Isochronous dynamical systems

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This is a terse review of recent results on isochronous dynamical systems, namely systems of (first-order, generally nonlinear) ordinary differential equations (ODEs) featuring an open set of initial data (which might coincide with the entire set of all initial data), from which emerge solutions all of which are completely periodic (i.e. periodic in all their components) with a fixed period (independent of the initial data, provided they are within the isochrony region). A leitmotif of this presentation is that ‘isochronous systems are not rare’. Indeed, it is shown how any (autonomous) dynamical system can be modified or extended so that the new (also autonomous) system thereby obtained is isochronous with an arbitrarily assigned period $T$, while its dynamics, over time intervals much shorter than the period $T$, mimics closely that of the original system, or even, over an arbitrarily large fraction of its period $T$, coincides exactly with that of the original system. It is pointed out that this fact raises the issue of developing criteria providing, for a dynamical system, some kind of measure associated with a finite time scale of the complexity of its behaviour (while the current, standard definitions of integrable versus chaotic dynamical systems are related to the behaviour of a system over infinite time).

Keywords: nonlinear dynamical systems; periodic dynamical system; isochronous dynamical system; integrable dynamical system; chaotic behaviour; deterministic chaos

1. Introduction

A dynamical system is characterized by a set of $N$ first-order (generally nonlinear) ordinary differential equations (ODEs),

$$X'_n = F_n(X), \quad X' = F(X).$$  \hspace{1cm} (1.1)

Notation. Here (and hereafter, unless otherwise indicated), $N$ is an arbitrary positive integer, indices such as $n, m$ run over the integers from 1 to $N$, the $N$ quantities $X_n \equiv X_n(t)$ are the dependent variables, $\tau$ is the independent variable, primes denote differentiation with respect to the independent variable $\tau$, underlined symbols denote $N$-vectors and the $N$ functions $F_n(X)$ are supposed to be given. For simplicity, we assume that they are assigned so that, for any

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This paper is dedicated to Robin Bullough (1929–2008) in memoriam.

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set of initial conditions $X(0)$, this system of ODEs admits a unique solution $X(\tau)$ that is non-singular for all (real) values of the independent variable $\tau$. But we shall also consider this dynamical system for complex values of the independent variable $\tau$ (hence also for complex values of the dependent variables $X_n$), and we, therefore, always assume that the given functions $F_n(X)$ depend analytically on their variables; likewise, the solutions $X(\tau)$ depend analytically on the independent variable $\tau$, as it is essential to provide an unambiguous definition of the $\tau$-derivatives $X'_n \equiv X'_n(\tau)$, see equation (1.1).

**Definition 1.1.** A dynamical system is isochronous if there exists an open set of initial data (which might coincide with the entire set of all initial data) from which emerge solutions, all of which are completely periodic (i.e. periodic in all their components) with a fixed period (independent of the initial data, provided they are within the isochrony region).

The investigation of isochronous dynamical systems in the context of what is now considered ‘mathematical physics’ has a long history going back to such eminent scientists as Galileo, Newton, Huygens, Jacobi and Poincaré, and it received a further boost in the last century—mainly focused on the special case with $N = 2$ (see, for instance, the review paper [1])—in connection with the 16th Hilbert problem. In the present review, we ignore all these important investigations and focus exclusively on some recent developments that occurred over the last one to two decades. The main idea of these findings is rather elementary. Consider, as a variant of equation (1.1), the new dynamical system

$$\dot{x}_n = \dot{t} F_n(x), \quad \dot{t} = \dot{t} F(x),$$

where we denote as $x_n \equiv x_n(t)$ the $N$ dependent variables, as $t$ the independent variable (which we now interpret as ‘time’; and the superimposed dot denotes differentiation with respect to this variable). Let us also assume (for simplicity) that the function $\tau(t)$ vanishes at $t = 0$,

$$\tau(0) = 0.$$ (1.3)

It is then plain, from a comparison of equation (1.2) with equation (1.1), that the solution $x(t)$ of this system (1.2) characterized by initial data $x(0)$ is related to the solution $X(t)$ of the system (1.1) with the same initial data,

$$x(0) = X(0),$$ (1.4)

by the simple formula

$$x(t) = X[\tau(t)].$$ (1.5)

And it is moreover evident that, if the function $\tau(t)$ is periodic in $t$ with an arbitrarily assigned period $T$,

$$\tau(t \pm T) = \tau(t),$$ (1.6a)

then the solution $x(t)$ of the system (1.2) inherits this property of periodicity,

$$x(t \pm T) = x(t).$$ (1.6b)

One therefore concludes that the system (1.2) is isochronous.

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We have thereby seen an elementary trick to manufacture, starting from the quite generic dynamical system (1.1)—whose behaviour might be quite complicated (‘chaotic’)—a new system, (1.2), which is isochronous. The price to pay is that the new system (1.2) is not autonomous. But this ‘defect’ can be by-passed; different ways to achieve this goal are detailed in the following three sections (§§§2, 3 and 4), whose presentations follow more or less the chronological evolution of these findings. In the subsequent two sections (§§5 and 6), we outline tersely two related topics: how to manufacture systems that are asymptotically isochronous rather than isochronous, and the possibility to exploit isochronous systems to arrive at Diophantine results. Section 7 outlines some implications of these findings, in particular, the desirability implied by them to invent criteria providing some kind of measure associated with a finite time scale of the complexity of the behaviour of a dynamical system. This will require some creativity: indeed, according to current notions, the current, standard definitions of integrable versus chaotic dynamical systems are related to the behaviour of a system over infinite time.

2. A technique involving complexification to modify certain dynamical systems so that they become isochronous

The presentation in this section is quite terse because this material has already been reviewed in the monograph [2]. In particular, we only convey here the main idea of this approach in the simplest context, referring to the monograph for more general treatments and for many examples and applications.

Recall that the functions $F_n(X)$ appearing in the right-hand side of equation (1.1) are (assumed to be) analytical in their arguments and assume moreover that they feature the grading property

$$F_n(\lambda X) = \lambda^r F_n(X),$$

where $r$ is a rational number different from unity,

$$r = \frac{p}{q} \quad \text{and} \quad p \neq q,$$

with $p$ and $q$ two coprime integers. Then, introduce the following change of (dependent and independent) variables:

$$z_n(t) = \exp\left(\frac{i\omega t}{r-1}\right) X_n(\tau) = \exp\left(\frac{iqa t}{p-q}\right) \quad (2.2a)$$

and

$$\tau = \frac{\exp(i\omega t) - 1}{i\omega}. \quad (2.2b)$$

Here and throughout, $i$ is the imaginary unit, $i^2 = -1$, and $\omega$ is a positive constant to which we associate the period

$$T = \frac{2\pi}{\omega}. \quad (2.3)$$
It is then plain (via the grading property (2.1)) that the new dependent variables \( z_n(t) \) evolve according to the new (autonomous) dynamical system

\[
\dot{z}_n - i \frac{q}{p - q} \omega z_n = F_n(z).
\]  (2.4a)

And note that, since the definition (2.2b) entails \( \tau(0) = 0 \), the initial data for the original system (1.1) and the new system (2.4a) coincide,

\[
z_n(0) = X_n(0). \]  (2.4b)

On the other hand, it is clear (see equation (2.2b)) that, when the time \( t \) varies over one period \( T \), the (complex) variable \( \tau(t) \) makes correspondingly, in the complex \( \tau \)-plane, a complete tour over the circle \( C \) whose diameter lies on the imaginary axis, with one end at the origin \( \tau = 0 \) and the other end at the point \( \tau = 2i/\omega = iT/\pi \) (the diligent reader will draw the relevant figure). It is therefore evident (see equation (2.2)) that the solution \( z(t) \) of the new dynamical system (2.4a) is completely periodic in \( t \) with period \( |p - q| T \),

\[
z_n(t \pm |p - q| T) = z_n(t), \]  (2.5)

provided the corresponding solution \( X_n(\tau) \) of the original dynamical system (1.1) has no branch point in the disc of the complex \( \tau \)-plane whose boundary is the circle \( C \). But, the standard Cauchy problem for the dynamical system (1.1) guarantees that the solution of this system is holomorphic inside a circle of the complex \( \tau \)-plane centred at the origin \( \tau = 0 \), provided the diameter of this circle is sufficiently small. Actually, the size of this circle is only determined—for an autonomous system such as equation (1.1)—by the initial data \( X_n(0) \), and can be made arbitrarily large by an appropriate choice of these initial data. There is, therefore, certainly an open set of initial data \( X_n(0) = z_n(0) \) (see equation (2.4b)) implying that the radius of this circle exceeds the diameter \( 2/\omega = T/\pi \) of the circle \( C \) (the diligent reader will again draw the relevant figures), hence that the corresponding solution \( X(\tau) \) is holomorphic in \( \tau \) inside the circle \( C \), hence that the corresponding solution \( z(t) \) of the new dynamical system (2.4a) is completely periodic, with period \( |p - q| T \), see equation (2.5). It is thereby seen that the new dynamical system (2.4a) is isochronous.

A limitation of this procedure to generate dynamical systems is that it is only applicable to dynamical systems satisfying the grading property (2.1) (for some extensions, see [2]). Moreover, even when the original system (1.1) is real, the new dynamical system (2.4a) is complex. This is not a qualitative change since, by introducing as new dependent variables, the real and imaginary parts of the complex dependent variables \( z_n, z_n = x_n + iy_n \), one reobtains in any case a real dynamical system; however, when the original system (1.1) is real, this entails that the corresponding (real) isochronous dynamical system manufactured in this manner features twice as many dependent variables as the original model.

One final observation. If the original dynamical system (1.1) was integrable, then all its solutions \( X_n(\tau) \) might well be, as functions of the complex variable \( \tau \), free of branch points in the entire complex \( \tau \)-plane (the Painlevé property). Then, all the solutions—with arbitrary initial data—of the new dynamical system (2.4a)
would be isochronous. And note that this conclusion remains valid (up to minor adjustments), even if the Painlevé property is weakened by admitting the presence of rational branch points, as is indeed the case for some well-known integrable dynamical systems.

3. How to extend any dynamical system so that it becomes isochronous

This treatment has also already been reviewed in the monograph [2]; however, some of the results reported in this section were obtained after the monograph had been published (see, in particular, [3,4]), and some of the examples presented herein are, to some extent, new.

The starting point is again the generic dynamical system (1.1). We now follow the treatment reported in §1, but we replace the system (1.2) with the following equivalent system:

$\dot{x}_n = \phi F_n(x), \quad \dot{x} = \phi F(x). \quad (3.1)$

The equivalence is guaranteed because we shall impose that

$\phi = \tau, \quad (3.2a)$

implying, see equation (1.3),

$\tau(t) = \int_0^t dt' \phi(t'). \quad (3.2b)$

But we shall not express $\phi$ directly as a function of the independent variable $t$, since we now want the dynamical system (3.1) to be autonomous: hence, $\phi$ must only be a function of the dependent variables $x_n$, its time dependence being then implied by the time dependence of these variables. On the other hand, the time dependence of $\phi$ must guarantee that the function $\tau(t)$ defined by equation (3.2) is periodic in $t$ with period $T$, see equation (1.6a).

A natural way to achieve this goal is by extending the dynamical system (3.1) via the introduction of additional dependent variables: various ways to do so are reported below, in this section. A different approach is to manufacture a function $\phi$ depending only on the dependent variables $x_n$, yet having the desired property (to yield via (3.2) a periodic function $\tau(t)$); in this case, we will consider the transition to the new, isochronous dynamical system (3.1), rather a modification than an extension of the dynamical system (1.1). This second method shall require some limitation on the class of dynamical systems (1.1) to which it is applicable, but—as we show in the next section where we treat this approach—it is nevertheless applicable to a case of major physical relevance.

(a) A trivial assignment

The most trivial way to implement the programme outlined above is to add two additional dependent variables, $\xi_1$ and $\xi_2$, to the $N$ dependent variables $x_n$, and correspondingly to complement the dynamical system (3.1) with the two
additional, trivial equations of motion,
\[ \dot{\xi}_1 = \omega \xi_2 \quad \text{and} \quad \dot{\xi}_2 = -\omega \xi_1, \quad (3.3a) \]
which of course imply
\[ \xi_1(t) = A \sin(\omega t + a) \quad \text{and} \quad \xi_2(t) = A \cos(\omega t + a), \quad (3.3b) \]
where \( A \) and \( a \) are two constants determined in an obvious manner by the two initial values \( \xi_1(0) \) and \( \xi_2(0) \). Then, by setting, say,
\[ \phi = \xi_1, \quad (3.4) \]
implying (see equations (3.2) and (3.3b))
\[ \tau(t) = \frac{A}{\omega} [\cos(a) - \cos(\omega t + a)], \quad (3.5) \]
our goal is achieved.

An additional step (see [3]) that might be conveniently implemented is to perform the additional change of dependent variables
\[ \xi_j = y_j G_j(x), \quad j = 1, 2, \quad (3.6) \]
with the two functions \( G_j(x) \) assigned at one’s convenience (perhaps with the convenient restriction, see below, that they be positive definite for all values of the variables \( x_n \)). In this manner, one concludes that the class of dynamical systems characterized by the following system of \( N + 2 \) ODEs:
\[ \dot{x}_n = y_1 G_1(x) F_n(x), \quad (3.7a) \]
\[ \dot{y}_1 = \omega y_2 \frac{G_2(x)}{G_1(x)} - y_1^2 \sum_{n=1}^{N} \left[ \frac{\partial G_1(x)}{\partial x_n} F_n(x) \right] \quad (3.7b) \]
and
\[ \dot{y}_2 = -y_1 \frac{G_1(x)}{G_2(x)} \left\{ \omega + y_2 \sum_{n=1}^{N} \left[ \frac{\partial G_2(x)}{\partial x_n} F_n(x) \right] \right\}, \quad (3.7c) \]
with the \( N + 2 \) functions \( F_n(x), G_1(x) \) and \( G_2(x) \) of the \( N \) dependent variables \( x_n \), largely arbitrary (up to the minor restrictions mentioned above), is isochronous with period \( T \), see equation (2.3), for arbitrary initial conditions. Indeed, its general solution reads
\[ x(t) = X(\tau(t)), \quad (3.8a) \]
\[ y_1(t) = \frac{A \sin(\omega t + a)}{G_1[x(t)]} \quad \text{and} \quad y_2(t) = \frac{A \cos(\omega t + a)}{G_2[x(t)]}. \quad (3.8b) \]
Here, \( A \) and \( a \) are two arbitrary constants, \( X(\tau) \) is the general solution of the original system (1.1) and \( \tau(t) \) is given by the explicit formula (3.5).

Let us emphasize that this conclusion requires essentially no restriction on the original system (1.1); it also holds if that system features a very complicated (chaotic) evolution.

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It is moreover clear from equation (3.5) that, for any generic value $\tilde{t}$ of $t$ (such that $\sin(\omega \tilde{t} + a)$ does not vanish),

$$
\tau(t) = \tilde{\tau} + B(t - \tilde{t}) + O \left( \frac{(t - \tilde{t})^2}{T} \right),
$$

(3.9a)

where

$$
B = A \sin(\omega \tilde{t} + a) \quad \text{and} \quad \tilde{\tau} = \frac{A}{\omega} [\cos(a) - \cos(\omega \tilde{t} + a)].
$$

(3.9b)

Hence, up to a constant rescaling of the time variable (by a factor $B$ which need not be positive) and an irrelevant time shift, the part of the dynamics of the new model (3.7) characterizing the evolution of the $N$ dependent variables $x_n$, see equations (3.7a) and (3.8a), reproduces, around a generic time $\tilde{t}$, the dynamics of the original model (1.1). Of course, this statement is only applicable to time intervals that are much shorter than the period $T$ of the isochronous new model (3.7), indeed, the coincidence about the two dynamics holds up to corrections $O(|t - \tilde{t}|/T)$. We will discuss some implications of these findings in the last section.

Let us complete this subsection by exhibiting an example (the alert reader will manufacture others). As an instance of the original model, we take the well-known Lorenz model [5], which is a prototypical chaotic system. It corresponds to equation (1.1) with $N = 3$, and

$$
F_1(X) = -\alpha (X_1 - X_2), \quad F_2(X) = \beta X_1 - X_2 - X_1 X_2 \quad \text{and} \quad F_3(X) = X_1 X_2 - \gamma X_3.
$$

Here, $\alpha$, $\beta$ and $\gamma$ are three largely arbitrary constants.

We then set, for instance, $G_1(x) = G_2(x) = 1 + x_1^2$. The new isochronous system (3.7) then reads as follows:

$$
\dot{x}_1 = -y_1 (1 + x_1^2)\alpha (x_1 - x_2),
$$

(3.10a)

$$
\dot{x}_2 = y_1 (1 + x_1^2)(\beta x_1 - x_2 - x_1 x_2),
$$

(3.10b)

$$
\dot{x}_3 = y_1 (1 + x_1^2)(x_1 x_2 - \gamma x_3),
$$

(3.10c)

$$
\dot{y}_1 = \omega y_2 + 2\alpha y_1^2 x_1 (x_1 - x_2)
$$

(3.10d)

and

$$
\dot{y}_2 = -y_1 [\omega + 2\alpha y_2 x_1 (x_1 - x_2)].
$$

(3.10e)

The diligent reader might try and solve numerically this model, to check its isochrony.

Let us end this subsection by emphasizing that the results presented herein (and in the rest of this paper) justify the statement that ‘isochronous dynamical systems are not rare’. (A referee suggested that, to justify the statement that ‘isochronous dynamical systems are not rare’, it is appropriate to also mention that ‘there exist other methods which can also lead to isochronous systems, for example, direct non-local transformations, which can include even non-autonomous systems, see, for example [6] and references therein’.)
(b) A less trivial assignment

In the preceding subsection, we indicated a rather trivial method to extend any dynamical system so that the new system thereby generated is isochronous with an arbitrarily assigned period $T$. Trivial as this technique is, it does yield a class of isochronous systems, which is rather flexible, see equation (3.7), and which has moreover the remarkable property to entail a dynamics that mimics closely the dynamics of the original model, over time spans very short with respect to the (arbitrarily assigned) period $T$. In this subsection, we describe another type of extension, which yields a new system that is again isochronous with an arbitrarily assigned period $T$ and which moreover, over a fraction $(1 - \varepsilon)$ of that period (with $\varepsilon$ arbitrary in the interval $0 < \varepsilon < 1$), features a dynamics that is exactly identical to that of the original model. The idea that underlies this result was already used in [4], but here we introduce a new variant of it, which requires the introduction of only one additional dependent variable rather than two (and of course the corresponding introduction of one additional ODE). (The diligent reader will also explore how this variant could be applied in the preceding subsection.)

The first step of this development is to introduce the following simple ODE:

$$\dot{\xi} = \frac{\omega}{2}(1 + \xi^2),$$  \hfill (3.11a)

the general solution of which is

$$\xi(t) = \tan \left[ \pi \left( \frac{t}{T} - \eta \right) \right].$$  \hfill (3.11b)

Here, $\omega$ is again an arbitrary positive constant, to which we associate as usual the period $T$, see equation (2.3), and $\eta$ is a constant in the interval $0 \leq \eta < 1$, the value of which is determined by the initial value $\xi(0)$.

The next step is to extend the original system (1.1) by adding to it the ODE (3.11a) and by again replacing equation (1.1) with equation (1.2) where however we now set

$$\phi(\xi; \varepsilon) = 1 - K(\varepsilon) \theta[S(\xi; \varepsilon)] \exp[-S^{-2}(\xi; \varepsilon)],$$  \hfill (3.12a)

$$S(\xi; \varepsilon) = \frac{1}{1 + \xi^2} - \cos^2 \left( \frac{\pi \varepsilon}{2} \right)$$  \hfill (3.12b)

and

$$K(\varepsilon) = \left( 2 \int_0^{\varepsilon/2} dx \exp \left\{ - \left[ \cos^2(\pi x) - \cos^2 \left( \frac{\pi \varepsilon}{2} \right) \right]^{-1} \right\} \right)^{-1}.$$  \hfill (3.12c)

Here, $\varepsilon$ is a number that can be assigned at our convenience (in the open interval $0 < \varepsilon < 1$), while $\theta(x)$ is the standard step function, $\theta(x) = 1$ for $x \geq 0$, $\theta(x) = 0$ for $x < 0$.

Note that the (real) function $\phi(\xi; \varepsilon)$ is not analytical (owing to the presence of the discontinuous step function), yet it is not only continuous, but in fact infinitely differentiable, because where the argument $S(\xi; \varepsilon)$ of the step function vanishes, all derivatives of $\phi(\xi; \varepsilon)$ are well defined and continuous (indeed, they all vanish).
We then note that, via equation (3.11b), the definition (3.12b) yields
\[
S[\xi(t); \varepsilon] \equiv S(t) = \cos^2 \left[ \pi \left( \frac{t}{T} - \eta \right) \right] - \cos^2 \left( \frac{\pi \varepsilon}{2} \right), \tag{3.13a}
\]
implying that \( S(t) \) is periodic with period \( T \),
\[
S(t \pm T) = S(t), \tag{3.13b}
\]
and that it vanishes at \( t = T_\pm \mod(T) \),
\[
T_\pm = \left( \eta \pm \frac{\varepsilon}{2} \right) T \quad \text{and} \quad S(T_\pm) = 0. \tag{3.13c}
\]
It is moreover easily seen that the integral over one period \( T \) of
\[
\phi[\xi(t); \varepsilon] \equiv \phi(t) = 1 - K(\varepsilon) \theta[S(t)] \exp[-S^{-2}(t)], \tag{3.14a}
\]
with \( S(t) \) given by equation (3.13a) and \( K(\varepsilon) \) given by equation (3.12c), vanishes,
\[
\int_0^T dt \phi(t) = 0. \tag{3.14b}
\]
Hence, the function \( \tau(t) \), see equation (3.3), is itself periodic in \( t \) with period \( T \), see equation (1.6a), implying isochrony of the extended system ((1.2) with equations (3.12) and (3.11a)). And it is moreover easily seen that, within its period \( T \), the function \( \tau(t) \) has the following explicit expressions: if \( \eta \geq \varepsilon/2 \), hence \( 0 \leq T_- < T_+ \), then
\[
\tau(t) = t, \quad \text{for} \quad T_- - T \leq t \leq T_\mod(T), \tag{3.15a}
\]
and
\[
\tau(t) = t - \Psi(t) + \Psi(T_-), \quad \text{for} \quad T_- \leq t \leq T_+ \mod(T); \tag{3.15b}
\]
if instead \( \eta \leq \varepsilon/2 \), hence \( T_- \leq 0 < T_+ \), then
\[
\tau(t) = t - \Psi(T_+) + \Psi(0), \quad \text{for} \quad T_+ \leq t \leq T_- + T \mod(T), \tag{3.16a}
\]
and
\[
\tau(t) = t - T - \Psi(t - T) + \Psi(0), \quad \text{for} \quad T_- + T \leq t \leq T_+ + T \mod(T). \tag{3.16b}
\]
In these expressions,
\[
\Psi(t) = K(\varepsilon) \int_0^t dt' \exp \left\{ - \left[ \cos^2 \left[ \pi \left( \frac{t'}{T} - \eta \right) \right] - \cos^2 \left( \frac{\pi \varepsilon}{2} \right) \right]^{-2} \right\}, \tag{3.17a}
\]
implying, via equations (3.12c) and (3.13c),
\[
\Psi(T_+) - \Psi(T_-) = T. \tag{3.17b}
\]
Note that the last formula entails that, in both cases, these explicit expressions of \( \tau(t) \) imply that this function is periodic in \( t \) with period \( T \), see equation (1.6a), and that it is continuous (indeed, infinitely differentiable) in spite of it being given by different formulae in different intervals of the variable \( t \). Moreover, these explicit expressions of \( \tau(t) \) demonstrate that \( \tau(t) \) coincides with \( t \) (up to an irrelevant constant shift, see equations (3.15a) and (3.16a)) in a fraction of length \((1 - \varepsilon) T = T_+ - T_- \) of its period \( T \) (which in the first case, \( \eta \geq \varepsilon/2 \), includes the
origin \( t = 0 \); in the second, it does not—unless \( \eta = \varepsilon/2 \) implying \( T_\eta = 0 \), in which case, the two expressions coincide). As for the remaining part of its period, of length \( \varepsilon T \), the qualitative behaviour of \( \tau(t) \) as a function of \( t \) is easily described: at both ends of this sector it grows a bit (as implied by its continuity and by the fact that, in the complementary part of its period, its derivative with respect to \( t \) is just unity), while in the middle part of this sector, it decreases. The overall periodicity of \( \tau(t) \) (with period \( T \)) requires that its overall decrease in the sector of length \( \varepsilon T \) compensate exactly its increase in the complementary sector of length \((1 - \varepsilon)T\), which itself amounts to \((1 - \varepsilon)T\) (since in this sector, \( \tau(t) \) coincides with \( t \) up to an additive constant). Hence, the decrease in the middle part of the interval of length \( \varepsilon T \) shall be quite sharp if \( \varepsilon \) is very small so that this interval is very short.

The general solution of this extended model is provided by equation (3.11b)—which is periodic with period \( T \), although it is of course also singular—and by the formula (1.5), with \( X(\tau) \) being the general solution of the original model (1.1), and \( \tau(t) \) being given by the explicit expressions reported above, see equations (3.15) and (3.16) with equations (3.17) and (3.12c).

A more general variant of this extended model is obtained by replacing the dependent variable \( \xi \) with

\[
y = \xi G(x),
\]

where \( G(x) \) is an arbitrary function (preferably positive definite, see equation (3.20) below). This more general model is, of course, also isochronous with period \( T \), and its general solution is again characterized by the formula (1.5), with \( X(\tau) \) being the general solution of the original model (1.1), and \( \tau(t) \) being given by the explicit expressions reported above, see equations (3.15) and (3.16) with equations (3.17) and (3.12c), where the value of the number \( \eta \) is now determined by the initial value of the new dependent variable \( y \) whose time evolution is now given by the formula

\[
y(t) = \tan \left[ \pi \left( \frac{t}{T} - \eta \right) \right] G[x(t)].
\]

The equations of motion of the \( N \) dependent variables \( x_n \) are again given by equation (3.1) with equation (3.12) (where, however, \( \xi \) must be replaced by \( y \) via equation (3.18)), while the ODE determining the time evolution of \( y \) now reads as follows:

\[
y = \left[ G(x) \right]^{-1} \left\{ \frac{\omega}{2} [y^2 + G^2(x)] + y \phi \sum_{n=1}^{N} \frac{\partial G(x)}{\partial x_n} F_n(x) \right\},
\]

where \( \phi \) is again given by equation (3.12) with \( \xi \) replaced by \( y \) via equation (3.18).

4. How to modify a Hamiltonian system so that it becomes isochronous

In this section, we show how to modify a dynamical system so that the new dynamical system thereby obtained is isochronous. The basic idea is the same as that described above, but we now use the term modify rather than extend because the new system shall have the same dynamical variables as the original system (in particular, no additional variables are introduced). The treatment is
always restricted to autonomous systems. And we focus on Hamiltonian systems because, in this context, a neat formulation can be provided of the restriction on the original model that is sufficient for the applicability of our approach (see the assumption detailed below). Moreover, Hamiltonian systems are very important in physics, see, for instance, the example described below; it is therefore interesting to introduce in the Hamiltonian context a modification technique yielding a new system that is isochronous and retains its Hamiltonian character (while the techniques described above need not guarantee the second property).

Let us recall that a Hamiltonian system features $2N$ dependent variables, generally denoted as $N$ canonical variables $q_n$ and $N$ canonical momenta $p_n$. This dynamical system is characterized by the following $2N$ equations of motion:

$$
\dot{q}_n = \frac{\partial H(p, q)}{\partial p_n} \quad \text{and} \quad \dot{p}_n = -\frac{\partial H(p, q)}{\partial q_n},
$$

(4.1)

where $H(p, q)$ is the (assumedly given) Hamiltonian function. It is then plain that the time derivative of any function $F(p, q)$ of the dependent variables is provided by the formula

$$
\dot{F}(p(t), q(t)) = [F(p, q), H(p, q)],
$$

(4.2a)

where the Poisson bracket on the right-hand side of this formula is defined as follows:

$$
[F(p, q), G(p, q)] = \sum_{n=1}^{N} \left[ \frac{\partial F(p, q)}{\partial q_n} \frac{\partial G(p, q)}{\partial p_n} - \frac{\partial F(p, q)}{\partial p_n} \frac{\partial G(p, q)}{\partial q_n} \right].
$$

(4.2b)

This of course implies that the Hamiltonian $H(p, q)$ itself is a constant of motion,

$$
H(p(t), q(t)) = H(p(0), q(0)).
$$

(4.3)

And since the addition of a constant to the Hamiltonian function does not change the dynamics, we can if need be (see below) assume, without significant loss of generality, that the initial value of the Hamiltonian vanishes,

$$
H(p(0), q(0)) = 0.
$$

(4.4)

But note that this requires that one fixes, for each choice of initial data, a specific value of the constant to be added to the Hamiltonian in order that the above formula holds. And while the value of this constant has no effect on the dynamics produced by the Hamiltonian $H(p, q)$, it generally affects the dynamics yielded by the modified Hamiltonians described below.

**Assumption 4.1.** We now assume that there exists a function $\Theta(p, q)$ of the dependent variables such that

$$
[\Theta(p, q), H(p, q)] = 1,
$$

(4.5a)

which implies, see equation (4.2), that the time evolution of this quantity under the dynamics induced by the Hamiltonian $H(p, q)$ is trivially simple,

$$
\Theta(p(t), q(t)) = \Theta(p(0), q(0)) + t.
$$

(4.5b)
As detailed by the following two theorems, and moreover shown by the physically important example displayed below, this assumption allows the construction of modified Hamiltonians \( \tilde{H}(\tilde{p}, \tilde{q}; \omega) \) that are isochronous with period \( T \) (see equation (2.3)), yielding dynamics such that

\[
\tilde{p}(t \pm T) = \tilde{p}(t) \quad \text{and} \quad \tilde{q}(t \pm T) = \tilde{q}(t). \tag{4.5c}
\]

Note that the tilde has been superimposed over the dynamical variables merely to emphasize that these variables are now assumed to evolve according to the dynamics determined by the modified Hamiltonian \( \tilde{H}(\tilde{p}, \tilde{q}; \omega) \).

But before proceeding with the illustration of these findings, let us emphasize that the class of Hamiltonians \( H(p, q) \) allowing the introduction of a function \( \Theta(p, q) \) satisfying (4.5) is quite large; indeed, recent papers have described procedures to identify suitable functions \( \Theta(p, q) \), see [7–9]. It appears therefore justified to assert that ‘isochronous Hamiltonians are not rare’.

**Theorem 4.2.** The modified Hamiltonian

\[
\tilde{H}_1(\tilde{p}, \tilde{q}; \omega) = u(\tilde{p}, \tilde{q})H(\tilde{p}, \tilde{q}), \tag{4.6a}
\]

where

\[
u(\tilde{p}, \tilde{q}) = 1 + i\omega\Theta(\tilde{p}, \tilde{q}), \tag{4.6b}\]

is isochronous.

Proof (see ch. 5 of [2]). The first observation is that, via equations (4.2) and (4.5a), one notes that this Hamiltonian, \( \tilde{H}_1(\tilde{p}, \tilde{q}; \omega) \), causes the two quantities \( H(\tilde{p}, \tilde{q}) \) and \( u(\tilde{p}, \tilde{q}) \) to evolve in time as follows:

\[
\dot{H}(\tilde{p}(t), \tilde{q}(t)) = -i\omega H(\tilde{p}(t), \tilde{q}(t)) \tag{4.7a}
\]

and

\[
\dot{u}(\tilde{p}(t), \tilde{q}(t)) = i\omega u(\tilde{p}(t), \tilde{q}(t)), \tag{4.7b}
\]

so that

\[
H(\tilde{p}(t), \tilde{q}(t)) = H(\tilde{p}(0), \tilde{q}(0)) \exp(-i\omega t) \tag{4.8a}
\]

and

\[
u(\tilde{p}(t), \tilde{q}(t)) = u(\tilde{p}(0), \tilde{q}(0)) \exp(i\omega t). \tag{4.8b}
\]

Next, one notes that, by restricting attention to Hamiltonians (or, equivalently, to initial data) such that \( H(\tilde{p}(0), \tilde{q}(0)) \) vanishes (see equation (4.4)), the first of these two formulae implies

\[
H(\tilde{p}(t), \tilde{q}(t)) = 0. \tag{4.9}
\]
The next step is to investigate the evolution of the dependent variables \( \tilde{q}_n \) and \( \tilde{p}_n \) yielded by the Hamiltonian \( \tilde{H}_1(\tilde{p}, \tilde{q}; \omega) \), see equation (4.6). Via equation (4.2) and (4.5a), and taking advantage of the restriction (4.9) and of the explicit time evolution of \( u(\tilde{p}(t), \tilde{q}(t)) \), see equation (4.8b), one gets

\[
\dot{\tilde{q}}_n = u(\tilde{p}(0), \tilde{q}(0)) \exp(i\omega t) \frac{\partial H(\tilde{p}(t), \tilde{q}(t))}{\partial \tilde{p}_n} \tag{4.10a}
\]

and

\[
\dot{\tilde{p}}_n = -u(\tilde{p}(0), \tilde{q}(0)) \exp(i\omega t) \frac{\partial H(\tilde{p}(t), \tilde{q}(t))}{\partial \tilde{q}_n}. \tag{4.10b}
\]

And finally, a comparison of these equations of motion with the original equations of motion (4.1) implies that

\[
\tilde{q}(t) = q[\tau(t)] \quad \text{and} \quad \tilde{p}(t) = p[\tau(t)], \tag{4.11}
\]

with

\[
\tau(t) = \frac{u(\tilde{p}(0), \tilde{q}(0))[\exp(i\omega t) - 1]}{i\omega}, \tag{4.12}
\]

entailing, in addition to the coincidence of the initial conditions for the original and modified models,

\[
\tilde{q}(0) = q(0) \quad \text{and} \quad \tilde{p}(0) = p(0), \tag{4.13}
\]

that the solutions yielded by the modified Hamiltonian \( \tilde{H}_1(\tilde{p}, \tilde{q}; \omega) \) are completely periodic with period \( T \), see equation (2.3),

\[
\tilde{q}(t \pm T) = \tilde{q}(t) \quad \text{and} \quad \tilde{p}(t \pm T) = \tilde{p}(t). \tag{4.14}
\]

This conclusion holds provided the initial conditions yield solutions \( q(t), p(t) \) of the original Hamiltonian \( H(p, q) \) (assumedly analytical), which are holomorphic functions of the complex variable \( \tau \) in an adequately large circular disc centred at the origin of the complex \( \tau \)-plane (see the analogous discussion in the preceding §2, after equation (2.5)). This concludes the proof of theorem 4.2. ■

Note however that, if the original Hamiltonian \( H(p, q) \) is real, the modified Hamiltonian \( \tilde{H}_1(\tilde{p}, \tilde{q}; \omega) \), see equation (4.6), is certainly complex, hence the dependent variables \( \tilde{q}_n, \tilde{p}_n \) are themselves complex. Of course (as discussed in the preceding §2), real dependent variables can be recovered by considering as new dependent variables, the real and imaginary parts of \( \tilde{q}_n \) and \( \tilde{p}_n \); but this causes a doubling of the dependent variables, so from this point of view, one might argue that the time evolution associated with the new Hamiltonian \( \tilde{H}_1(\tilde{p}, \tilde{q}; \omega) \) amounts rather to an extension than a modification of the original evolution associated with the original Hamiltonian \( H(p, q) \).

The approach based on the following theorem 4.3 does not suffer from this defect, indeed in this case, the entire treatment is restricted to real variables.

**Theorem 4.3.** The modified Hamiltonian

\[
\tilde{H}_2(\tilde{p}, \tilde{q}; \omega) = \frac{1}{2} \{[H(\tilde{p}, \tilde{q})]^2 + \omega^2[\Theta(\tilde{p}, \tilde{q})]^2\} \tag{4.15}
\]

is isochronous.

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The proof of this theorem 4.3 is analogous but not quite identical to the proof of theorem 4.2, see [10]. Note, however, that it cannot be excluded that the evolution yielded by this modified Hamiltonian is singular; in such a case, the isochrony holds in the same sense as the statement that the time evolution described by the function \( \tan(\omega t) \) is periodic (but singular).

Let us end this section by exhibiting how a specific Hamiltonian of great physical interest can be modified so that its modified version is isochronous. Our starting point is the most general (classical and non-relativistic) many-body problem, with the only—very reasonable—restriction that it be, overall, translation invariant. This system is characterized by the Hamiltonian

\[
H(p, q) = \frac{1}{2} \sum_{n=1}^{N} p_n^2 + V(q) \tag{4.16a}
\]

or, equivalently,

\[
H(p, q) = \frac{P^2}{2N} + h(y, x), \tag{4.16b}
\]

with the potential \( V(q) \) quite arbitrary, except for the (physically very reasonable) restriction

\[
V(q + q_0) = V(q). \tag{4.16c}
\]

Note that, in equation (4.16c), \( q_0 \) is an (arbitrary) scalar, not an \( N \)-vector like \( q \) (so the same translation is made on each component of \( q \)). Moreover, in equation (4.16b) and below, we denote by \( P \) and \( Q \) the total momentum and the centre-of-mass coordinate, respectively,

\[
P = \sum_{n=1}^{N} p_n \quad \text{and} \quad Q = \frac{1}{N} \sum_{n=1}^{N} q_n, \tag{4.16d}
\]

and by \( y_n \) and \( x_n \) the momenta and coordinates of the particles in the centre-of-mass system,

\[
y_n = p_n - \frac{P}{N} \quad \text{and} \quad x_n = q_n - Q, \tag{4.16e}
\]

and

\[
h(y, x) = \frac{1}{2} \sum_{n=1}^{N} y_n^2 + V(x) = \frac{1}{4N} \sum_{n,m=1}^{N} (p_n - p_m)^2 + V(q) \tag{4.16f}
\]

is essentially the Hamiltonian characterizing the motion in the centre-of-mass system.

The notation used here refers to one-dimensional space, and we, moreover, assumed all the particles to have the same mass (set to unity). Both these restrictions are merely for notational simplicity; the extension to particles with different masses and to spaces with arbitrary dimensions (including the ‘realistic’ case of three-dimensional ambient space) is trivial. And let us emphasize that the number of particles \( N \) is arbitrary, as well as the potential \( V(q) \) (except for the restriction (4.16c))—although it is generally preferable to assume, for simplicity, that the forces do guarantee that the solution yielded by the Hamiltonian \( \tilde{H} \)
exist for all time, as would, for instance, be the case for Lennard-Jones two-body forces, possibly also with Coulomb or gravitational forces, and perhaps also with an arbitrary confining potential (possibly multiplied by an exceedingly small coupling constant). It is, moreover, generally instructive to keep in mind that three values of \( N \) are particularly interesting: \( N = 2 \) (when the problem can be solved exactly; for an explicit example, see §5.5.3 of [2]); \( N \approx \exp(30) \) (corresponding to a macroscopic body: for this kind of values, the Hamiltonian (4.16) subtends much of macroscopic physics, including standard statistical mechanics and thermodynamics); and \( N \approx \exp(85) \) (a classical—non-relativistic non-quantal—model of the Universe!).

We then introduce (following, up to minor modifications, the approach suggested by theorem 4.3) the following modified Hamiltonian:

\[
\tilde{H}(p, q; \omega) = \frac{1}{2} \left[ (P + h(y, x))^2 + \omega^2 Q^2 \right].
\]

It can then be shown (see, for instance, [2,10]) that the dynamics yielded by this Hamiltonian has the following two properties: (i) it is isochronous with period \( T \), see equation (2.3) and (ii) starting from generic initial data, the time evolution yielded by this modified Hamiltonian \( \tilde{H} \) coincides, over time intervals much smaller than \( T \), with that yielded by the original, ‘physical’ Hamiltonian \( H \)—up to a rescaling of time corresponding to its multiplication by a constant parameter, and to corrections of order \( t/T \). The relevant mechanism is quite analogous to that discussed in §3a, except that now the modified model contains no additional variables; the role played by the two additional variables \( \xi_1 \) and \( \xi_2 \) in §3a is instead played here by the centre-of-mass variables \( P \) and \( Q \).

5. Asymptotically isochronous systems

Definition 5.1. A dynamical system is asymptotically isochronous (with period \( T \)) if its general solution \( \mathbf{x}(t) \) approaches asymptotically some function \( \mathbf{x}_{\text{per}}(t) \) itself completely periodic with period \( T \),

\[
\lim_{t \to \infty} ||\mathbf{x}(t) - \mathbf{x}_{\text{per}}(t)|| = 0 \quad (5.1a)
\]

and

\[
\mathbf{x}_{\text{per}}(t \pm T) = \mathbf{x}_{\text{per}}(t) \quad (5.1b)
\]

The symbol \( ||\mathbf{x}|| \) indicates some norm of the \( N \)-vector \( \mathbf{x} \). Of course, the function \( \mathbf{x}_{\text{per}}(t) \) is not uniquely defined.

This definition of asymptotically isochronous dynamical system was introduced in the context treated herein in [11]. Note that this notion does not entail the existence of a limit cycle, which refers, by definition, to a periodic solution of a dynamical system whose trajectory is isolated; this need not be the case of asymptotically isochronous dynamical systems as defined above.

Let us now recall that—as indicated in §1—the main idea reviewed in this paper goes essentially as follows. Given a dynamical system characterized by a general solution \( \mathbf{X}(\tau) \)—where \( \tau \) is the independent variable and the \( N \)-vector \( \mathbf{X} \) is the dependent variable, and let us again assume, for simplicity, that this general solution exists for all (real) values of the independent variable \( \tau \)—introduce a new
(modified or extended) dynamical system whose general solution features an $N$-vector dependent variable $\mathbf{x}(t)—$where $t$ (time) is the new independent variable—

which is related to the general solution $\mathbf{X}(\tau)$ by the formula (see equation (1.5))

$$\mathbf{x}(t) = \mathbf{X}[\tau(t)].$$

Then, if the function $\tau(t)$ is periodic with a period $T$,

$$\tau(t \pm T) = \tau(t),$$

see equation (1.6a), this property is inherited by the general solution $\mathbf{x}(t)$,

$$\mathbf{x}(t \pm T) = \mathbf{x}(t),$$

see equation (1.6b), thus entailing that the new dynamical system is isochronous.

Likewise, if the function $\tau(t)$ is asymptotically isochronous, namely there exists some (obviously not unique) function $\tau_{\text{per}}(t)$ that is periodic with a period $T$,

$$\tau_{\text{per}}(t \pm T) = \tau_{\text{per}}(t),$$

and $\tau(t)$ approaches it asymptotically,

$$\lim_{t \to \infty} |\tau(t) - \tau_{\text{per}}(t)| = 0,$$

then this property is inherited by the general solution $\mathbf{x}(t)$, see equation (5.1) with

$$\mathbf{x}_{\text{per}}(t) = \mathbf{X}[\tau_{\text{per}}(t)],$$

in as much as (5.2b) generally entails

$$\lim_{t \to \infty} ||\mathbf{X}[\tau(t)] - \mathbf{X}[\tau_{\text{per}}(t)]|| = 0.$$

Hence, in this case, the new dynamical system is asymptotically isochronous.

It is then clear how the treatment of §3a can be modified to yield asymptotically isochronous systems rather than isochronous systems. It is sufficient to extend the original dynamical system by introducing three (rather than two) additional dependent variables $\xi_1, \xi_2, \xi_3$ satisfying the rather trivial system of three ODEs

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = \xi_3 \quad \text{and} \quad \dot{\xi}_3 = -p\omega^2 \xi_1 - \omega^2 \xi_2 - p\xi_3,$$

where $\omega$ and $p$ are positive constants, whose general solution reads

$$\xi_1(t) = A \frac{\sin(\omega t)}{\omega} + B \cos(\omega t) + C \exp(-pt),$$

$$\xi_2(t) = A \cos(\omega t) - \omega B \sin(\omega t) - pC \exp(-pt)$$

and

$$\xi_3(t) = -\omega A \sin(\omega t) - \omega^2 B \cos(\omega t) + p^2 C \exp(-pt),$$

where $A, B, C$ are arbitrary constants. Then, provided $\omega$ and $p$ are positive, by setting

$$\phi = \sum_{j=1}^{3} c_j \xi_j,$$

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with three arbitrary constants \( c_j \), and proceeding as in §3a, one extends any dynamical system to a new system that is then asymptotically isochronous. The diligent reader will have no difficulty to fill in the details of this procedure, and may have some fun in manufacturing explicit models after having obtained the analogue of equation (3.7).

6. Diophantine findings

In this section, we outline tersely a simple technique allowing the derivation of Diophantine findings from isochronous dynamical systems.

Assume that the dynamical system characterized by the ODEs

\[ \dot{x}_n = 2\pi i f_n(x) \]  

is isochronous with period \( 2\pi \), and that, inside the isochrony region, it features an equilibrium configuration,

\[ x_n(t) = \bar{x}_n \quad \text{and} \quad f_n(\bar{x}) = 0. \]

Then, clearly, the fact that the oscillatory behaviour of this system in the immediate neighbourhood of this equilibrium configuration is isochronous with period \( 2\pi \) implies that the \( N \) eigenvalues of the \( N \times N \) matrix

\[ f_{nm} = \left. \frac{\partial f_n(x)}{\partial x_m} \right|_{x=\bar{x}} \]

are all (generally different) integer numbers: a Diophantine result!

For a review of Diophantine results and conjectures (as well as other developments, such as the identification of named polynomials having Diophantine zeros) arrived at in this manner see appendix C of [2], the references therein, and the more recent papers [12–17].

7. Outlook

Above (see in particular §3), we have shown how, given any (autonomous) dynamical system, it is possible to manufacture another (also autonomous) dynamical system that has the property to be isochronous with an arbitrarily assigned period \( T \), yet whose dynamics mimics closely that of the original system over time spans short with respect to \( T \), or even coincides exactly with the original dynamics over an arbitrarily large fraction (of course, less than unity) of \( T \). The main mechanism to manufacture such systems is by causing the general solution of the new model, \( \bar{x}(t) \), to be related to the general solution of the original model, \( X(\tau) \), as follows (see equation (1.5)):

\[ \bar{x}(t) = X[\tau(t)], \]  

with \( \tau(t) \) periodic in \( t \) with period \( T \), see equation (1.6a). Let us now suppose that the dynamics of the original system is well defined for all (real) values of its dependent variable \( \tau \) and that its motions are confined but chaotic. (This may well
happen in important physical models, see, for instance, the remarks in the paragraph following (4.16f)). Then, the typical trajectory of the original system in its $N$-dimensional space (note that we must assume that $N > 2$) shall meander endlessly in an unpredictable way. The corresponding trajectory of the new system (characterized by the same initial data, see equation (1.4)) is then exactly superimposed over the trajectory of the original system, but only covers a finite piece of it; and the dynamics of the new system corresponds to travelling over that piece of the trajectory back and forth periodically, as dictated by the periodic function $\tau(t)$.

On the other hand, the new system, being isochronous, is not chaotic; it is integrable, indeed superintegrable, in fact, one might even say more than superintegrable. (A superintegrable system is identified by the property to feature the maximal number of (functionally independent) constants of motion compatible with its dynamics not being frozen, i.e. in our context, $N - 1$. The fact that isochronous systems are superintegrable is presumably well known; for a formal proof, see the appendix of [18]. The fact that all confined solutions of superintegrable systems are completely periodic is obvious, since the $N - 1$ constants of motion imply that the evolutions of all dependent variables are slave to the evolution of a single one of them; and the confined evolution of a system with a single degree of freedom is necessarily periodic. However, the period may depend on the initial conditions, hence superintegrable systems need not be isochronous).

We thus see that, given an arbitrary chaotic system, one can manufacture another dynamical system that is superintegrable but features a dynamics that mimics closely, or is even exactly identical, to the dynamics of the original model over a finite, but \textit{a priori} arbitrarily long, time interval. This finding seems to us to have a somewhat paradoxical connotation. Of course, it does not conflict with the mathematical correctness of the current definitions of chaotic versus integrable (or superintegrable) behaviour, which are related to the behaviour of a dynamical system over infinite time. But it does suggest that it would be desirable—if quite non-trivial—to invent criteria providing some kind of measure of the degree of complexity of the behaviour of a dynamical system over a finite time scale. Such notions might then be applicable both to integrable and to chaotic systems.

This predicament calls to mind an analogy, which we proffer while being well aware that analogies do not make science. For a long time in mathematics, infinite series were only considered respectable if, in some sense, they could be guaranteed to converge. Eventually, ways were found—without, of course, decreasing the importance of the various notions of convergence—to make also divergent series mathematically respectable, indeed quite useful: think, for instance, of asymptotic series and of the information they may provide on the behaviour of a function over different scales of its argument.

So, we consider that an implementation of the above suggestion is an interesting task for future scholars.

Although I am solely responsible for the contents of this paper, I wish to emphasize that essentially all the findings reported herein have been obtained in collaboration with François Leyvraz, whom I also wish to thank for a critical reading of a first draft of this text entailing significant improvements.
References


