Dynamical phenomena induced by bottleneck

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We study a microscopic follow-the-leader model on a circle of length \( L \) with a bottleneck. Allowing large bottleneck strengths we encounter very interesting traffic dynamics. Different types of waves—travelling and standing waves and combinations of both wave types—are observed. The way to find these phenomena requires a good understanding of the complex dynamics of the underlying (nonlinear) equations. Some of the phenomena, like the ponies-on-a-merry-go-round solutions, are mathematically well known from completely different applications. Mathematically speaking we use Poincaré maps, bifurcation analysis and continuation methods beside numerical simulations.

Keywords: traffic flow; numerical bifurcation analysis; Hopf bifurcation; Neimark–Sacker bifurcation; standing waves; travelling waves

1. Introduction

In recent decades many authors have studied traffic models for vehicular traffic as summarized in the overview articles on this topic (Brackstone & McDonald 1999; Helbing 2001; Nagel et al. 2003; Klar & Wegener 2004). The main purpose for these studies is to understand complex traffic-flow phenomena and eventually to influence or even to control traffic flow.

There exist a large number of traffic-flow models. A possible classification is to consider microscopic, kinetic and macroscopic models. In microscopic models the dynamics of the single drivers are described. Kinetic models mimic the Boltzmann equation in gas dynamics and deal with probability distributions. Finally, macroscopic models describe ‘macroscopic’ quantities like traffic density and traffic-flow velocity. Microscopic models have advantages from the modelling point of view, whereas macroscopic models have advantages in their simple description and in simulations. However, the study of the interesting (nonlinear) phenomena is highly challenging in all the modelling approaches. Here, we focus on a widely studied class of microscopic models, the so-called follow-the-leader models.

A classical problem studied for microscopic models is the traffic dynamics on a circular road. Many interesting phenomena like the formation of stop-and-go waves can be observed in real experiments (Sugiyama et al. 2008). Some of the phenomena can be reproduced easily by simple microscopic models

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Dedicated to Hans Troger.

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using periodic boundary conditions. Unfortunately, in many cases the nonlinear
dynamics of a simple model is not studied carefully enough to encounter many
or most of the dynamics of the model. Contrary to that, more and more
complex models have been proposed. Our approach—already applied in our
previous papers (Gasser et al. 2004; Seidel et al. 2009)—is to take very simple
(nonlinear) microscopic traffic-flow models and to study deeply the (nonlinear)
dynamics of such a model. Surprisingly, we have learnt that already the very
simple models show very rich dynamics, which can be interpreted easily in
the traffic-flow context. A key finding in the mathematical analysis of such
simple models is bifurcations that lead to stable and unstable periodic solutions
(in relative velocities and headways), coexisting multiple periodic and quasi-
stationary solutions, etc. From a macroscopic viewpoint (considering the velocity
and the inverse of the headway as the density), the quasi-stationary solutions are
very simple constant-velocity solutions, whereas the periodic solutions correspond
to downstream-travelling density and velocity waves. Moreover, the well-known
inverse lambda structure in the flux–density (so-called fundamental) diagram can
be recovered.

The results mentioned above were mostly obtained for a circular road with
identical or non-identical drivers. In the presence of a bottleneck (which could
be due to roadworks on the circular road), the dynamics becomes even more
interesting and the corresponding analysis changes dramatically. The reason for
this is that the bottleneck-free case is very special and the resulting mathematical
problem ends in a classical stability analysis for an equilibrium point of a system
of autonomous ordinary differential equations (ODEs). This is not true any more
in the presence of a bottleneck. In Seidel et al. (2009) a new mathematical
approach to this problem was presented. With this approach, new types of
solutions were found. Macroscopically, these new solutions correspond to standing
(density and velocity) waves or to more complex interactions of travelling waves
and standing waves. Mathematically these solutions are known as rotational or
quasi-rotational solutions. In the special case of identical drivers, they become the
famous ponies-on-a-merry-go-round solutions (POMs; see Aronson et al. 1991)
or quasi-POMs.

In Seidel et al. (2009) the main issue was to introduce this new approach
for studying the dynamics in the case of a bottleneck reducing the maximal
velocity in a limited part of the circle. The reduction size of the maximal
velocity is characterized by some parameter $\epsilon$. Some simulations were presented
to underline the potential of the method. However, the main attention was
given to the case of a bottleneck with small $\epsilon$. In this paper, we focus on
general (large) $\epsilon$, which leads to new problems not only from the computational
point of view. Our numerical investigation shows that a bottleneck induces
very interesting complex dynamics. We are able to classify some fundamental
patterns (standing waves, travelling waves) and to identify more complex
phenomena as combinations of these fundamental patterns. We note the
interesting situation where we have different coexisting patterns. Depending
on the initial data or on perturbations of the traffic situation, one of these
patterns is selected and appears. We mention that there are results about traffic
phenomena induced by a bottleneck based on a much more complex stochastic
three-phase multi-lane theory in Kerner (2008). We will comment on some
analogies in §3.
The paper is organized in the following way. In §2, we summarize the existing results in this direction. In §3, we present many new numerical results with interesting macroscopic visualizations showing discrete versions of the speed $v(\xi, t)$ as a function of position $\xi$ and time $t$. We give interpretations of the results in a traffic-flow context. Finally, in §4 we add a conclusion.

2. Theoretical setting

We study the situation of $N$ cars on a circular road of length $L$. A widely used car-following model describing such a situation is the well-known optimal velocity model introduced by Bando et al. (1995). Here, we use a generalization of the standard optimal velocity model. Let $x_j = x_j(t), t \geq 0$, be the distance the $j$th car has covered at time $t$. Then the model reads

$$\ddot{x}_j = \frac{1}{\tau_j} [V_{j,\epsilon}(\xi_j, x_{j+1} - x_j) - v_j], \quad j = 1, \ldots, N, \quad x_{N+1} = x_1 + L. \quad (2.1)$$

The circular road is represented by the fact that $x_{N+1} = x_1 + L$, i.e. the $N$th car is following the first car. Then we have to explain the optimal velocity function $V_{j,\epsilon}$. Let $V_j = V_j(d_j)$ be the optimal velocity function of the $j$th car as a function of the headway $d_j = x_{j+1} - x_j$, which is the distance between the $j$th car and the $(j + 1)$th car in front. The optimal velocity function expresses the velocity that the $j$th car is aiming to achieve, according to the distance to the car in front. This function is assumed to be

$$V_j : [0, \infty) \to [0, \infty), \quad \text{smooth and strictly monotone increasing},$$

$$V_j(0) = 0,$$

$$\lim_{d_j \to \infty} V_j(d_j) = V_{j,\text{max}}. \quad (2.2)$$

An example is given in figure 1. In Seidel et al. (2009), a bottleneck (caused for example by roadworks) was introduced by extending the optimal velocity function to

$$V_{j,\epsilon}(\xi, y) = (1 - \epsilon e^{-\left(\xi - L/2\right)^2}) V_j(y). \quad (2.3)$$

The (position) variable $\xi$ is defined by

$$0 \leq \xi \leq L, \quad \xi = x \mod L, \quad (2.4)$$

where $x$ denotes a position and $y$ a headway. The bottleneck is centred around the position $\xi = L/2$ and it acts by reducing the maximal velocity. The parameter $\epsilon \geq 0$ describes the ‘strength’ of the bottleneck (figure 2). Model (2.1) says that every driver aims to reach his optimal velocity, which depends on the headway and on the position with respect to the bottleneck. In the last decade various simplified versions of this model have been studied. In the general case of non-identical drivers, every single car obeys its own optimal velocity law. In the examples we study in §3, identical drivers are assumed. We recall some results for the various versions of identical and non-identical drivers with and without a bottleneck. An overview is given in table 1.

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Figure 1. An example of an optimal velocity function.

Figure 2. A region of reduced maximal optimal velocity $V_{j,\varepsilon,max}$.

Table 1. Overview on solutions and bifurcations.

<table>
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<tr>
<th>solution</th>
<th>bifurcation</th>
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<td>quasi-stationary</td>
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<td>drivers, no bottleneck</td>
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<td>Hopf</td>
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<td>non-identical drivers, with</td>
<td>rotation, standing</td>
<td>Neimark–Sacker, quasi-rotation, interacting</td>
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<td>bottleneck</td>
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<td>Neimark–Sacker, quasi-POM, interacting</td>
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<td>bottleneck</td>
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<td>standing and travelling waves</td>
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(a) Non-identical drivers without bottleneck

We recall equation (2.1) in the case $\varepsilon = 0$ and write it as a first-order (nonlinear) system

\[
\begin{align*}
\dot{x}_j &= v_j \\
\dot{v}_j &= \frac{1}{\tau_j} \left[ V_j(x_{j+1} - x_j) - v_j \right], \\
& j = 1, \ldots, N, \ x_{N+1} = x_1 + L.
\end{align*}
\]
Although this model—and especially the corresponding version with identical drivers in §2b—is very simple, it has become an important tool in the description of traffic flow on a circular road. This is due to the fact that there exist simple solutions—called quasi-stationary solutions—that can be observed in real experiments on a circular road (Sugiyama et al. 2008). Quasi-stationary solutions (with superscript 0) are given by

\[ v^0_j(t) = v^0, \]

and

\[ x^0_j(t) = x^0_1(0) + \sum_{k=1}^{j-1} d^0_k + tv^0, \quad j = 1, \ldots, N, \quad (2.6) \]

where \( v^0 \) is the same (constant) velocity of all cars and \( d^0_j, \quad j = 1, \ldots, N, \) represent the (constant) headways satisfying

\[ v^0 = V_j(d^0_j), \quad j = 1, \ldots, N, \quad \sum_{j=1}^{N} d^0_j = L. \quad (2.7) \]

The terminology ‘quasi-stationary solution’ is due to the fact that this solution itself is (obviously) not stationary, but the corresponding velocities \( v^0 \) and headways \( d^0_j \) are. Even more, the velocities \( v^0 \) and headways \( d^0_j \) are stationary solutions of the corresponding model for the relative velocities and the headways.

The question whether or not a quasi-stationary solution can be observed in reality is related to the stability as a solution of the nonlinear ODE system (2.5) with \( 2N \) equations. However, a related (linear) stability analysis is only possible in special cases (see Gasser et al. 2004, 2007).

(b) Identical drivers without bottleneck

Now we assume that all drivers are identical, i.e. that the optimal velocity functions \( V_j = V, \quad j = 1, \ldots, N, \) and the relaxations times \( \tau_j = \tau, \quad j = 1, \ldots, N, \) are equal for all cars. Again, the case \( \varepsilon = 0 \) stays for no bottleneck. Then we have

\[
\begin{cases}
\dot{x}_j = v_j \\
\dot{v}_j = \frac{1}{\tau} [V(x_{j+1} - x_j) - v_j]
\end{cases}, \quad j = 1, \ldots, N, \quad x_{N+1} = x_1 + L. \quad (2.8)
\]

In fact, this is the model presented originally by Bando et al. (1995). It was studied by various authors (see Seidel et al. 2009 and references therein).

As already mentioned, this model has become an important tool in the description of traffic flow on a circular road. Many phenomena discovered in real experiments on a circular road setting (Sugiyama et al. 2008) can be described by the simple model (2.8). Since here \( d^0_j = d^0 = L/N \) for \( j = 1, \ldots, N, \) the quasi-stationary solutions are given by

\[ x^0_j(t) = (j - 1) \frac{L}{N} + tV \left( \frac{L}{N} \right) + x^0_1(0), \quad j = 1, \ldots, N. \quad (2.9) \]
Also, it is well known from the literature that for our model the quasi-stationary solutions are asymptotically stable if \( V'(L/N) < 1/[1 + \cos(2\pi/N)] \) (see Huijberts 2002; Gasser et al. 2004). When the parameters are such that the critical value \( V'(L/N) = 1/[1 + \cos(2\pi/N)] \) is reached, then a qualitative change in the dynamics occurs. This is called a bifurcation. In the traffic context, the critical parameter corresponds to a critical mean density on the circular road. When the critical density is exceeded, simple quasi-stationary solutions can no longer be observed. In our case a so-called Hopf bifurcation occurs. A Hopf bifurcation generates periodic solutions for parameters close to the critical one (for delay models see Igarashi et al. 2001; Gasser et al. 2004; Orosz et al. 2004, 2005). These bifurcating periodic solutions that we will call Hopf-periodic are travelling waves showing the well-known oscillations in headway and velocity such that the congestion travels upstream. They can be observed in real experiments (Sugiyama et al. 2008) and are sometimes named stop-and-go waves—even though in most of the Hopf-periodic solutions no real stop (vanishing velocity) appears. Therefore the Hopf-periodic solutions are a very important class of non-trivial solutions that can be observed in reality. Mathematically—by analysing the related Lyapunov exponent—the stability of the Hopf-periodic solutions can be determined, too. In the case of non-stationary solutions there are various definitions of stability. The one we are talking about here is the so-called orbital stability.

However, the bifurcation results are of the local type, valid only in a small neighbourhood of the critical parameter values. Using special numerical tools (path-following methods) one can study the global bifurcation diagram. Then many additional non-trivial (stable and unstable) periodic solutions can be found. Interesting phenomena like coexistence of multiple periodic and quasi-stationary solutions have been discovered. Details on the global bifurcation analysis can be found in Gasser et al. (2004).

(c) Non-identical drivers with bottleneck

In the case of a bottleneck we use the general model (2.1) for non-identical drivers with optimal velocity function (2.3). Here quasi-stationary solutions of type (2.9) no longer exist and the standard tools are no longer applicable. In other words, the situations in §2a,b were mathematically very special since the quasi-stationary solutions formed a stationary point in the system for the relative velocities and the headways. Therefore standard methods for the stability analysis of stationary points of autonomous ODE systems could be applied. In the more general case with a bottleneck, the situation is different, since we do not know what the generalization of a quasi-stationary solution is. Rewriting the system in terms of relative velocities and headways does not show any advantage since there are no interesting stationary points. Therefore in this case a completely different approach has to be used. It turns out that so-called rotation solutions are the right objects at which to look (see Seidel et al. 2009).

A rotation solution with orbital period \( T \) and rotation number \( k \in \mathbb{Z} \) of equation (2.1) is defined by

\[
x_j(t + T) = x_j(t) + kL, \quad v_j(t + T) = v_j(t), \quad j = 1, 2, \ldots, N,
\] (2.10)
where $T$ and $k$ are assumed to be minimal. We will restrict ourselves to the special, but most important, case $k = 1$. From a traffic point of view a rotation solution is nothing but a standing wave for the velocity or the headways.

We see that, for $\varepsilon = 0$, our quasi-stationary solutions are (trivial) rotation solutions with orbital period $T := L/v^0$, where $v^0$ is the common velocity of the drivers. But observe that the Hopf-periodic solutions (travelling waves) in general are not rotation solutions with orbital period $T$. They always satisfy

$$x_j(t + T) = x_j(t) + L, \quad j = 1, 2, \ldots, N,$$

with an orbital length $L$. If $L$ and $L$ are commensurate, formally the Hopf-periodic solutions are rotation solutions with possibly large orbital periods and rotation numbers $k$.

Now we rewrite our problem as a fixed point problem. The time-$T$ map $\Phi^T$ is defined as follows. Assume that $x(0) = (x_1, x_N, v_1, \ldots, v_N)$ is the state of our system at time $t = 0$ and that $x(t)$ is the solution of the corresponding initial value problem. Then

$$\Phi^T(x(0)) = x(T).$$

Setting $\Lambda := (L, \ldots, L, 0, \ldots, 0)$, rotation solutions ($k = 1$) satisfy (rewriting equation (2.10))

$$\Phi^T(x(t)) = x(t + T) = x(t) + \Lambda \text{ for all } t.$$ 

Therefore, rotation solutions are fixed points of the map $Q$ defined by

$$Q(x) = \Phi^T(x) - \Lambda.$$ 

This means that in the case of a bottleneck, instead of a quasi-stationary solution, we have rotation solutions solving the fixed point problem (2.14). Note that when looking for rotation solutions we do not know the period $T$ a priori. For mathematically interested readers, we mention that we will not consider the map $Q$ itself, but a related Poincaré map $P$. A Poincaré map looks for discrete times whenever car number 1 passes the position $x = 0$ (we could also take any other position and any other car).

Again, the question whether a solution can be observed in real traffic situations or in experiments is related to the stability of the solution. The corresponding stability concept for rotation solutions was discussed in Seidel et al. (2009). We note that—similar to the bottleneck-free case—when reaching a critical density the rotation solutions may lose their stability. This is due to another type of bifurcation, a so-called Neimark–Sacker bifurcation. In the bottleneck-free case—when passing the critical parameter values—a Hopf-periodic solution bifurcates whereas a Neimark–Sacker bifurcation leads to so-called quasi-rotation solutions. We will see that, contrary to Hopf-periodic or rotation solutions, it is not so easy to identify quasi-rotation solutions. We will see that they seem to be combinations of standing and travelling waves.

As a consequence, in the $(L, \varepsilon)$-plane, we conjecture parameter regions where rotations and quasi-rotations exist that at $\varepsilon = 0$ coincide with the quasi-stationary solutions and the Hopf-periodic solutions, respectively. Between these two
In Seidel et al. (2009), we showed the existence of rotation solutions for small $\epsilon > 0$ by considering the case of a small bottleneck as a perturbation of the bottleneck-free case. Hopf-periodic solutions are perturbed to quasi-rotations.

**Identical drivers with bottleneck**

We restrict and simplify the setting to the case of identical drivers. This is done by adding an additional symmetry condition to a rotation solution, namely

$$x_j \left( t + \frac{T}{N} \right) = x_{j+1}(t) \text{ for all } t, \ j = 1, 2, \ldots, N. \quad (2.15)$$

This means that in the case of identical drivers all cars behave in the same way except for a time shift of $T/N$ between two cars. Rotation solutions satisfying equation (2.15) are known as POMs solutions (see Aronson et al. 1991; Seidel et al. 2009). It turns out that the method to find rotation solutions presented in §2c can be simplified considerably. The additional condition (2.15) allows the use of so-called reduced Poincaré maps $\pi$, and the computation of POMs can be based on $\pi$ in a very efficient way. While the Poincaré map looks for discrete times whenever car number 1 passes the position $x = 0$, the reduced Poincaré map lists the whole configuration at discrete times whenever any car passes the position $\xi = 0$. This gives a denser discrete time grid on which the dynamics is evaluated.

Mathematically, POMs correspond to fixed points and quasi-POMs to invariant curves of $\pi$ that bifurcate in Neimark–Sacker points of $\pi$. Again, quasi-stationary solutions correspond to POMs and Hopf-periodic solutions to quasi-POMs for $\epsilon = 0$.

Our numerical analysis in §3 is performed for the model of identical drivers and is based on the use of reduced Poincaré maps. More theoretical details can be found in Seidel et al. (2009).

**Traffic flow and ‘flocking’**

Here we point out the following alternative viewpoint of the dynamics of the microscopic traffic-flow models. The single drivers can be seen as individual self-propelled agents. Each agent follows simple rules that in our case involve only the agent (driver) in front. The dynamics we observe is the result of the collective motion of all the individual agents. There is no central control of the dynamics. In the case of (asymptotically) stable quasi-stationary solutions, all agents (drivers) tend to the same asymptotic velocity. For a stable POM solution, all drivers approach the same velocity profile (at least for somehow ‘close’ initial profiles). In population dynamics this phenomenon is referred to as ‘flocking’ (or ‘herding’ etc.). For more details see Cucker & Smale (2007). In this sense on the circular road we can observe a one-dimensional version of ‘flocking’.

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3. Numerical results

We consider the general model (2.1) for identical drivers,

\[
\begin{aligned}
\dot{x}_j &= v_j \\
\dot{v}_j &= \frac{1}{\tau} \left[ V_\varepsilon(\xi_j, x_{j+1} - x_j) - v_j \right],
\end{aligned}
\]

with

\[
V_\varepsilon(\xi, y) = (1 - \varepsilon e^{-(\xi - L/2)^2}) V(y).
\]

We restrict our attention to \(N = 10\), \(\tau = 1\) and to the Bando optimal velocity function

\[
V(y) = v_{\text{max}} \frac{\tanh a(y - 1) + \tanh a}{1 + \tanh a},
\]

with \(a = 2\) and \(v_{\text{max}} = 1\).

We know that for \(\varepsilon = 0\) there exist two Hopf bifurcation points with respect to \(L\), namely \(L_1^H = 5.890\) and \(L_2^H = 14.109\). The larger value \(L_2^H\) is more interesting, since it is connected with the first loss of stability of the quasi-stationary solutions for increasing traffic densities \(N/L\). Moreover, we know that the quasi-stationary solutions (being special POMs) are unstable for \(L_1^H < L < L_2^H\).

It is impossible to give a complete survey about the dynamics of the model for all parameter pairs \((L, \varepsilon)\). We will mainly present some results for two fixed values of \(L\), namely \(L = 13 < L_2^H\) and \(L = 18 > L_2^H\).

The dynamics of POMs and quasi-POMs will be visualized in three ways using suitable projections.

— Speed of a single car as a function of length (Lagrangian description). For a POM we will encounter an \(L\)-periodic pattern.

— Macroscopic view (Eulerian description). This is obtained by colouring all trajectories according to the speed of the corresponding car. Hereby, we obtain a discrete version of the speed \(v(\xi, t)\) as a function of position \(\xi\) and time \(t\). One single trajectory is drawn. For a POM, \(v(\xi, t)\) is independent of \(t\). For a quasi-POM, we have the interesting observation that \(v(\xi, t)\) is periodic in \(t\). However, this observation is still without proof.

— More mathematically, the orbit is under the reduced Poincaré map, mainly showing the limit set. For a POM, the limit set is just a point. For a quasi-POM, we will encounter closed invariant curves.

\(\text{(a) Ponies-on-a-merry-go-round solutions (standing waves)}\)

We are interested in how the POMs change with \(\varepsilon\) for fixed \(L\). To this end we numerically continue POM branches as a function of the bottleneck strength \(\varepsilon\) using the characterization of POMs as fixed points of (reduced) Poincaré maps.

From the theory in Seidel et al. (2009) we know that, for fixed \(L\), a POM branch parametrized by \(\varepsilon \geq 0\) emanates from the quasi-stationary solution \((\varepsilon = 0)\). We will path-follow POM branches also for larger values of \(\varepsilon\). Some results are visualized in figure 3. As expected we encounter Neimark–Sacker bifurcation points, but also—less expected—folds.
In figure 3 (vertical axis) a POM is characterized by the average speed \( v_M := L/T \) with \( T \) being the orbital period; \( v_M \) is proportional to the space-averaged flow. The horizontal coordinate is the bottleneck strength \( \varepsilon \). For \( \varepsilon = 0 \) the trivial POM coincides with the quasi-stationary solution where all cars have the same speed \( v_M = V(L/N) \). The numerical continuation is not influenced by the stability of the POMs. Of obvious interest are those (bifurcation) parameters \( \varepsilon \) where stability is lost or gained. We encounter two qualitatively different bifurcations: Neimark–Sacker points (as in figure 3a denoted by N) and fold points (as in figure 3c denoted by F1 and F2).

In figure 3, folds can be found on the POM branches. For \( L = 18 \) there are two folds, for \( \varepsilon_1 := 0.22 \) and for \( \varepsilon_2 := 0.313 \); two other folds are very close together at \( \varepsilon \approx 0.41 \). Observe that the S-shape of the POM branch with two neighbouring folds is associated with the coexistence of two stable POMs for the same parameter set. We will see that the wave speeds of these two stable POMs in the vicinity of the bottleneck differ significantly. This is already indicated by the corresponding different average speeds. In the mentioned theory of Kerner (2008) the POMs seem to correspond to the so-called congested traffic phase.

(i) Bifurcation diagrams in \( L \) and \( \varepsilon \)

A bifurcation diagram in a parameter plane shows bifurcation curves. By this it contributes to the information about possible dynamics for a fixed pair of parameters.
In our traffic model, we expect Neimark–Sacker and fold curves showing the dependence of the bifurcation parameters $\varepsilon$ on the circle length $L$. Figure 4 contains Neimark–Sacker (grey) and fold curves (black) in the $(L, \varepsilon)$ parameter plane. The Neimark–Sacker curve emanates in the Hopf point $(0, L^H_2)$.

Though these curves deliver only local information, we guess, supported by numerical simulations, that quasi-POMs live in the grey-shaded region, where POMs are unstable. In the other parameter domains we expect stable POMs. In the black-shaded areas, due to the S-shape of the POM branches in figure 3, we expect two stable POMs and one unstable POM.

(ii) POMs for $L = 18$

Figure 5 visualizes different stable POMs for $L = 18$ and various $\varepsilon$ values. They can be computed directly by Newton’s method as fixed points of the reduced Poincaré maps—in contrast to the quasi-POMs that we get only by simulation, see §3b. From figure 3c we conclude that, for some values $\varepsilon$, the corresponding POMs are not unique. For example, there are two coexisting stable POMs and one unstable POM for $\varepsilon = 0.3$, a value between the two fold values $\varepsilon_1 = 0.22$ and $\varepsilon_2 = 0.313$ (figure 5c–f). These two stable POMs are qualitatively very different. This is already indicated by the difference of their average speeds $v_M$.

Remarkably, the decrease of speed induced by the bottleneck is considerably large only for the POMs in the last two rows of figure 5. Here traffic jams occur downstream of the bottleneck, while for the POMs in the first two rows of figure 5 the minimal speed occurs upstream of the bottleneck (the black circles in figure 5). For $\varepsilon = 0.3$ both types of stable POMs exist.

(b) Quasi-POMs for $L = 13$

In this section we fix $L = 13$. We know from figure 4 and particularly from figure 3a that there is a wide parameter range where POMs emanating from the quasi-stationary solutions for $\varepsilon = 0$ are unstable. Here we expect quasi-POMs.

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Figure 5. $N = 10$. Stable POMs for $L = 18$ and different $\varepsilon$. Speed versus length (left-hand side) and macroscopic view (right-hand side) with a trajectory of a single car in white. The position of the bottleneck and its size are indicated by black circles. Note that for $\varepsilon = 0.3$ there are two different stable POMs. (a,b) $\varepsilon = 0.2$, $v_M = 0.94$; (c,d) $\varepsilon = 0.3$, $v_M = 0.91$; (e,f) $\varepsilon = 0.3$, $v_M = 0.78$; and (g,h) $\varepsilon = 0.4$, $v_M = 0.72$.

We present three different visualizations of quasi-POMs (figures 6–9). Again each quasi-POM is associated with the average speed $v_M$, where the average is taken over a suitable large time interval.

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Figure 6. \(N = 10, L = 13\). Quasi-POMs and their average speeds \(v_M\) for \(0 \leq \varepsilon \leq 0.25\). Speed versus length (left-hand side), Hopf-periodic dynamics and macroscopic views (right-hand side). The position of the bottleneck is indicated by black circles. (a,b) \(\varepsilon = 0, v_M = 0.67\); (c,d) \(\varepsilon = 0.2, v_M = 0.65\); and (e,f) \(\varepsilon = 0.24, v_M = 0.65\).

Quasi-POMs are special solutions that show non-periodic dynamic behaviour for \(t \to \infty\) when considering the trajectories of individual vehicles. Theoretically, the quasi-POM type irregularity can be identified by closed invariant curves of (reduced) Poincaré maps as shown in figure 9. But we found another method of identification, namely the time periodicity of the macroscopic function \(v(\xi, t)\) in the macroscopic views in figures 6–8 (right side).

In figure 3a there is a Neimark–Sacker bifurcation for \(\varepsilon \approx \varepsilon_N := 0.347\), and the POMs are stable for \(\varepsilon > \varepsilon_N\) and unstable for \(\varepsilon < \varepsilon_N\). Hence one could expect quasi-POMs for \(\varepsilon < \varepsilon_N\).

Moreover, we know that for \(\varepsilon = 0\) the quasi-stationary solution is unstable. The Hopf point at \(L_H^2\) is responsible for the occurrence of a stable headway- and speed-periodic solution appearing as a travelling wave. In our new context...

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This solution is a quasi-POM. Its travelling wave dynamics is visualized in figure 6a, b. The corresponding invariant curve can be seen in figure 9a. We expect that this special quasi-POM for $\epsilon = 0$ is perturbed to quasi-POMs for $\epsilon > 0$.

Indeed, for various values of $\epsilon$ with $0 < \epsilon < \epsilon_N$, we found quasi-POMs by simulation. Figures 6–8 show how the macroscopic speed pattern is changed due to increasing $\epsilon$ from $\epsilon = 0$ (travelling wave) to $\epsilon = 0.35$ (POM). There is an interaction of the travelling wave with the bottleneck. For $\epsilon \in [0, 0.24]$ the travelling wave structure of the Hopf-periodic wave persists. For larger bottleneck strength ($\epsilon \geq 0.25$) the travelling wave structure does not exist anymore. There appears more than one congestion upstream of the bottleneck and rather free-flowing traffic downstream.

Similar to POMs for $L = 18$, the coexistence of two different stable dynamics was found for $L = 13$. Looking more thoroughly at the last two rows in figure 8, we see that two qualitatively different quasi-POMs coexist for $\epsilon = 0.25$, one of which—the second—seems to be the ‘Neimark–Sacker successor’ of quasi-POMs emanating in $\epsilon_N$, and the other the ‘Hopf successor’ of quasi-POMs emanating in $\epsilon = 0$. One should also compare the corresponding invariant curves in figure 9c, d. The quasi-POM visualized in figure 9d seems to have less dramatic dynamics since the headways are farther away from zero than that in figure 9c. On the other hand, this quasi-POM has a slightly lower average speed $v_M$ than the other. Obviously, in figure 7d on each round a car trajectory (in white) in general
Figure 8. $N = 10, L = 13$. (Quasi-)POMs. Continuation of figure 7. (a, b) $\epsilon = 0.3, v_M = 0.59$; (c, d) $\epsilon = 0.33, v_M = 0.57$; and (e, f) POM for $\epsilon = 0.35, v_M = 0.56$.

passes two congestions upstream of the bottleneck while the trajectory in figure 7b crosses only one jam area (dark red) in each round, which is in general ‘stronger’ than the jam areas in figure 6.

Again, let us mention a possible analogy to the theory of Kerner (2008). The quasi-POMs with travelling wave character in figure 6 seem to be realizations of the so-called jam phase whereas the ‘fixed’ (at the bottleneck) quasi-POMs in figures 7c, d and 8 make part of the congested phase. A possible passage from a ‘fixed’ quasi-POM to a travelling wave quasi-POM (by changing the density) would correspond to the so-called pinch effect in the Kerner theory. However, let us underline that the results of Kerner are based on a much more complex stochastic multi-phase and multi-lane theory.
Figure 9. $N = 10$, $L = 13$, $0 \leq \varepsilon \leq 0.33$. Visualization of quasi-POMs in figures 6–8 by invariant curves of the reduced Poincaré map for car no. 4, $L = 13$ and different values of $\varepsilon$. The unstable POMs are marked. (a) $\varepsilon = 0$, $v_M = 0.67$; (b) $\varepsilon = 0.2$, $v_M = 0.64$; (c) $\varepsilon = 0.25$, $v_M = 0.64$; (d) $\varepsilon = 0.25$, $v_M = 0.61$; (e) $\varepsilon = 0.3$, $v_M = 0.59$; and (f) $\varepsilon = 0.33$, $v_M = 0.56$.

(i) **Invariant curves**

To analyse the type of irregularity of a certain dynamics one has to look at the orbit of the (reduced) Poincaré map. If the limit set of the orbit is a closed invariant curve, we have the dynamics of a quasi-POM. A Neimark–Sacker bifurcation leads to such bifurcating invariant curves. Figure 9 shows projections of the invariant curves on the speed–headway plane of the fourth car, counted from the observation place at the measuring point ($\xi = 0$). Since there are $N = 10$ cars, we expect that the fourth car is rather close to the centre of the bottleneck, hence having a more interesting dynamics than cars farther away.
There is still no powerful numerical tool to continue invariant curves of quasi-POMs as a function of parameters. If this were available, we would guess that there might be an S-shaped branch of quasi-POMs with respect to $\varepsilon$ connecting the quasi-POMs of Neimark–Sacker type and that of Hopf type and possessing two folds near $\varepsilon = 0.25$.

(c) Other values of $L$

Up to now we have chosen mainly $L = 13$ and $L = 18$ and $0 \leq \varepsilon \leq 0.5$. Our dynamical simulations yielded POMs and quasi-POMs, nothing else. This is different for smaller values of $L$ where we guess there is more complex dynamics.

Figure 10a shows a chaotic-like pattern for $L = 8$—no time periodicity is observed. Figure 10b shows a quasi-POM that may be due to a period-doubling process in increasing $\varepsilon$ from $\varepsilon = 0$ to $\varepsilon = 0.3$ for $L = 10$.

(d) Fundamental diagrams

Studying fundamental diagrams (i.e. flux–density diagrams) in a microscopic optimal velocity model seems to be an uninteresting topic. The main input in the model is the optimal velocity function and we may expect a density–flow relation according to the optimal velocity function.

Now we mimic a traffic measurement by performing numerical simulations. We fix a measurement position $\xi = 0$ on the circular road and whenever a car passes this measurement position we take its velocity and the inverse of the headway (as an approximation for the traffic density). Note that our measurement position is on the opposite side of the bottleneck on the circular road. Since in a fundamental diagram we have to vary the density, we perform the same measurement by varying the length $L$ of the circular road (and keeping fixed the number of cars).

Suppose the dynamics on the circular road with identical drivers and no bottleneck corresponds to a quasi-stationary solution (constant headway $d^0 = L/N$ and constant velocity $v^0 = V(d^0)$). Let us denote the corresponding density

\[ \text{Figure 10. } N = 10. \text{ Macroscopic visualization of two complex dynamics for } L = 8 \text{ and } L = 10. \ (a) \ L = 8, \varepsilon = 0.4, v_M = 0.28. \text{ Chaos? } (b) \ L = 10, \varepsilon = 0.3, v_M = 0.45: \text{ quasi-POM. Result of period doubling?} \]
by $q^0 = 1/d^0$. Then the corresponding flux $f^0 = f^0(q^0)$ is given by the optimal velocity function

$$f^0(q^0) = q^0 V\left(\frac{1}{q^0}\right).$$

(3.4)

This corresponds to the well-known flux–density curves similar to the red curves in figure 11.

However, this is only true for quasi-stationary solutions in the case of identical drivers with no bottleneck. In the bottleneck-free case we know that the quasi-stationary solution is unstable in a certain density interval. We know that in the unstable density interval the dynamics is driven by a Hopf-periodic solution and the measurement data will not lie on the red curves. In fact, in Seidel et al. (2009) we showed that in general we recover the famous inverted lambda structure in the flux–density diagram.

Here we study the flow–density diagrams for the case of a (strong) bottleneck. In the case of identical drivers we expect POM solutions (instead of quasi-stationary solutions). Those POMs are stable in a certain density region. Since POMs are standing waves and we measure at a fixed position on the circular road, the resulting curve is very similar to the curves obtained for identical drivers with no bottleneck (quasi-stationary solution). The POMs are the red curves in the flux–density diagram in figure 11. In general, the dynamics also involves quasi-POMs. The corresponding flux–density diagram becomes more

Figure 11. Fundamental diagrams for $N = 10$ and $L = 4$ to 20 at the measure point $\xi = 0$. Flow–density curve for the quasi-stationary solutions (red). (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.3$; (c) $\varepsilon = 0.4$; and (d) $\varepsilon = 0.5$. 

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complicated. In figure 11 possible results for different values of $\varepsilon$ are presented. We see that the resulting structures have partly lost their similarity to the inverse lambda structure. The complex dynamics involved in this simulation shows us that the resulting flux–density diagrams are not easy to interpret. We conclude that only in the case of simple dynamics do measurements result in known structures of the fundamental diagram.

4. Conclusions

In this paper, we study traffic dynamics on a circular road with a bottleneck. For this setting in Seidel et al. (2009) a new mathematical approach was presented. This approach enables us to classify complex traffic dynamics on a circular road and therefore to improve the understanding of complex phenomena.

We are now able to interpret phenomena in the case of a bottleneck using analogies to the bottleneck-free case. This leads to travelling waves, standing waves and interactions of those two wave types, which can be interpreted in the traffic-flow context. Also coexisting stable patterns were observed in certain parameter regimes. The macroscopic viewpoint on the microscopic results helps considerably to interpret the dynamics.

We believe that this phenomenon persists for traffic models on the line. We made the interesting observation that quasi-POMs correspond to macroscopic functions $v(\xi, t)$ that are periodic in $t$. This is very helpful when classifying solution patterns. We will prove this conjecture in a forthcoming paper.

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