An experimental and theoretical study of the mixing characteristics of a periodically reoriented irrotational flow

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The minimum-energy method to generate chaotic advection should be to use an irrotational flow. However, irrotational flows have no saddle connections to perturb in order to generate chaotic orbits. To the early work of Jones & Aref (Jones & Aref 1988 Phys. Fluids 31, 469–485 (doi:10.1063/1.866828)) on potential flow chaos, we add periodic reorientation to generate chaotic advection with irrotational experimental flows. Our experimental irrotational flow is a dipole potential flow in a disc-shaped Hele-Shaw cell called the rotated potential mixing flow; it leads to chaotic advection and transport in the disc. We derive an analytical map for the flow. This is a partially open flow, in which parts of the flow remain in the cell forever, and parts of it pass through with residence-time and exit-time distributions that have self-similar features in the control parameter space of the stirring. The theory compares well with the experiment.

Keywords: chaotic advection; irrotational flow; potential flow; mixing; minimum energy; Hele-Shaw

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One contribution of 10 to a Theme Issue ‘Experiments in complex and excitable dynamical systems’.
1. Introduction

For decades, it has been appreciated that the motions of fluid particles of planar incompressible flows correspond to the orbits of a Hamiltonian dynamical system: with $\psi$, the stream function of the flow and $x$, $y$ the space coordinates, the conservative dynamical system is

$$\frac{dx}{dt} = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{dy}{dt} = \frac{\partial \psi}{\partial x},$$

(1.1)

and the physical space of the fluid directly visualizes the conjugate variable space of the dynamical system. See, e.g. Aref (2002) for a history of the development of chaotic advection. For applications, we are not only interested in how to generate and use chaotic advection, but also in how to generate chaos in the simplest and most energy-efficient ways. What is the minimum-energy method to generate chaotic advection in an experimental flow? Answering this question is essential in designing energy-efficient mixers, heat exchangers and chemical reactors. See, e.g. Metcalfe (2010) for a review of applications of fluid chaos. In this paper, we will reconcile two seemingly incompatible facts in an approach to answer this question and explore some of the experimental and theoretical mixing characteristics of a minimum-energy chaotic flow.

Perturbing elliptic points of a steady flow produces the classic route to chaos in Hamiltonian systems of destroying Kolmogorov–Arnold–Moser (KAM) resonant torii, replacing them with pairs of elliptic and hyperbolic periodic orbits that embed chaotic orbits between intersecting stable and unstable manifolds (Ott 2002). On the other hand, Kelvin’s minimum-energy theorem says that of all incompressible flows in a given domain, the flow having the smallest kinetic energy is the unique irrotational flow compatible with the boundary conditions (Batchelor 1967). Lacking points of non-zero circulation—elliptic points or more generally saddle connections—leaves nothing to perturb to generate chaotic orbits in irrotational flows; and, indeed nearly all chaotic advection studies have used flows with some vorticity. Is chaos in fluids incompatible with minimum-energy use? Or can we find ways to generate minimum-energy chaotic advection using irrotational flows? To answer these questions, we will follow the pioneering but neglected work on potential flow chaos of Jones & Aref (1988) with the addition of our own ideas of using periodic reorientation of flows to generate chaotic advection with irrotational experimental flows.

2. A periodically reoriented irrotational flow

Little work exists on mixing and transport in irrotational flows. Jones & Aref (1988) made an early study of a pulsed source–sink flow in the open plane. Their motivation was mainly to contrast with the (still) dominant choice of viscous flows, in order to demonstrate the purely kinematic nature of chaotic advection. Potential flow mixing was revived in one of the earliest designs of a micromixer (Evans et al. 1997) and later in a pulsed-flow design of a DNA chip with improved hybridization time (Stremler & Cola 2006).
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Our experimental irrotational flow is a dipole potential flow in a disc-shaped Hele-Shaw (HS) cell. HS flow occurs in the gap between rigid, parallel plates when the gap size $b$ is much smaller than the lateral extent $R$ of the plates, and HS flow is effectively two-dimensional (Batchelor 1967). If $x$ and $y$ are the horizontal coordinates, then the two-dimensional, steady, incompressible HS velocity field is

$$ u(x, y) = -\frac{b^2}{12\mu} \nabla P(x, y), \quad (2.1) $$

where $\nabla P$ is the horizontal pressure gradient. A HS flow in a disc with one inlet and one outlet $180^\circ$ apart generates a dipole flow, streamlines for which are shown as black lines in figure 1a. Call the stream function of the dipole flow—or more generally of any multi-well potential flow—the base flow $\psi$. For a source and sink at $\{r, \theta\}=(1, \pi/2)$ and $(1, -\pi/2)$ respectively, the stream function is

$$ \psi = \arctan \left( \frac{2r \cos \theta}{1 - r^2} \right). \quad (2.2) $$
Let $\psi$ operate for a time $\tau$, after which stop the flow and reorient it through an angle $\Theta$; restart the flow and repeat the sequence. Figure 1b shows three successive reorientations of $\Theta = 2\pi/3$ by superposing successive sets of streamlines. We will call this stirred potential flow the \textit{rotated potential mixing} (RPM) flow, and we will find that it leads to chaotic advection and transport in the disc in a manner similar to that of previous work on reoriented flows (Metcalfe et al. 2006; Speetjens et al. 2006; Lester et al. 2008, 2009).

To see the route to chaos more clearly, consider the time-averaged Hamiltonian $H$ of this periodically reoriented flow (PRF),

$$H = \frac{1}{N} \sum_{n=0}^{N-1} \psi(r, \theta - n\Theta).$$

(2.3)

In the $\tau \to 0$ limit, $H$ is steady and integrable; figure 1c shows contours of $H$ for $\Theta = 2\pi/3$. ($H$ for irrational $\Theta$ is a set of concentric circles.) Note that the PRF converts the irrotational flow into a Hamiltonian flow with a central elliptic point. Increases in $\tau$ now do perturb this central elliptic island, leading to chaotic orbits. However, a substantial point of difference with the RPM experimental flow is that it is open: fluid can and does enter and leave the domain through the inlet and outlet wells. While we do not pursue it further here, note that conceptually the RPM flow could be closed by making the inlet and outlets symmetry boundaries, i.e. material leaving the sink is re-injected at the source. Jones & Aref (1988) noted how the choice of re-injection protocol (delayed, reflected, stochastic) generates different Lagrangian topologies; although, for the RPM flow, bifurcations of the central elliptic island are not affected by re-injection protocol (Lester & Metcalfe 2008). We also note that there are engineering problems in porous media, where the inlets and outlets are connected for recirculation and where getting the reinjection model right could be crucial (Metcalfe et al. 2008).

As an example of topological structure, two orbits of the RPM flow for $(\tau, \Theta) = (0.19, \pi/5)$ are shown in figure 1d. The large crosses are the two initial conditions. The thick (thin) lines denote the orbit for the flow (rotation) part of the map. The dark orbit leaves through the sink, and we have chosen to re-inject it at the source, while the light orbit never exits the disc but circulates forever in a KAM island. This is remarkable because for $\psi$ by itself, all points in the disc exit after a finite time. The discontinuities in the orbits are caused by rotating the points instead of the flow. In an experiment, the orbits are continuous and the flow is rotated. In this sense, figure 1d—and subsequent computational figures—can be thought of as being viewed from a reference frame rotating with the flow. Note that the dipole is merely the simplest configuration of RPM flow in a disc. As many wells as desired can be used as inlets or outlets, and neither need the domain to be a disc.

3. Experiment

The experimental RPM flow consists of a HS cell with a rotating manifold to open and close pairs of source–sink wells located at the edge of the cell, as shown in figure 2, where the left panel shows an exploded view of the assembly and the right panel shows a schematic of the HS cell. The cell has an aluminium base
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Figure 2. Experimental version of the RPM flow. (a) Exploded view. (b) Schematic of the stirred HS cell.

plate and a glass top plate separated by a steel shim, which sets the gap size of thickness $b = 120 \mu$m. The cell’s fluid region is circular with radius $R = 60 \text{ mm}$. At the perimeter of the cell, there are 360 holes of diameter 0.6 mm spaced at 1° intervals. Each hole can act as a source or sink. Below the base plate is a rotating manifold with a rubber gasket glued to its top. The gasket has two holes spaced 180° apart, such that the gasket seals all wells leading to the fluid space, except for the opposing pair coinciding with the position of the gasket holes. The manifold is rotated by a stepper motor to open and close source–sink pairs and orient the dipole flow. There are two reservoirs, one feeds the source and the other collects the fluid from the sink. The source flow is driven by a constant pressure high-pressure liquid chromatography pump. The source reservoir is the space under the manifold, and the sink reservoir is a recessed band around the circumference of the manifold. O-rings seal the reservoirs from each other and from the cell. In this way, entry and exit of fluid need not be through rotating connections. Solenoid valves at the fluid entry and exit points ensure flow stops in the cell during reorientation. The assembly is covered with a box containing a circular UV light tube and a hole in the top allowing camera access.

The working fluid was a glycerin–water mixture (50–70% glycerin); the initial fluid in the cell had a pink fluorescent dye added. A charge-coupled device camera with a resolution of 1600 × 1200 pixels captured images after flow was stopped,
i.e. after every $\tau$ interval. A computer program controlled the manifold rotation, solenoid valves and triggered the camera on input of $\tau$, $\Theta$ and the number of flow iterations to perform.

4. Theory

The RPM flow is fully described by the control parameters $\tau$ and $\Theta$. As time dependence is simulated by switching among symmetric copies of the base flow, any base-flow configuration and any of its symmetries can be used to stir and drive transport within the disc. Importantly, different symmetries will affect the transport properties, particularly the presence and bifurcation structure of islands, which are regions of the flow with restricted or non-existent fluid exchange with other regions of the flow. In this section, we will first outline the pertinent symmetry theory and then will make several testable predictions to be compared with the experiment in §5.

Dynamically closed flows are much more widely studied than dynamically open flows. Closed advection is largely governed by the symmetries of the base flow and boundary conditions (Ottino et al. 1992; Speetjens et al. 2006). Open advection is largely governed by filamentary unstable manifolds (Tél et al. 2005). As the RPM flow exhibits transport characteristics of both closed and open flows, we may call it a partially open flow. We first discuss transport in the closed part of the RPM flow and then the open part.

Transport in stirred flow is generically governed by stable and unstable manifolds of hyperbolic periodic points of the flow. Points approach stable manifolds as they move forward in time and approach unstable manifolds as they move backward in time. Periodic elliptic points are important because they have disconnected (possibly large) island regions around them that sequester or hinder (through cantori) material transport and reaction activity (Ottino et al. 1992; Speetjens et al. 2006). Islands are regions of rotation without much deformation that have KAM surfaces separating them from the rest of the flow. At $\tau = 0$, the PRF has a single elliptic point at the origin for all $\Theta$ (except for the trivial case of $\Theta = 0$), and the associated island takes up the entire disc. As $\tau$ rises from zero, where does the origin elliptic point go, how does it bifurcate along the way and how does the area of the associated island change? Do new elliptic points and islands arise?

(a) Symmetry

Let $\hat{Y}_t$ be the continuous-flow solution to the kinematic equation, $\dot{x} = u(x, y)$ describing passive particle transport due to equation (2.1),

$$x(t) = \hat{Y}_t(x(0)).$$

This flow can be reduced to a map that operates in the dipole frame of reference, i.e.

$$Y = R^{-1} \int_0^\tau \hat{Y}_t \, dt,$$
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where $R$ is the rotation operator

$$R : \theta \rightarrow \theta + \Theta. \quad (4.3)$$

As the Lagrangian topology is invariant under rotations, then analysis of the stroboscopic map $Y$ in the dipole frame is equivalent to that of the continuous map $\hat{Y}_t$ in the laboratory frame. The advantage of such an approach is that the system in the dipole frame is $\tau$-periodic. If $x_n$ is the location of a fluid particle in the dipole frame at time $t = n\tau$, then the map

$$x_{n+1} = \hat{Y}(x_n) \quad (4.4)$$

describes evolution of the dynamical system for integer values of $\tau$. If the stroboscopic map associated with the base flow over time $\tau$ is denoted $Y_0$, then from equation (4.2), $Y = R^{-1}Y_0$.

From the domain geometry and boundary conditions, the base flow contains two symmetries: a reflection symmetry along the $y$-axis and a reflection-reversal symmetry along the $x$-axis,

$$Y_0 = S_0 R_0 S_0 \quad \text{and} \quad S_0 : \theta \rightarrow \pi - \theta, \quad (4.5)$$

$$Y_0 = S_1 R_0^{-1} S_1 \quad \text{and} \quad S_1 : \theta \rightarrow -\theta. \quad (4.6)$$

Substitution of equation (4.5) into equation (4.2) yields

$$Y = R^{-1} S_0 R_0 S_0 = R^{-1} S_0 R S_0, \quad (4.7)$$

and using $R = S_0 R^{-1} S_0$ and $S_0^2 = I$, then

$$S_0 R S_0 = R^2 R, \quad (4.8)$$

where $R^2$ maps $\Theta \rightarrow -\Theta$. Therefore, the Lagrangian topology is symmetric between positive and negative $\Theta$. Substitution of the symmetry

$$S_2 = R^{-1} S_1 \iff S_1 = R S_2 \quad \text{and} \quad S_2 : \theta \rightarrow \theta - \Theta \quad (4.9)$$

into equations (4.6) and (4.2) yields

$$Y = R^{-1} S_1 R_0^{-1} S_1 = S_2 R^{-1} S_2, \quad (4.10)$$

and so $Y$ also satisfies the reflection-reversal symmetry $S_2$, further constraining coherent structures in the Lagrangian topology to evolve symmetrically about $\theta = -\Theta/2$.

A symmetry axis immediately says that all period one (P1) points must lie on the symmetry axis. Consider a line of initial conditions advected by $Y$. $R^{-1}$ rotates the line counter-clockwise without deformation, then $\psi$ advects the line towards the sink, stretching it because the velocity field is not radially uniform. The intersection of the initial line and its first iterate gives the P1 points. As $\tau$ is increased above zero for any given $\Theta$, the origin elliptic point must move along the symmetry axis.
Figure 3. Plot of residence time $T_{\text{res}}$ in the dipole flow from injection at the source on streamline $\psi$ to exiting at the sink.

As fluid particles travel along streamlines of constant $\psi$ during advection, Lester & Metcalfe (2008), using equation (2.2) for the stream function, derived an expression for the advection time $t_{\text{adv}}$ along a streamline as

$$t_{\text{adv}}(\theta, \psi) = \text{sgn}(\psi) \csc^2 \psi \left( \cot \psi \arctan \left[ \frac{\sin \theta \cot \psi}{\sqrt{1 + \cos^2 \theta \cot^2 \psi}} \right] \right.$$ 

$$+ \sin \theta \sqrt{1 + \cos^2 \theta \cot^2 \psi} - |\cot \psi| (\theta + \cos \theta \sin \theta) \right).$$

(4.11)

Symmetries of the flow are preserved; $t_{\text{adv}}$ is odd with respect to both $\theta$ and $\psi$. As $t_{\text{adv}}(0, \psi) = 0$, $t_{\text{adv}}(\theta_1, \psi)$ is the time for a particle to be advected from $\theta = \theta_1$ to $\theta = 0$ along streamline $\psi$. From equation (4.11), the residence time $T_{\text{res}}$ of a fluid particle injected at the source on streamline $\psi$ to exit at the sink is

$$T_{\text{res}}(\psi) \equiv \text{sgn}(\psi) \left[ t_{\text{adv}}\left(\frac{\pi}{2}, \psi\right) - t_{\text{adv}}\left(-\frac{\pi}{2}, \psi\right) \right] = \csc^2 \psi (2 - \cot \psi) \left(\frac{\pi}{2} - \psi\right).$$

(4.12)

which is plotted in figure 3. $T_{\text{res}}$ ranges from $2/3$ for the centreline trajectory to $T_{\text{res}}(\pm \pi/2) = 2$ for the outermost streamlines. A fluid particle at initial position $\theta_1$ advected for time $\tau$ is then positioned at $\theta_2$ such that

$$t_{\text{adv}}(\theta_2, \psi) + \frac{T_{\text{ref}}}{2} = t_{\text{adv}}(\theta_1, \psi) + \frac{T_{\text{ref}}}{2} + \tau \text{ mod } T_{\text{ref}},$$

(4.13)

which gives the advection map $Y_0$ for $t = \tau$,

$$Y_0 : \{\theta_1, \psi\} \longrightarrow \{\theta_2, \psi\}.$$  

(4.14)

The rotational map $R$ is given by

$$R : \{\theta, \psi\} \longrightarrow \left\{ \theta + \Theta, \arctan \left( \frac{\cos(\theta + \Theta)}{\cos \theta} \tan \psi \right) \right\}.$$  

(4.15)

The composite map $Y = R^{-1} Y_0$ fully describes the advection of fluid particles for integer multiples of time $\tau$ in the dipole frame of reference.
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Figure 4. The parameter space boundary for the existence of islands in the RPM flow and selected Poincaré sections as described in the text.

(b) Stability of the fixed point

The stability of period-$p$ points of the stroboscopic map $Y^p(x_0) = x_0$ directly controls the local transport dynamics of a given flow. Non-degenerate periodic points of Hamiltonian systems are either elliptic or hyperbolic: elliptic points are locally stable and represent non-mixing islands, which fluid cannot traverse. Hyperbolic points are locally unstable and possess stable and unstable manifolds, which may intersect to form heteroclinic or homoclinic connections, a recognized signature of chaos. As elliptic points and their associated islands impede good mixing, the bifurcation of elliptic points to hyperbolic is a necessary but not sufficient condition for complete global mixing. As $P1$ fixed points must reside on the same streamline for all times, $P1$ fixed points satisfy

$$\tau = 2 \, t_{\text{adv}} \left( \frac{\Theta}{2}, \psi \right).$$

Consequently, the fixed point that occurs at the origin in $\mathcal{D}$ for $\tau = 0$ and all $\Theta$ moves along the $\theta = -\Theta/2$ axis of symmetry in the negative-$\psi$ direction. This fixed point reaches the separating streamline $\psi = -\pi/2$ at $\tau = 2t_{\text{adv}}(-\Theta/2, -\pi/2)$ and disappears at larger $\tau$, and there is at most one fixed point of the flow, which occurs for $\tau \leq 2t_{\text{adv}}(-\Theta/2, -\pi/2)$.

(c) Predictions: island existence, island size and residence times

Several predictions are possible with the symmetry theory and analytical map, $Y$. We begin with the existence of the central fixed point. It disappears above a value of $\tau = \tau_e$, that is a function of $\Theta$ given by

$$\tau_e = 2 \sin \left( \frac{\Theta}{2} \right),$$

which is shown in figure 4 as the curve rising from $\tau = 0$. Below the $\tau_e$ line, a fixed point exists in the disc, while above the line, the fixed point does not exist. Figure 4 also illustrates the symmetry results with a selection of Poincaré maps.
As the RPM flow is open (points can and do leave through the sink), these are non-standard Poincaré maps that only show the closed parts of the flow, except for the first few iterates as open flow points move to the sink. Poincaré maps of the disc are either centred on the parameter point to which they belong or are displaced along a line whose end is on the parameter point to which they belong. Maps below \( \tau_e \) show the last 60 of 100 iterates of \( Y \). The solid line marks the symmetry axis and the initial conditions of the map. The rotation of the symmetry axis with \( 2\Theta \) and the movement of the P1 elliptic point towards the boundary with increasing \( \tau \) is clear. The maps for values above \( \tau_e \) show the first three to five iterates of \( Y \); after this, all points have exited. To the right of the point labelled \( \Theta_{\text{crit}} \) is an example of something the symmetry analysis does not reveal. Within the curve labelled \( \tau_u \) and \( \tau_1 \), the elliptic point undergoes a period-doubling bifurcation, as \( \tau \) increases above \( \tau_1 \) to a hyperbolic point, but retains a KAM boundary to form a non-elliptic isolated mixing region (Bresler et al. 1997). As \( \tau \) increases further to approach \( \tau_u \), the now hyperbolic point undergoes an inverse cascade back to an elliptic point.

On the \((\tau, \Theta)\) plane, figure 5 plots an estimate of the area, as a fraction of the total disc area, of points remaining in the disc forever. This mainly estimates the island size. The size estimate is calculated by randomly placing 10 000 initial points in the disc and advecting them for a total of 100 times greater than the emptying time for the un-reoriented flow. The fraction of points left after this time is the estimate of the island area. This calculation was done on a \(360 \times 100\) grid in the parameter space \(0.02 \leq \tau \leq 2\) and for every degree in \(\Theta\). Colour in

Figure 5. Island size. In the \(\tau-\Theta\) plane, colours indicate the log of the number of points estimated to remain in the disk for all time.
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Figure 6. (a) Exit-time $t_e$ distribution for the non-rotated dipole flow. The maximum $t_e$ defines the total area exchange time $t_{ex}$. The inset shows the spatial distribution of exit times colour-coded according to $\log(t_e)$. (b) Frequency distribution of exit times $t_e$ normalized to the area exchange time $t_{ex}$ at $(\tau, \Theta) = (0.4, 4\pi/15)$. (inset) The spatial map of exit times coloured by $\log(t_e)$ according to the colour scale. Black are points that do not exit after $10t_{ex}$ and correspond to an island.

Figure 5 indicates the log of the number of points remaining in the disc. Red is a large island size; pink is a small island size; and magenta is zero points remaining. There is a good bit of structure in the plot still to be understood, but the island existence boundary of equation (4.17) is clearly shown, and there seems to be ‘tongues’ of low island size associated with integer values of $\Theta/\pi$.

Figure 6 shows an example of how stirring modifies the exit-time $t_e$ distribution of fluid initially in the disc. For the non-rotating dipole flow, figure 6a shows the exit-time frequency distribution calculated by randomly placing around 3000 points in the disc and running the map $Y(\tau = 1, \Theta = 0)$ until all the points have

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exited at one area exchange time $t_{ex}$. The inset plots the initial points in the disc coloured by $\log(t_{eq})$ according to the scale in the figure. With no rotation, the spatial pattern of exiting is an orderly progression with points near the sink exiting first and those starting in the low-velocity regions near the boundary and the source exiting last. Figure 6b shows an example of an exit-time distribution under stirring. There is no re-injection, and after $10t_{ex}$ we declare the remaining points to be in an island. The frequency distribution only shows points that did exit. Several things are noteworthy. The exit order has some simplicity. Points in a scallop shape near the sink leave similar to the un-reoriented flow. Then, new points are rotated near the sink and they leave in their turn. However, points in the boundary between scallops have markedly longer exit times. Instead of exiting with their neighbours, these veins flow around the island first before exiting. They are 2–4% of the total and account for the toe in the frequency distribution above 0. Points that do not exit after $10t_{ex}$ are coloured black and correspond to an island.

Tél et al. (2005) review reaction and transport in open chaotic flows. By open, Tél et al. mean a flow that is relatively uniform far upstream and far downstream from a region of chaotic interaction or stirring. This allows treatment of transport in the flow as a chaotic scattering problem. A key theoretical concept is exit-time distributions from which one can derive exit rates. The orbits of points with the longest exit times can be followed to approximate the filamentary manifolds controlling transport in the chaotic scattering region. On the other hand, the chemical engineering world would treat reaction and transport in the RPM flow in the same manner as a through flow reactor and would generate residence-time distributions (Nauman & Buffham 1983). We show an example of each approach for advection in the RPM flow, emphasizing somewhat the parametric variation, i.e. how residence or exit times change over the entirety of the control parameter space $(\tau, \Theta)$ of the flow.

Figure 7a shows the distribution of residence times $t_r$ of a point released at the source over a grid of about 1 million points in $(\tau, \Theta)$ space for the RPM flow. The point is released $10^{-7}$ below the source to ensure it is on the central streamline. For the un-reoriented flow $\psi$, a point released at the source moves down the central streamline and exits through the sink after a time $t_{r0} = 2/3$, c.f. equation (4.12). The inset plots the frequency distribution on a log-scale showing the distribution’s power-law tails. A fit of the tails—grey lines in the figure—gives exponents of $0.222 \pm 0.006$ for the short-time tail and $-0.275 \pm 0.002$ for the long-time tail. Stirring expands the residence-time distribution by three orders of magnitude both above and below the unstirred residence time. Figure 7b shows the $(\tau, \Theta)$ parameter space colour-coded by $\log(t_r/t_{r0})$. There is clear structure, including what seems to be self-similar resonance ‘tongues’ in the lower left corner of the space.

5. Comparison of experiment and theory

Figure 8 shows a comparison of the experimental and theoretical RPM flow for $\tau = 0.1$ and $\Theta = \pi/4$, i.e. below the island existence boundary. In the top row, the experimental initial condition is a uniform distribution of red fluid in the disc. Clear fluid flows from the source well and displaces the dyed fluid. The bottom line shows simulations at the same conditions. The numbers on the bottom give
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Figure 7. (a) Normalized frequency distribution of residence times $t_r$ scaled to the residence time without reorientation of an orbit initiated at the source singularity ($t_{r0} = 133.342$). Inset plots the log of the frequency to show the power-law tails of the distribution, which have different slopes as indicated. Median($t_r/t_{r0}$) = 1.30. (b) Distribution of residence times of an orbit initiated at the source singularity over the $\tau$-$\Theta$ parameter space with $\tau$ scaled to the residence time without reorientation ($t_{r0} = 133.342$). The grey scale bar gives the log of the residence time $t_r$ scaled to $t_{r0}$.

the number of flow and reorientation iterations. For iterations above 13, the red island persists unchanged. Compare with figure 6b. Note what appears to be filamentary manifolds developing and thinning.

Figure 9 shows the island existence boundary decorated with several experiments at the points indicated by the red dots. Below the line, islands of some size exist, though closer to the boundary island, size becomes smaller. Compare with figure 5. Above the line, all the dyed fluid initially in the cell eventually leaves.

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Figure 8. (a–d) RPM flow experiments and (e–h) simulations at the indicated number of iterations for $\tau = 0.1$ and $\theta = \pi/4$.

Figure 9. Island existence boundary with experiment photographs at the parameter values indicated by dots. Below the boundary, islands persist. Above the boundary, all fluid initially in the cell eventually exits.
6. Conclusions and discussion

We have described an experiment and theoretical model for advection of a periodically reoriented irrotational flow in a disc. We have described results for a dipole flow, but the model extends to any number and configuration of sources and sinks. Periodic reorientation creates fixed points in the dynamical system of the irrotational flow whose destruction leads to chaos in the flow, while retaining the minimum-energy character of the irrotational flow. The RPM flow is also an example of a partially open flow that has characteristics of both dynamically closed and dynamically open flows.

Using symmetry, we have derived the character and extent of the closed (island) parts of the flow and where, in the stirring control parameter space, isolated regions can exist. Experiments and computations verified the theory. In the open part of the flow, exit-time distributions and experiments show the filamentary manifolds predicted to control transport. Residence-time distributions for fluid entering the disc at the source show power-law tails and a self-similar structure in parameter space. Neither the power-laws in the frequency distribution, nor the self-similar parameter space structure, nor the island size structure currently have a complete theoretical understanding.

While we have not focused here on applications, we note both that HS flow is mathematically equivalent to porous-media flow with a uniform permeability, and that many porous media flows can be well approximated by superposition of a number of dipole flows. And so these experiments and theory may yield direct application in geophysical transport problems (Metcalfe et al. 2008; Ord et al. 2009).

This work has been supported by CSIRO’s Minerals Down Under flagship and Complex Systems Science initiative. Laboratory work by P. Kulkarni at CSIRO Highett was funded by a Minerals Down Under/iVEC internship and by the Research Foundation of the City University of New York. Tony Kilpatrick, Tony Swallow, Robert Stewart and Dean Harris provided technical support at various stages of building the experiment. Some of the computations used CSIRO’s Advanced Scientific Computing parallel cluster, and we are grateful to Gareth Williams for helping adapt code.

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