nabla \phi \cdot v
\]

and the minimum in equation (2.4) is achieved at the Legendre transform of \( \nabla \phi \). We thus recover Hamilton’s equation (1.7).

The following reformulation will clarify the connection between admissibility and Bogaevsky’s original construction for the Burgers equation. Let \( v_i = \nabla_p H(t, x, p_i) \) be the velocity corresponding to the limit momentum \( p_i \) and observe that \( p_i = \nabla_v L(t, x, v_i) \). The Young inequality implies that

\[
H_i = H(t, x, p_i) = p_i \cdot v_i - L(t, x, v_i)
\]

and therefore equation (2.5) assumes the form

\[
\hat{L}(v) = \max_i [L(t, x, v) - L(t, x, v_i) - \nabla_v L(t, x, v_i) \cdot (v - v_i)].
\]

The quantity in square brackets is known as the Bregman divergence between vectors \( v \) and \( v_i \), a specific measure of their separation with respect to the convex function \( L \) (Bregman 1967). When \( L(t, x, v) = |v|^2/2 \), the Bregman divergence reduces to (half) the squared distance between the two vectors; hence, the admissible velocity \( v^* \) exactly coincides with the centre of the smallest ball containing all \( v_i \), and Bogaevsky’s result is recovered.

Finally, we discuss the physical meaning of the function \( \hat{L} \). Consider an infinitesimal movement from \((t, x)\) with velocity \( v \). Obviously,

\[
\phi(t, x) + L(t, x, v) \, dt - \phi(t + dt, x + v \, dt) \geq 0.
\]

It is easy to see that, in the linear approximation in \( dt \),

\[
\phi(t, x) + L(t, x, v) \, dt - \phi(t + dt, x + v \, dt) = \hat{L}(v) \, dt.
\]

Hence, the unique admissible velocity \( v^* \) minimizes the rate of the difference in action between the true minimizers and trajectories of particles on shocks. In other words, the trajectory on a shock cannot be a minimizer, but it does its best to keep its surplus action growing as slowly as possible.

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3. The vanishing viscosity limit

In the preceding section we constructed a canonical vector field, \( \mathbf{v}^* = \nabla_p H(t, \mathbf{x}, \mathbf{p}^*) \), corresponding to a given viscosity solution \( \phi \) of the Cauchy problem (equations (1.1) and (1.8)). The basis of this construction, the admissibility condition, appears as a natural consistency condition between velocities and supergradients. This condition, together with the variational principle (2.4), guarantees the uniqueness of the admissible pair \((\mathbf{v}^*, \mathbf{p}^*)\).

The vector field \( \mathbf{v}^*(t, \mathbf{x}) \) can be decomposed into a union of smooth tangent vector fields defined on connected pieces of smooth shock surfaces of various dimensions as well as on the domain where \( \phi \) is differentiable; so the dynamics in the latter domain or inside any piece of a smooth shock surface is given locally by a smooth flow. But globally, the field \( \mathbf{v}^* \) is discontinuous, and it is not immediately clear if there exists an overall continuous flow of trajectories \( \gamma \) that is compatible with \( \mathbf{v}^* \) in the sense that \( \dot{\gamma}(t + 0) = \mathbf{v}^*(t, \gamma) \). Even less obvious is the uniqueness of such a flow.

In order to answer these questions affirmatively, we employ the vanishing viscosity limit for the parabolic regularization

\[
\frac{\partial \phi^\mu}{\partial t} + H(t, \mathbf{x}, \nabla \phi^\mu) = \mu \nabla^2 \phi^\mu, \quad \mu > 0,
\]

of the Hamilton–Jacobi equation (1.1). For sufficiently smooth initial data \( \phi^\mu(t = 0, \mathbf{y}) = \phi_0(\mathbf{y}) \), equation (3.1) has a globally defined strong solution, which is locally Lipschitz with a constant independent of \( \mu \). Moreover, as \( \mu \downarrow 0 \), \( \phi^\mu \) converges to the unique viscosity solution \( \phi \) corresponding to the same initial data. Proof of these facts may be found, for example, in Lions (1982), where they are established for \( \phi_0 \in C^{2,\alpha} \).

Consider now the differential equation

\[
\dot{\gamma}^\mu(t) = \nabla_p H(t, \gamma^\mu, \nabla \phi^\mu(t, \gamma^\mu)) \quad \text{and} \quad \gamma^\mu(0) = \mathbf{y}.
\]

For \( \mu > 0 \) this equation has a unique solution \( \gamma_y^\mu \), which continuously depends on the initial location \( \mathbf{y} \). Fix a point \((t, \mathbf{x})\) with \( t > 0 \) and for all sufficiently small \( \mu > 0 \) pick trajectories \( \gamma^\mu \) such that \( \gamma^\mu(t) = \mathbf{x} \). The uniform Lipschitz property of solutions \( \phi^\mu \) implies that the curves \( \gamma^\mu \) are uniformly bounded and equicontinuous on some interval containing \( t \). Hence, there exists a curve \( \dot{\gamma} \) and a sequence \( \mu_i \downarrow 0 \) such that \( \lim_{\mu_i \downarrow 0} \gamma^\mu_i = \dot{\gamma} \) uniformly. Note that all \( \gamma^\mu_i \) and \( \dot{\gamma} \) are also Lipschitz with a constant independent of \( \mu \) and that \( \dot{\gamma}(t) = \mathbf{x} \). Furthermore let \( \tilde{\mathbf{v}} \) be a limit point of the ‘forward velocity’ of the curve \( \dot{\gamma} \) at \((t, \mathbf{x})\), i.e. for some sequence \( \tau_k \downarrow 0 \) let

\[
\tilde{\mathbf{v}} = \lim_{\tau_k \downarrow 0} \frac{1}{\tau_k} [\dot{\gamma}(t + \tau_k) - \dot{\gamma}(t)].
\]

Of course neither the curve \( \dot{\gamma} \) nor the velocity \( \tilde{\mathbf{v}} \) are \textit{a priori} defined uniquely. Nevertheless, it turns out that \( \tilde{\mathbf{v}} \) must satisfy the admissibility condition with respect to the solution \( \phi \) and therefore it coincides with \( \mathbf{v}^* \). Also, trajectories of the flow \( \gamma^\mu \) converge as \( \mu \downarrow 0 \) to segments of integral trajectories of the vector field \( \mathbf{v}^* \) on smooth shock surfaces, establishing the uniqueness of the limit flow \( \dot{\gamma} \). Moreover, the limit flow is \textit{coalescing}: if two trajectories intersect at time \( t \), they coincide for all \( t' > t \). All these statements follow from the following fact.

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Lemma 3.1. The flow defined by equation (3.2) for sufficiently small \( \mu > 0 \) collapses the neighbourhood of the shock surface that passes through \((t, x)\) and is tangent to \((1, v^*)\) onto this surface.

Here is a sketch of the proof. Let \( v^* \) be the admissible velocity corresponding to \((t, x)\), \( p^* \) the corresponding momentum, and define

\[
H^* = p^* \cdot v^* - \min_i (p_i \cdot v^* - H_i).
\]  

(3.3)

The full time derivative of the function

\[
\psi^\mu(t + \tau, x + \xi) = \phi^\mu(t + \tau, x + \xi) - p^* : \xi + H^* \tau
\]

along an integral trajectory \( \gamma^\mu \) of equation (3.2), which passes through \( x + \xi \) at time \( t + \tau \), is given by

\[
\frac{d\psi^\mu(t + \tau, \gamma^\mu(t + \tau))}{d\tau} = \frac{\partial \phi^\mu}{\partial t} + H^* + (\nabla \phi^\mu - p^*) \dot{\gamma}^\mu.
\]

Convergence of viscosity solutions implies convergence of their superdifferentials (to see this, it is enough to replace in equation (2.1) the function \( \phi \) with a sequence of functions \( \phi_n \) converging pointwise). Therefore, limit points of \((\partial \phi^\mu / \partial t, \nabla \phi^\mu)\) belong to \( \partial \phi(t, x) \) for all \((t, x)\), and for sufficiently small \( \mu > 0 \), \( \tau > 0 \) and \( \xi \) there exist \((-H^*, p^*) \in \partial \phi(t, x)\) and \( v^* = \nabla_p H(t, x, p^*)\) such that, to the linear approximation, \( \nu^\mu = v^* + \cdots \)

\[
\frac{d\psi^\mu}{d\tau} = -H^* + H^* + (p^\mu - p^*) \cdot v^* + \cdots \]

(3.4a)

\[
= p^\mu \cdot v^* - H^* - \min_i (p_i \cdot v^* - H_i)
\]

(3.4b)

\[
+ (p^\mu - p^*) \cdot (v^\mu - v^*) + \cdots.
\]

(3.4c)

Lines (3.4b) and (3.4c) contain non-negative quantities, which are positive if \( p^\mu \neq p^* \). To see this for (3.4c), observe that by the strict convexity \( H(t, x, p^\mu) > H(t, x, p^*) + (p^\mu - p^*) \cdot v^* \) and \( H(t, x, p^*) > H(t, x, p^*) + (p^* - p^*) \cdot v^* \), and add these two inequalities.

On the other hand, the superdifferential of the function

\[
\lim_{\mu \downarrow 0} \psi^\mu(t + \tau, x + \xi) = \phi(t + \tau, x + \xi) - p^* : \xi + H^* \tau
\]

at \((\tau = 0, \xi = 0)\) contains zero because \( p^* \) corresponds to an admissible velocity. Therefore, in the linear approximation the point \( \xi = 0 \) and other points of the shock surface of \( \psi \) tangent to \((1, v^*)\) are local maxima of \( \psi \) up to terms of higher order, and the full time derivative of \( \psi \) along the curve \( \xi = v^* \tau \) vanishes. As \( d\psi^\mu / dt \) is positive for trajectories that start outside the shock, we see that the flow (3.2) collapses them asymptotically on the shock surface. This completes the argument.

The complete details of this proof and the rigorous derivation of uniqueness of the limit flow \( \dot{\gamma} \) will be given in a forthcoming article (Khanin & Sobolevski in preparation). Here we just formulate the main result.

Theorem 3.2. Let \( \phi \) be a viscosity solution to the Cauchy problem for the Hamilton–Jacobi equation (1.1) with initial data (1.8). There exists a unique flow \( \gamma_y \) of continuous trajectories such that \( \gamma_y(0) = y \), \( \dot{\gamma}_y(t + 0) \) is defined for all \( t, y \)

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and coincides with the admissible velocity \( v^*(t, \gamma_y(t)) \) given by solution to the convex minimization problem for equation (2.4). The trajectory \( \gamma_y \) continuously depends on \( y \). After a trajectory \( \gamma_y \) comes to a shock, it stays inside the shock manifold for all later times. The flow is coalescing: if two trajectories \( \gamma_y' \) and \( \gamma_y'' \) coincide at time \( t \), then \( \gamma_y'(t') = \gamma_y''(t') \) for all \( t' > t \).

4. Concluding remarks

Starting from a viscosity solution \( \phi \) to the Hamilton–Jacobi equation, we have constructed a unique continuous coalescing flow \( \gamma_y \) compatible with the admissible velocity field \( v^* \) defined in §2 in the sense that \( \dot{\gamma}_y(t + 0) = v^*(t, \gamma_y) \). This flow is a natural extension of the smooth flow defined by Hamilton’s equation (1.7). We conclude with a few observations concerning this construction.

1. Recall an important observation made in the original work of Bogaevsky: pieces of the shock manifold, irrespective of their dimension, are classified into restraining and non-restraining depending on whether \( p^* \) belongs to the interior or the boundary of the convex polytope formed by projection of \( \partial \phi(t, x) \) on the \( p \) space. Particles stay on restraining shocks but leave non-restraining shocks along pieces of shock manifold of lower codimension corresponding to faces of the boundary containing \( p^* \). Shocks of codimension one are always restraining; in particular, this is the case in the one-dimensional setting, which makes the construction of the coalescing flow \( \gamma_y \) trivial, as remarked in §1. Interestingly, this classification of shocks, introduced in Bogaevsky (2004) (‘acute’ and ‘obtuse’ superdifferentials of \( \phi \)), seems to have been overlooked by physicists before.

2. Note that the construction of the admissible velocity \( v^* \) is purely kinetic: when the Lagrangian is ‘natural’, i.e. has the form \( L(t, x, v) = K(v) - U(t, x) \), the value \( v^* \) is the same for all choices of the potential term \( U(t, x) \) as long as the kinetic energy \( K(v) \) is fixed. We owe this observation to P. Choquard.

3. Seen as a family of continuous maps of variational origin from initial coordinates \( y \) to current coordinates \( x \), the flow \( \gamma_y \) is clearly relevant for optimal transportation problems (Gangbo & McCann 1996; Villani 2009). An interesting problem suggested by B. Khesin is to study the extremal properties of this flow. Indeed, it is known from Khesin & Misiolek (2007) that, before the first shock formation, the flow \( \gamma_y \) is an action-minimizing flow of diffeomorphisms, while the first shock formation time \( t^* \) marks a conjugate point in the corresponding variational problem. According to the suggested view, the flow constructed above may be seen as a kind of saddle-point, rather than a minimum, for a suitable transport optimization problem.

4. Another natural context in which to place our construction is that of differential inclusions (e.g. Aubin & Cellina 1984). The flow constructed here may be seen as a solution of the differential inclusion

\[
\dot{\gamma} \in \nabla_p H(t, \gamma, \text{Pr}_p \partial \phi(t, \gamma)),
\]  

(4.1)
where \( \Pr_p \) is the \( p \) projection of the superdifferential \( \partial \phi \). In comparison with standard constructions of the theory of differential inclusions, the flow \( \gamma_y \) solves equation \((4.1)\) in a stronger sense: the forward derivative \( \dot{y}_y(t + 0) \) exists everywhere. Also, a simple modification of lemma 3.1 gives a proof of uniqueness for inclusion (equation \((4.1)\)).

5. The flow \( \gamma_y \) was constructed as a limit of a parabolic regularization, and it was noticed above that the limit of \( (-\partial \phi^\mu/\partial t, \nabla \phi^\mu) \) belongs to \( \partial \phi(t, x) \) for any \((t, x)\). This statement can be refined if one considers the values of \( \nabla \phi^\mu \) along trajectories of the flow (equation \((3.2)\)). Namely, arguments of §3 imply that, under the successive limits \( \mu \downarrow 0 \) and \( \tau \downarrow 0 \), the gradient \( (\partial \phi^\mu/\partial t, \nabla \phi^\mu) \) taken at \((t + \tau, \gamma^\mu(t + \tau))\) converges to \((-H^*, p^*)\). Therefore

\[
\lim_{\tau \downarrow 0} \lim_{\mu \downarrow 0} \mu \nabla^2 \phi^\mu(t + \tau, \gamma^\mu(t + \tau)) = \lim_{\tau \downarrow 0} \lim_{\mu \downarrow 0} \left[ \frac{\partial \phi^\mu}{\partial t} + H(t, \gamma^\mu(t + \tau), \nabla \phi^\mu) \right] \\
= H(t, x, p^*) - H^* \\
= \min_i (p_i \cdot v^* - H_i) - L(t, x, v^*) \\
= \hat{L}(v^*) = \min_v \hat{L}(v),
\]

where we have taken into account formulae \((3.3)\) and \((2.4)\). In other words, the minimum of the convex minimization problem \((2.4)\) coincides with the value of the ‘dissipative anomaly’ in the parabolic regularization \((1.4)\) of the Hamilton–Jacobi equation \((1.1)\).

6. Observe also that convergence of superdifferentials makes it possible to use other smoothing procedures for \( \phi \) (e.g. taking convolution with a standard mollifier), giving the same limit \( \gamma_y \). However, one can imagine the following completely different regularization of the discontinuous velocity field \( \nabla_p H(t, x, \nabla \phi(t, x)) \). Physically speaking, this regularization may be characterized by a ‘zero Prandtl number’ in contrast with the previous class of regularizations featuring an ‘infinite Prandtl number’.

Consider the stochastic equation

\[
d\gamma' = \nabla_p H(t, \gamma', \nabla \phi(t, \gamma')) \, dt + \epsilon \, dW(t),
\]

where \( W \) is the standard Wiener process. The corresponding stochastic flow is well defined in spite of the fact that \( \nabla \phi \) does not exist everywhere: whenever the trajectory \( \gamma' \) hits shocks, noise in the second term will instantaneously steer it in a random direction away from them.

Assume that, as \( \epsilon \downarrow 0 \), the stochastic flow \( \gamma_y' \) tends to a limit flow \( \tilde{\gamma}_y \), which is also forward differentiable. It is easy to see that, because of the averaging, the forward velocity \( v^!(t, \tilde{\gamma}) := \tilde{\gamma}_y(t + 0) \) must belong to the convex hull of \( v_j, j \in I(v^!) \). Namely, one has

\[
v^! = \sum_{j \in I(v^!)} \pi_j v_j, \quad \pi_j \geq 0 \quad \text{and} \quad \sum_{j \in I(v^!)} \pi_j = 1, \quad (4.2)
\]

where the velocities \( v_j(t, x) \) are Legendre transforms of the corresponding momenta \( p_j = \nabla \phi_i(t, x) \) at a singular point \((t, x)\). The coefficients \( \pi_i \) are equal to the asymptotic values of the shares of time that a trajectory \( \gamma' \) spends in
each of the domains where $\phi = \phi_j$. Condition (4.2) above is another compatibility condition, in a certain sense dual to the admissibility condition in §2. For want of a better term, let us call such a velocity $v^\dagger$ *self-consistent*.

The self-consistent velocity is a convex combination of velocities seen by an infinitesimal observer leaving $(t, x)$ with velocity $v^\dagger$. Compare this with the definition of admissible momentum $p^\ast$, which is a convex combination of momenta seen by a similar observer moving with velocity $v^\ast$. When $H(t, x, p) = |p|^2/2$ and $v = p$, self-consistent velocities and admissible velocities coincide. It is however clear that in the case of a general nonlinear Legendre transform $v^\dagger \neq v^\ast = \nabla_p H(t, x, p^\ast)$.

In view of the analogy between self-consistent velocities and admissible momenta, it is tempting to conjecture that the admissible velocity is also unique. Generally speaking, this statement is wrong, although it holds in one and two space dimensions. In higher dimensions there exist Hamiltonians and sets of limiting momenta for which there is more than one admissible velocity (Khanin & Sobolevski in preparation). It is an interesting problem nevertheless to see whether a limiting flow still exists in the limit of weak noise in spite of the non-uniqueness of the admissible velocity. This problem bears a certain similarity with the problem of limit behaviour for one-dimensional Gibbs measures in the zero-temperature limit, in the case of non-unique ground states.

This work is supported by the NSERC Discover Grant and the Russian Foundation for Basic Research, project RFBR-CNRS–07–01–92217; the second author is supported in part by the French Agence Nationale de la Recherche, project ANR–07–BLAN–0235 OTARIE. We acknowledge the hospitality of Observatoire de la Côte d’Azur (Nice, France), where part of this work has been performed, as well as the support of the French Ministry for National Education.

We are sincerely grateful to Jérémie Bec, Patrick Bernard, Ilya Bogaevsky, Yann Brenier, Philippe Choquard and Boris Khesin for numerous valuable discussions. It is our special pleasure to express gratitude to Uriel Frisch, who made crucial contributions to the current revival of interest in the ‘Burgers turbulence’ and related fields, and whose 70th birthday is in 2010.

References


*Phil. Trans. R. Soc. A* (2010)
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