Nanophysics in graphene: neutrino physics in quantum rings and superlattices

BY H. A. FERTIG1,* AND LUIS BREY2

1 Department of Physics, Indiana University, Bloomington, IN 47405, USA
2 Instituto de Ciencia de Materiales de Madrid (CSIC), Cantoblanco 28049, Spain

Electrons in graphene at low energy obey a two-dimensional Dirac equation, closely analogous to that of neutrinos. As a result, quantum mechanical effects when the system is confined or subjected to potentials at the nanoscale may be quite different from what happens in conventional electronic systems. In this article, we review recent progress on two systems where this is indeed the case: quantum rings and graphene electrons in a superlattice potential. In the former case, we demonstrate that the spectrum reveals signatures of ‘effective time-reversal symmetry breaking’, in which the spectra are most naturally interpreted in terms of effective magnetic flux contained in the ring, even when no real flux is present. A one-dimensional superlattice potential is shown to induce strong band-structure changes, allowing the number of Dirac points at zero energy to be manipulated by the strength and/or period of the potential. The emergence of new Dirac points is shown to be accompanied by strong signatures in the conduction properties of the system.

Keywords: graphene; nanophysics; quantum rings; superlattices

1. Introduction

The ability to fabricate solid-state systems with nanoscale dimensions has engendered an explosion of progress in electronic systems, both by allowing the creation of ever smaller and faster devices and in deepening our fundamental understanding of how electrons may behave and organize themselves. Such systems are typically understood by beginning with non-interacting electrons in some effective geometry or potential, and then deducing their behaviour when various further complications are added in (disorder, interactions etc.) Many of the most basic properties of such systems can be ascertained by solving simple non-relativistic quantum problems.

*Author for correspondence (hfertig@indiana.edu).

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Very recently, the fabrication of graphene (Geim & Novoselov 2007; Castro Neto et al. 2009) has opened a window on a completely new paradigm. Graphene is a single layer of carbon atoms laid out in a honeycomb lattice, which differs from conventional two-dimensional electron gas (2DEG) systems in that the low-energy physics is governed by a Dirac Hamiltonian that is closely analogous to that of massless neutrinos. Manipulating this system, by cutting it into narrow ribbons or other small geometries, or by imposing potentials that vary at very small length scales, allows one to explore an ultrarelativistic system under circumstances largely unreachable for truly massless particles. Several studies to date have focused on effectively zero-dimensional systems, such as quantum dots or quantum rings (Abergel et al. 2008; Bahamon et al. 2009; Luo et al. 2009; Potas et al. 2009; Wurm et al. 2009, 2010; Molitor et al. 2010). Electronic wave functions in graphene have been found to have a variety of behaviours owing to the precise nature of the boundaries (Brey & Fertig 2006a, b) of such systems, or of the junctions connecting different parts of the system (Iyengar et al. 2008). Manipulation at the nanoscale may also be implemented by the application of electric fields via gate geometries, subjecting the system to potentials varying on a very short length scale. Much recent work has focused on p–n junctions (Cheianov et al. 2007; Beenakker 2008; Zhang & Fogler 2008; Stander et al. 2009; Yoshioka & Higashibata 2009) and p–n–p junctions (Pereira et al. 2006; Velasco et al. 2009), whose behaviour in graphene is distinct from that of conventional 2DEGs because of the absence of a gap in the spectrum. Very recently, studies of graphene in superlattice potentials have demonstrated the possibility of ‘band-structure engineering’ of graphene for both one-dimensional (Park et al. 2008a, b; Esmailpour et al. 2009) and two-dimensional (Pletikosic et al. 2009; Tiwari & Stroud 2009) periodicities.

In this article, we review recent progress on two graphene systems with nanoscale geometries, quantum rings and superlattices. We will see that in the former case, a judicious choice of ribbons and the junctions joining them leads to very surprising effects, indicating an effective presence of magnetic flux through the rings, yielding spectra that reflect effective (but not real) breaking of time-reversal symmetry (Luo et al. 2009). Superlattice potentials, which may be viewed as an infinite series of p–n junctions, turn out to profoundly affect the electronic band structure, in particular inducing the generation of new Dirac points, whose presence may be inferred in transport experiments (Brey & Fertig 2009; Park et al. 2009).

2. Graphene rings

Graphene rings have been studied by several groups. Nanoribbons closed into short nanotubes have spectra that are sensitive to the precise boundary termination of the ring, and may behave in a complicated way as the width is varied (Nakamura et al. 2004; Yoshioka & Higashibata 2009). Some studies adopt simplified boundary conditions allowing for studies using the Dirac equation, so that one may construct spectra from those of the individual valleys (Recher et al. 2007), or treat many-body effects (Abergel et al. 2008). Tight-binding studies of flat graphene rings involve different edge terminations and corner geometries (Recher et al. 2007; Bahamon et al. 2009; Wurm et al. 2009, 2010), and reveal
spectra that are sensitive to both. Some experimental studies have recently been reported (Russo et al. 2008; Molitor et al. 2010) in which Aharonov–Bohm oscillations in graphene rings are reported. This demonstrates that sufficiently large phase coherence lengths may be reached to allow quantum coherence effects to be observed, although current sample geometries are not sufficiently controlled to allow direct comparison with the theoretical studies of idealized models.

(a) Chirality in armchair nanoribbons

Graphene rings constructed from metallic armchair ribbons turn out to offer particularly intriguing behaviour. The reason is that the effective Hamiltonian of armchairs supports an unusual symmetry involving interchange of the sublattice and valley indices of the electrons, allowing the electron states in the lowest electric subband to possess a chirality: states with currents in opposite directions have different quantum numbers associated with this symmetry (Luo et al. 2009). These states are present even at zero energy for ribbons of appropriate width (Brey & Fertig 2006b), so that the ribbons are metallic when undoped. Another important property is that 60° corner junctions may be constructed from such ribbons (Iyengar et al. 2008), which are perfectly transmitting at low energy. For conventional metallic ribbons connected by perfectly transmitting junctions, one expects to find a zero-energy state in the spectrum. The graphene junctions, however, introduce a phase shift as electrons pass through them, with a resulting spectrum that looks much like magnetic flux has been threaded through the ring. This ‘effective time-reversal symmetry breaking’ (ETRSB) is a hallmark of the Dirac equation that governs the electron dynamics at low energy.

The relevant symmetry in this problem can be characterized in terms of a 4 × 4 matrix,

\[ T = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \] (2.1)

which acts on the electron wave functions in the ribbon represented by 4-vectors, \( \Psi_{p_n p_y} = (\phi^{A,K*}, \phi^{B,K*}, \phi^{A,K'*}, \phi^{B,K'*})^\dagger \). In these wave functions, the first two entries represent amplitudes for the wave functions on the \( A \) and \( B \) sublattices in the \( K \) valley, and the second two entries are the corresponding amplitudes for the \( K' \) valley. The quantum number \( p_n \) is a transverse momentum characterizing the electron subband, and \( p_y \) is the momentum along the ribbon. One may easily confirm that \( T \) commutes with the Hamiltonian, and the armchair boundary condition (Brey & Fertig 2006a) respects this symmetry as well (Luo et al. 2009). In general, one must admix wave functions with positive and negative values of \( p_n \) to construct eigenstates of \( T \); however, the \( p_n = 0 \) subband is special in that \( \Psi_{p_n=0, p_y} \) is an eigenstate of \( T \). In this case, one finds

\[ T\Psi_{0,p_y} = \text{sgn}(\varepsilon)\text{sgn}(p_y)\Psi_{0,p_y}, \] (2.2)

Thus, for metallic nanoribbons, the eigenvalue of \( T \) in the lowest subband \( (p_n = 0) \) is tied to the direction of current \( \text{sgn}(p_y) \), in a way that is highly analogous to the connection between current direction and valley index for zigzag nanoribbons, with the eigenvalue of \( T \) playing the role of valley index.

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(b) Phase jumps at corner junctions

Figure 1a illustrates a typical armchair graphene ring that implements the interesting properties discussed above. An important property of the corner junctions in this geometry is that, in addition to perfectly transmitting at low energy, they induce a phase jump in the wave functions as an electron passes through them. This can be demonstrated explicitly by matching wave functions in the lowest subband where two arms of the armchair ribbon are joined (Luo et al. 2009). The result confirms the unit transmission found previously for these junctions (Iyengar et al. 2008), and also demonstrates that the wave functions jump by a factor of $\pm i$ as one passes through a junction, with the sign determined by the eigenvalue of the $T$ operator defined above.

This phase shift can also be viewed from the continuum perspective. Consider a long metallic armchair nanoribbon segment with its front and back ends identified to form a cylinder, effectively a short, fat nanotube. We may simulate the phase shift associated with the junction by adding a gauge field. Our Hamiltonian has the form ($\hbar = 1$)

$$H(p) = v_F \begin{pmatrix} 0 & -p_x + ip_y & 0 & 0 \\ -p_x - ip_y & 0 & 0 & 0 \\ 0 & 0 & 0 & p_x + ip_y \\ 0 & 0 & p_x - ip_y & 0 \end{pmatrix},$$

where $p_{x(y)} = (1/i)\partial_{x(y)}$. Writing $\Psi_{p_x,p_y} = e^{ix}\Psi'_{p_x,p_y}$, we can rewrite the Dirac equation for the wave functions in the form $H(p - A)\Psi'_{p_x,p_y} = e^{i\chi}\Psi'_{p_x,p_y}$, with $A = \partial_y \chi \hat{y}$ and $\chi = (\pi/2)T\Theta(y)$, where the junction between ribbons is located at $y = 0$. In this representation, $\Psi'_{p_x,p_y}$ is continuous across the junction, and the phase jump is fully implemented by the $e^{ix}$ factor.

The presence of the gauge field can be interpreted as being due to a pair of solenoids carrying magnetic flux in opposite directions, one above the plane of the ribbon, the other below. In this way, one can understand the problem of $n$-sided
rings constructed from metallic armchair ribbons, with these corner junctions, as being the same (near zero energy) as the problem of a ribbon closed into a cylinder (i.e. a short nanotube), with $2n$ flux tubes, half threaded through in one direction and half just outside it in the other direction. In this way, the system has properties illustrating ETRSB. It is important to recognize that real time-reversal symmetry is preserved for the system, because the effective flux runs in opposite directions for different eigenvalues of the operator $T$. As we shall see in detail below, the low-energy spectra of such rings as found from solutions of the tight-binding model behave precisely as if these phase jumps are present. The resulting spectra present properties that may be understood as signalling the ETRSB in graphene.

(c) Numerical results

The simplest ring system one can study using metallic armchair ribbons and the $60^\circ$ corner junctions discussed above is the hexagonal ring, as illustrated in figure 1a. The fact that the junctions are perfectly transmitting in the lowest subband might lead one to think that the low-energy spectrum is the same as that of a metallic armchair ribbon closed into a cylinder (i.e. a very short carbon nanotube). If this were the case, one would expect states at zero energy when no external magnetic flux threads the ring. Our discussion above, however, indicates that one needs to include the effective flux passing through the ring to understand the spectrum.

Figure 1b is the spectrum obtained from computing the eigenvalues of the tight-binding model near zero energy, as a function of flux $\Phi$ through the ring. Note in these calculations we include only the phase factors in the hopping matrix elements owing to a solenoid passing through the hole of the ring; magnetic flux through the individual plaquettes of the honeycomb lattice is not included. In general, the spectrum in this case is perfectly particle–hole symmetric, so only positive energy states are displayed.

The form of the spectrum is highly reminiscent of what has been seen previously in hexagonal rings with zigzag edges (Recher et al. 2007). In particular, it takes the form of two sets of spectra, each with a broken time-reversal symmetry such that the energies have a particular sign of slope near $\Phi = 0$. The spectra are effectively time-reverses of one another so that the spectrum as a whole has time-reversal symmetry; in particular, the spectrum evolves in the same way whether positive or negative flux is threaded through the hole. In the case of zigzag ribbons, the two sets of spectra are associated with the two valleys. In the present case, they are associated with the two eigenvalues of the matrix $T$. Note that the crossing of the energy states through zero at $\Phi = \pm \Phi_0/2$ may be understood as resulting from the sum of the effective fluxes owing to the corner junctions and that of the real field summing to an integral number of flux quanta.

The phase jumps associated with the corners may be demonstrated explicitly by a careful examination of wave functions. Figure 2 labels a set of sites around the ring, and figure 3 illustrates the wave functions on one of the sublattices for the four lowest positive energy levels at $\Phi = 0$. The jumps in amplitude associated with passing through the corners of the junctions are quite apparent. The amplitudes in the sides of the ring may be understood as linear combinations of pure plane waves, with momentum $\pm p_y$ along the ribbon directions, yielding
Figure 2. Hexagonal ring illustrating site labels for the examination of wave functions. Black dots indicate sites on the $A$ sublattice and grey dots are on the $B$ sublattice. The actual ring used in calculations has $r_a = 32.5a$ and $r_b = 38.5a$, with $a$ being the lattice constant of the underlying triangular lattice.

Figure 3. Wave functions at zero flux $\Phi$ for the four lowest eigenstates $n = 1, 2, 3, 4$ illustrated in figure 1b. (a) $n = 1$; $A$ sublattice; (b) $n = 2$; $A$ sublattice; (c) $n = 3$; $A$ sublattice; (d) $n = 4$; $A$ sublattice.
Table 1. Table showing values of $p_y$, $\theta_0$ and $m$ in a formula of the form $6(p_y L + \theta_0) = 2\pi m$ used in matching numerically generated wave functions with forms expected from wave-function continuity around a ring (see text).

<table>
<thead>
<tr>
<th>energy level</th>
<th>$p_y L$</th>
<th>$\theta_0$</th>
<th>$m$</th>
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<tr>
<td>1</td>
<td>$\pi/6$</td>
<td>$\pi/2$</td>
<td>2</td>
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<td>2</td>
<td>$-\pi/6$</td>
<td>$-\pi/2$</td>
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<td>3</td>
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<td>4</td>
<td>$-\pi/2$</td>
<td>$-\pi/2$</td>
<td>-3</td>
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<tr>
<td>5</td>
<td>$5\pi/6$</td>
<td>$\pi/2$</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>$-5\pi/6$</td>
<td>$-\pi/2$</td>
<td>-4</td>
</tr>
</tbody>
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purely real wave functions. Fits to wave functions with well-defined $|p_y|$ and phase jumps of $e^{i\theta_0} = \pm i$ yield excellent results (Luo et al. 2009); the results of such fits are detailed in table 1, which illustrates that the values of $p_y$ that are realized result from continuity of the wave functions when the phase jumps are included. Moreover, the results also directly support the expected sign of the phase jump, as determined by the eigenvalue of $T$.

The presence of six $\pi/2$ phase jumps (owing to the six corners of the hexagon) implies that $p_y = 0$ is not an allowed momentum for the electron wave function in this type of ring. Thus, there is no allowed zero-energy state, as would be expected for a metallic ribbon closed into an annulus, and its absence is imposed by the presence of phase factors that are very suggestive of effective magnetic flux threading the ring, as discussed above. Thus, the gap in the spectrum around $\varepsilon = 0$ may be interpreted as a signature of ETRSB.

One may examine how robust this behaviour is with respect to various perturbations of the ring geometry. Figure 1b clearly displays level crossings of states with different quantum numbers associated with the sixfold rotational symmetry of the structure. A generic perturbation breaking this symmetry opens small gaps at these crossings, but in most cases does not spoil the basic property that no zero-energy state is present when no (real) magnetic flux threads the ring. An important exception is the effect of removing atoms from the corners of the junctions, which in effect introduces a short length of zigzag edge. Such structures induce localized zero-energy states (Palacios et al. 2010), which are insensitive to flux (real or effective) through the ring. In general a collection of rings will have various defects, but one might expect that the energy-level statistics of such an ensemble will reflect the tendency for the rings to have a gap at zero energy.

3. Graphene in a periodic potential

The electronic structure of graphene may also be manipulated by external potentials. Of particular interest are periodic potentials, which in principle allow one to ‘engineer’ the band structure of the system. Moreover, because such potentials can be varied continuously, in principle, one can manipulate the system to tailor its conduction properties, opening many possibilities for electronic devices. Much recent work has focused on p–n junctions as a first
Figure 4. (a) Conductivity and (b) Fano factor, as functions of $V_0$ and for $G_0 = \pi/25a$, for two different graphene sample lengths containing 10 (dashed curves) and 20 (solid curves) periods of the periodic potential $2V_0 \cos G_0 x$. (c) The conductivity as a function of $V_0/G_0$ as obtained from different superlattice potentials. This result indicates that the conductivity only depends on $V_0/G_0$. Filled circles, $G_0 = \pi/25a$; filled squares, $G_0 = 3\pi/50a$; filled triangles, $G_0 = 2\pi/75a$.

realization of such systems (Huard et al. 2007; Özyilmaz et al. 2007; Williams et al. 2007; Stander et al. 2009; Young & Kim 2009). Periodic potentials may be induced by interaction with a substrate (Marchini & Wintterlin 2007; Pan et al. 2007; de Parga et al. 2008) or controlled adatom deposition (Meyer et al. 2008). Recently, the existence of periodic ripples in suspended graphene has been demonstrated (Bao et al. 2009); in a perpendicular electric field, this would also induce a periodic potential.
A one-dimensional periodic (i.e. superlattice) potential indeed has important effects on the transport properties of graphene. For example, this induces an anisotropy in the electron velocity around the Dirac point that ‘collimates’ the flow of electrons in such a structure (Park et al. 2008a,b). The effect is realized by a slowing of the electron velocity perpendicular to the superlattice axis, and for a strong enough potential, or a large enough period of the potential, this becomes so pronounced that the sign of the electron velocity becomes inverted, creating new Dirac points at zero energy in the band structure (Brey & Fertig 2009; Park et al. 2009). The emergence of these new Dirac points turns out to be controlled by the parameter $V_0/G_0$, where $V_0$ is the potential amplitude (assumed below to be a cosine) and $L = 2\pi/G_0$ is the period. The new Dirac points emerge whenever $J_0(4V_0/\hbar v_F G_0) = 0$, where $J_0$ is a Bessel function and $v_F$ the speed of the Dirac fermions in the absence of the potential. The total number of Dirac points (associated with a single valley and electron spin) is thus $2N + 1$, with $N$ the number of zeros of $J_0(x)$, with $|x| < 4V_0/\hbar v_F G_0$.

These emerging Dirac points have a clear signature in transport along the superlattice axis: conductance resonances appear at the values of $V_0/G_0$ where they first appear, as illustrated in figure 4a. Much like undoped graphene, in the limit of large width, the conductance scales as $L_y/L_x$ for most values of $V_0/G_0$, with $L_y$ the system width and $L_x$ its length. The system behaves diffusively, i.e. is characterized by a conductivity rather than a conductance. This interpretation is consistent with the computed Fano factor, illustrated in figure 4b. At the resonances, however, the conductance becomes independent of $L_x$, and the Fano factor indicates a more ballistic-like transport.

(a) Counting Dirac points

For such periodic structures, one may ascertain analytically how many Dirac points will appear at zero energy. For concreteness, we take our external potential to have the form $V(x) = 2V_0 \cos G_0 x$, and focus on the low-energy properties of electrons in a single valley and with a given spin. These are described by the massless Dirac Hamiltonian with a potential,

$$H = \hbar v_F (-i\sigma_x \partial_x + k_y \sigma_y) + V(x)\mathcal{I}, \quad (3.1)$$

where $\sigma_{x,y}$ are the Pauli matrices, and $\mathcal{I}$ is the identity matrix. The corresponding wave functions that this acts upon have two components, $\Phi_{A,B}$, which we assume have $k_y$ as a good quantum number.

A unitary transformation (Park et al. 2008b; Brey & Fertig 2009) allows the Hamiltonian to be recast in the form

$$H' = \hbar v_F \begin{pmatrix} -i \partial_x & -ik_y e^{i\alpha(x)} \\ ik_y e^{-i\alpha(x)} & i \partial_x \end{pmatrix}, \quad (3.2)$$

where $e^{i\alpha(x)} = \sum_{l=-\infty}^{l=\infty} J_l(4V_0/\hbar v_F G_0) e^{ilG_0 x}$, and $J_n$ is the $n$th Bessel function of the first kind. Dirac points at zero energy must satisfy $H'\phi = 0$; such solutions have
the property $\phi_A = \phi_B^*$. Writing $\phi_A = |\phi_A| e^{i\chi}$, one obtains equations for $\chi$ and $|\phi_A|$ (Brey & Fertig 2009):

$$k_y \sin(\alpha - 2\chi) + \partial_x \chi = 0 \quad (3.3)$$

and

$$|\phi_A| \propto \exp\left\{-k_y \int_0^x \cos[\alpha(x') - 2\chi(x') \} \, dx' \right\}. \quad (3.4)$$

Since $\phi$ is a Bloch state of the superlattice, it must obey the Bloch relation $\phi_{A,B}(x + L_0) = e^{ik_x L_0} \phi_{A,B}(x)$, with $k_x$ the crystal momentum. For a zero-energy state, only $k_x = 0$ is possible. We then require (i) $\chi(x + L_0) = \chi(x) + 2\pi m$ with $m$ an integer and (ii) $\int_0^{L_0} \cos[\alpha(x) - 2\chi(x)] = 0$. To see whether $\chi$ can satisfy these relations, it is helpful to recast equation (3.3) by writing $\tilde{\chi} = 2\chi - \alpha$, and $x \to t$, so that

$$-\partial_t \tilde{\chi} - \partial_t \alpha + 2k_y \sin \tilde{\chi} = 0. \quad (3.5)$$

This is the equation of motion for the position $\tilde{\chi}$ of an overdamped particle (with unit viscosity), subject to a periodic time-dependent force $\partial_t \alpha$ and a spatially periodic force $2k_y \sin \tilde{\chi}$. Despite the periodicity of the forces involved, the generic solution to this equation is not periodic. However, for certain parameters, periodic solutions can be found, which correspond to allowed zero-energy solutions of the Dirac equation in a periodic potential.

Equation (3.3) is nonlinear and an analytic solution is not readily available. However, one may generate approximate solutions perturbatively in $k_y$. Writing $\chi = k_y \chi^{(1)} + k_y^2 \chi^{(2)} + \mathcal{O}(k_y^3)$, one finds

$$\chi^{(1)} = - \int_0^x \, dx' \sin \alpha(x') + C^{(1)}$$

and

$$\chi^{(2)} = 2C^{(1)} \int_0^x \, dx_1 \cos \alpha(x_1) - \int_0^x \, dx_1 \int_0^{x_1} \, dx_2 \sin \alpha(x_2) + C^{(2)},$$

where $C^{(1,2)}$ are constants of integration. Explicitly performing the integrations for the above two equations, one finds that condition (i) can be satisfied if

$$k_y^2 \left[2C^{(1)}L_0 - \sum_{\ell \text{ odd}} J_\ell G_0 L_0 \right] = 2\pi m. \quad (3.6)$$

Here, $J_0$ and $J_\ell$ are Bessel functions evaluated at $4V_0/\hbar v_F G_0$. Since we have employed a small $k_y$ expansion, the only consistent solution is for $m = 0$. In this case, equation (3.6) fixes $C^{(1)}$, and the resulting $\chi$ (and the associated $\tilde{\chi}$) is periodic. Condition (ii) may then be implemented to fix the value of $k_y$ at which a zero mode appears:

$$\left( \frac{k_y}{G_0} \right)^2 = - \frac{J_0}{2 \sum_{\ell_1,\ell_2 \text{ odd}} J_{\ell_1} J_{\ell_2} J_{-\ell_1 - \ell_2}/\ell_1 \ell_2}. \quad (3.7)$$

Equation (3.7) predicts the presence of a zero mode whenever the right-hand side is positive. This turns out to occur for values of $x = 4V_0/\hbar v_F G_0$ just above the values of $x$ where $J_0(x) = 0$; the sign of the denominator on the right-hand side of equation (3.7) always works out such that $k_y^2 > 0$ in this situation. With
increasing $x$, the solution moves to larger $|k_y|$ until it diverges where the three Bessel function sum vanishes, which is always prior the next zero of $J_0(x)$. We note that since our approximation is only valid for small $k_y$, equation (3.7) cannot accurately predict the location of the zero-energy states well away from $k_y = 0$. However, since zero-energy states can only annihilate in pairs, once they emerge from the origin they should persist. This expectation is borne out by numerical studies, which we describe next.

(b) Numerical solutions of the Dirac equation

Our expectations about the new zero-energy states can be directly confirmed by numerically solving the Dirac equation in a periodic potential. To accomplish this, we represent the Hamiltonian $H = H_0 + 2V_0 \cos G_0 x$ in a plane-wave basis and diagonalize the resulting matrix for momenta $(k_x, k_y)$, with $-G_0/2 < k_x < G_0/2$. In figure 5, we plot, for different values of $V_0$, the lowest few energy eigenvalues as a function of $k_y$ for $k_x = 0$ and a superlattice potential of period $50a$, where $a$ is the lattice constant for pure graphene. As $V_0$ increases, the group velocity at the Dirac point decreases to zero, and thereafter two zero-energy states emerge from $k_y = 0$ as the group velocity of the $k_y = 0$ Dirac point becomes finite again. These are the new zero-energy states discussed above; we find that they emerge precisely when $J_0(4V_0/\hbar v_F G_0) = 0$. Upon further increase of $V_0$, the group velocity along $k_y$ at $k_y = 0$ becomes zero again and a new pair of zero-energy states emerge from $k_y = 0$, again precisely at the next zero of $J_0(4V_0/\hbar v_F G_0) = 0$. This pattern continues to repeat itself with increasing $V_0$. Further studies for different periodicities confirm the prediction that the emergence of these points depends only on the ratio $V_0/G_0$, precisely as discussed in the previous section.
Conductivity

Using transfer matrices, one may compute the conductivity through a graphene strip of length $L_x$, containing $N_p$ periods of the superlattice potential. Boundary conditions are taken to be periodic in the transverse direction, leading to transverse wave functions labelled by a momentum $k_y$; this is justified when the width of the strip, $L_y$, is much larger than its length. From the transmission probability of each mode, $T_{k_y}$, we obtain the conductance $G$ and the Fano factor $F$ (ratio of noise power and mean current):

$$G = 4 \frac{e^2}{h} \sum_{k_y} T_{k_y} \quad \text{and} \quad F = \frac{\sum_{k_y} T_{k_y} (1 - T_{k_y})}{\sum_{k_y} T_{k_y}} ,$$

where the factor 4 accounts for the spin and valley degeneracy. The conductivity is related to the conductance via geometrical factors, $\sigma = G \times L_y / L_x$. In what follows, we work in the limit $L_y \gg L_x$.

For pristine graphene, $V_0 = 0$, the conductivity is independent of $L_x$ and takes the value $\sigma_0 = 4 e^2 / \pi h$, and the Fano factor takes on the universal value $1/3$. This latter value is consistent with the apparent diffusive behaviour of pure, undoped graphene (Tworzydlo et al. 2006). Figure 4 shows the conductivity and the Fano factor as a function of $V_0$, for two graphene strips, respectively, containing 10 and 20 periods of a potential of the form $2 V_0 \cos G_0 x$, where $G_0 = \pi / 50 a$. For finite values of $V_0$, apart from some resonances, the system behaves diffusively ($F = 1/3$) and the conductivity is well defined. Interestingly, between the peaks, the overall scale increases with $V_0$, showing that the periodic potential tends to enhance the conductivity.

At certain values of $V_0$, one observes peaks in the conductance, for which the conductivity is not well defined and the Fano factor tends to zero. These resonances occur precisely whenever new zero-energy states emerge from the origin in $k$-space, and represent a direct experimental signature of their presence. We believe the resonances occur because the group velocity vanishes when a zero-energy state emerges, leading to a strong enhancement of the density of states. A further check that the resonances are associated with the zero-energy states is to see that they depend on the ratio $V_0 / G_0$; figure 4c demonstrates that this is the case not just for the resonances but for the entire conductance curve.

4. Conclusion

In this article, we have reviewed some recent progress in understanding nanostructured graphene. Graphene quantum rings were shown to offer a signature of ETRSB that is unique to the low-energy physics of graphene: for appropriately chosen corner geometries, the spectra are gapped at zero energy even when the ribbons from which they are constructed themselves are not. This behaviour may be naturally understood in terms of effective flux quanta enclosed by the ring, with direction dependent on the direction of current carried by a given wave function. Graphene in a periodic superlattice potential offers the possibility of band-gap engineering, in an in-principle controllable way:
the number of Dirac points at zero energy is controlled by the period and magnitude of the external potential. Appearances of these new Dirac points are accompanied by resonances in the conductance of the system. Clearly, graphene offers a new environment in which electronic properties may be tailored by nanoscale lithography and electric fields, yielding behaviours with no analogue in conventional low-dimensional electron systems. There is much yet to be explored in this class of problems and systems, with potential for further unique behaviours and device applications.

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