Nonlinear theory of slow light

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In the framework of the nonlinear $\Lambda$ model, propagation of solitons was analysed in atomic vapours and Bose–Einstein condensates. The complicated nonlinear interplay between fast and slow-light solitons in a $\Lambda$-type medium was shown to facilitate control of its optical transparency and formation of optical gates. An exact analytical description was given for the deceleration, stopping and revival of slow-light solitons in the experimentally relevant non-adiabatic regime. A stopping slow-light soliton imprints a localized immobile polarization pattern in the medium, which, as explicitly demonstrated here, can be used as a bit of readable optical memory. The whole process can be controlled with the background field and an auxiliary laser field. The latter regulates the signal velocity, while the slow-light soliton can be stopped by switching off the former. The location and shape of the imprinted memory bit were also determined. With few assumptions characteristic of slow light, the $\Lambda$ model was reduced to a simpler nonlinear model that also describes two-dimensional dilatonic gravity. Exact solutions could now be derived also in the presence of relaxation. Spontaneous decay of the upper atomic level was found to be strongly suppressed, and the spatial form of the decelerating slow-light soliton was preserved, even if the optical relaxation time was much shorter than the typical time scale of the soliton. The effective relaxation coefficient of the slow-light soliton was significantly smaller than that of an arbitrary optical pulse. Such features are obviously of great importance when this kind of system is applied, in practice, to information processing. A number of experimentally observable properties of the solutions reported were found to be in good agreement with recent experimental results, and a few suggestions are also made for future experiments.

Keywords: Bose–Einstein condensation; optical soliton; slow light

1. Introduction

In this article, we review some of the recent developments in the application of the inverse-scattering (IS) method to coherent pulse propagation through an active optical medium. Even though the appearance of soliton type of solutions

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is common in many branches of science, relevant applications of the IS method in quantum optics were reported very early, and in publications of R. K. Bullough with co-authors in particular, for example [1–4]. Recent progress in experimental techniques in a coherent control of light–matter interaction has revitalized this research topic, as it has opened up many new opportunities for interesting applications of practical importance. Experiments have been carried out on various types of material, such as cold sodium atoms [5,6], rubidium vapours [7–10], solids [11,12] and photonic crystals [13]. These experiments were based on good control over the absorption properties of the medium, and analysed effects of slow and superluminal light. Such a control could be realized via electromagnetically induced transparency (EIT), coherent population oscillations or other induced transparency techniques. Each material has specific advantages in the practical realization of a particular effect. For instance, cold atoms have a negligible Doppler broadening and small collision rate, which increase the ground-state coherence time. The experiments on rubidium vapours were carried out at room temperature, and application of complicated cooling methods were thus not required. Solids are obviously strong candidates for the realization of long-lived optical memory. Photonic crystals provide a broad range of ways to manipulate slow light. Interest in the physics of light propagation in atomic vapours and Bose–Einstein condensates (BEC) was strongly motivated by success in the storage and retrieval of optical information in these media [5–8,14,15].

Even though linear methods for analysing these phenomena, based on the theory of EIT [16], have been very effectively developed [17], interpretation of the recent experiments require a better, i.e. nonlinear, description of them [15]. Linear EIT theory assumes the probe field to be much weaker than the controlling field. To allow significant changes in the initial atomic state due to interaction with the optical pulse, one must, however, go beyond linear theory. In the adiabatic regime, when the fields change very slowly in time, approximate analytical solutions [18,19] and self-consistent solutions [20] have been found and applied to storage and retrieval processes [21]. Different EIT and self-induced transparency (SIT) solitons have been classified and numerically studied in the nonlinear regime for their stability [22]. As demonstrated by Dutton et al. [23], strong nonlinearity can result in interesting new phenomena. Recent experiments and numerical studies [15,24] have shown that the adiabatic condition can be relaxed, which allows for much more efficient control over the storage and retrieval of optical information.

In this article, we study the interaction of light with a gaseous active medium whose relevant energy levels are well approximated by the Λ model. This model well describes energy levels of, for example, a gas of sodium atoms. The level structure of the model is given in figure 1, where two hyperfine sub-levels of the sodium state $^3S_{1/2}$, with $F = 1$ and $F = 2$, can be identified as the $|1\rangle$ and $|2\rangle$ states, correspondingly [5]. The excited state $|3\rangle$ corresponds to the hyperfine sub-level $^3P_{3/2}$, with $F = 2$. We consider the case when the atoms are cooled down to microKelvin temperature in order to suppress the Doppler shift and to increase the coherence time of the ground state. The coherence time of sodium atoms at a temperature of 0.9 μK is of the order of 0.9 ms [6]. In the experiments, the laser pulses typically have a duration of microseconds, which is much shorter than the coherence time and much longer than the optical relaxation time of 16.3 ns.
Figure 1. $\Lambda$ model for the relevant energy levels of sodium atoms. The parameters of the model are $\omega_{12}/(2\pi) = 1772$ MHz, $\omega/(2\pi) = 5.1 \times 10^{14}$ Hz ($\lambda = 589$ nm) and $\Delta$ is a varying detuning from the resonance.

We consider here a situation in which a gas cell is illuminated by two circularly polarized optical beams co-propagating in the z-direction. One beam, denoted as channel $a$, is a $\sigma^-$-polarized field, and the other, denoted as $b$, is a $\sigma^+$-polarized field. In the slowly varying amplitude and phase approximation (SVEPA), these fields can be expressed in the form

$$E = e_a E_a e^{i(k_a z - \omega_a t)} + e_b E_b e^{i(k_b z - \omega_b t)} + \text{c.c.} \quad (1.1)$$

Here, $k_{a,b}$ are wavenumbers, while vectors $e_a, e_b$ describe the polarization of the fields. It is convenient to introduce two corresponding Rabi frequencies, such that

$$\Omega_a = \frac{2 \mu_a E_a}{\hbar} \quad \text{and} \quad \Omega_b = \frac{2 \mu_b E_b}{\hbar}, \quad (1.2)$$

where $\mu_a$ and $\mu_b$ are dipole moments of transitions in the respective channel ($a$ and $b$).

In the interaction picture and within the SVEPA, the Hamiltonian $H_\Lambda = H_0 + H_I$, which describes the interaction of a three-level atom with the fields, can be defined such that

$$H_0 = -\frac{\Delta}{2} D \quad \text{and} \quad H_I = -\frac{1}{2} \{(\Omega_a |3\rangle \langle 1| + \Omega_b |3\rangle \langle 2|) + \text{h.c.}\}, \quad (1.3)$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

Here, $\Delta$ is a varying detuning from the resonance, and we have set $\hbar = 1$. 

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The dynamics of the fields are described by the Maxwell equations

\[(\partial_t^2 - c^2 \partial_z^2)\Omega_a e^{i(k_a z - \omega_a t)} = -\frac{2\nu_a}{\omega_a} \partial_t^2 (\rho_{31} e^{i(k_a z - \omega_a t)})\]

and

\[(\partial_t^2 - c^2 \partial_z^2)\Omega_b e^{i(k_b z - \omega_b t)} = -\frac{2\nu_b}{\omega_b} \partial_t^2 (\rho_{32} e^{i(k_b z - \omega_b t)}),\]

where \(\nu_a = (n_A |\mu_a|^2 \omega_a)/\epsilon_0, \nu_b = (n_A |\mu_b|^2 \omega_b)/\epsilon_0, n_A\) is the density of atoms and \(\epsilon_0\) is the vacuum susceptibility. Here, \(\rho\) is the density matrix in the interaction representation. In typical experimental situations, the coupling constants \(\nu_a, \nu_b\) are almost the same. Therefore, we assume that \(\nu_a = \nu_b = \nu_0\). Hence, within the SVEPA, the wave equations are reduced to first-order partial differential equations (PDEs)

\[\partial_\zeta \Omega_a = i\nu_0 \rho_{31} \text{ and } \partial_\zeta \Omega_b = i\nu_0 \rho_{32}. \quad (1.4)\]

Equations (1.4) can be expressed in matrix form such that

\[\partial_\zeta H_I = \frac{1}{4} [D, \rho]. \quad (1.5)\]

In terms of these variables, the Liouville equation takes the form

\[\partial_\tau \rho = i \left[ \frac{\Delta}{2} D - H_I, \rho \right]. \quad (1.6)\]

Here, \(\zeta = z/c\) and \(\tau = t - z/c\). So as to work with dimensionless variables, we measure the time in units of the optical pulse length \(t_p = 1\ \mu s\), typical of experiments on slow-light phenomena [8]. We also divide the spatial coordinates by the length of the pulse slowed down in the medium, i.e. \(l_p = \nu_0 t_p \approx c (\Omega_0^2/2\nu_0) t_p\). Here, \(\Omega_0\) is a typical magnitude of the controlling field in an EIT experiment. We assume this field to be of the order of a few megahertz. In the following, we use \(\Omega_0 = 3\) as a representative value. This corresponds to a group velocity of several metres per second, depending on the density of the atoms. We also assume that the group velocity is \(10^{-7}\)c, so the spatial length of the pulse is 30\(\mu m\) and \(\zeta\) is normalized such that it is \(10^{-13}\)s. Then, with the chosen dimensionless variables, the coupling constant is \(\nu_0 = \Omega_0^2/2 = 4.5\). The retarded time \(\tau\) is measured in microseconds and the Rabi frequencies are normalized to megahertz.

The system of equations (1.5) and (1.6) is exactly solvable in the framework of the IS method [25–28]. This system of equations thus constitutes a compatibility condition for a certain linear system, namely

\[\partial_\tau \Phi = U(\lambda) \Phi = \frac{i}{2} \lambda D \Phi - i H_I \Phi \quad (1.7)\]

and

\[\partial_\zeta \Phi = V(\lambda) \Phi = \frac{i}{2} \frac{\nu_0 \rho}{\lambda - \Delta} \Phi. \quad (1.8)\]
Here, \( \lambda \) (in \( \mathbb{C} \)) is the so-called spectral parameter. Comparison of \( \Phi_{\tau} \) against \( \Phi_{\xi} \) leads to the zero-curvature condition \([25]\) \[
U_{\tau}(\lambda) - V_{\tau}(\lambda) + [U(\lambda), V(\lambda)] = 0,
\]
which holds identically with respect to the linearly independent terms in \( \lambda \). It is straightforward to check that the resulting conditions coincide with the nonlinear equations (1.5) and (1.6).

At this point, it is good to discuss the initial and boundary conditions that underlie the physical problem considered. We consider a semi-infinite \( (\zeta \geq 0) \) active medium with a pulse of light incident at point \( \zeta = 0 \) (the initial condition). This means that the subsequent evolution of the system is considered with respect to the space variable \( \zeta \), while the boundary conditions should be specified with respect to variable \( \tau \). In our case, we use, as the boundary conditions, the asymptotic values of the density matrix at \( \tau \to \pm \infty \). To solve the nonlinear dynamics as described by equations (1.5) and (1.6), the IS method considers the scattering problem for the linear system (1.7), while the auxiliary linear system (1.8) describes the evolution of the scattering data. The purpose of this work is to study, in particular, the essentially nonlinear interplay of the fields in the two channels. Therefore, we consider, for equation (1.7), the scattering problem of finite-density type (cf. [25] and references therein), i.e. \( \Omega_{a,b} \to \Omega^\pm_{a,b} \) as \( \tau \to \pm \infty \). There are also many other results one can derive for the \( \Lambda \) model by the IS method, for example Hioe & Grobe [29], Grobe et al. [18], Park & Shin [26] and Byrne et al. [27]. In this work, we choose to use an algebraic version of this method, i.e. the Darboux–Bäcklund (DB) transformations. The DB method does not require a full investigation of the initial-value problem, as a soliton solution of the problem can be directly implemented on a chosen background field. The resulting solution is, of course, consistent with the underlying initial-value problem. We use DB transformations in the spirit of Rybin et al. [30], Rybin [31], Rybin & Timonen [32] and Matveev & Salle [33], up to certain modifications.

In the particular case considered here, the system is assumed to be initially in a stationary state described by the following background solution:

\[
\Omega_a = 0, \quad \Omega_b = \Omega(\tau) \quad \text{and} \quad \rho = |\psi_{at}\rangle \langle \psi_{at}| = |1\rangle \langle 1|.
\]

Notice that the state \( |1\rangle \) is a ‘dark’ state for the controlling field \( \Omega(\tau) \). This means that the atoms do not interact with the field \( \Omega(\tau) \) created by the auxiliary laser. The configuration (1.9) above corresponds to a typical experimental set-up (e.g. [5,6,8]). The function \( \Omega(\tau) \) models the controlling field that governs the dynamics of the system. The time dependence of this function can result, for example, from modulation of the intensity of the auxiliary laser. In general, \( \Omega(\tau) \) can also depend on the spatial variable \( \zeta \). However, we do not specify such dependence explicitly in the formalism below, except for a simple case of linear phase shift discussed in §3.

The article is organized such that in §2, we describe the DB transformation for the \( \Lambda \) model. In §3, we then describe the mechanism of a transparency gate for the slow-light soliton. In §4, we discuss an exactly solvable example of manipulation of slow-light solitons, while §5 considers a similar problem for the case of a fairly arbitrary controlling field. In §6, we discuss the connection of slow-light solitons with dilatonic gravity, while §7 discusses slow-light solitons when the effects of relaxation are taken into account. Section 8 comprises conclusions and discussion.
2. Darboux–Bäcklund transformation for the \( \Lambda \) model

In order to describe the DB transformation for the \( \Lambda \) model, we first reformulate the linear system (equations (1.7) and (1.8)) in matrix form, viz.

\[
\partial_t \Psi = \frac{i}{2} D\Psi L - iH_I \Psi \quad (2.1)
\]

and

\[
\partial_z \Psi = \frac{i\nu_0}{2} \rho \Psi \mathcal{P}. \quad (2.2)
\]

Here, \( \Psi \) is a matrix composed of three linearly independent solutions of the linear system (equations (1.7) and (1.8)) corresponding to three (not necessarily different) values of the spectral parameter \( \lambda \), i.e. \( \lambda', \lambda'' \) and \( \lambda''' \). The matrix spectral parameter \( L \) is defined as

\[
L = \begin{pmatrix}
\lambda' & 0 & 0 \\
0 & \lambda'' & 0 \\
0 & 0 & \lambda'''
\end{pmatrix},
\]

while \( \mathcal{P}^{-1} = L - \Delta \cdot I \).

The \( N \)-fold (\( N \geq 1 \)) DB transformation can be formulated such that

\[
\Psi[N] = \sum_{n=0}^{N} (-1)^{n+1} \Xi_{N-n}(\Delta) \Psi \mathcal{P}^{-n}, \quad \Xi_0 = I. \quad (2.3)
\]

It is evident that the linear system (equations (2.1) and (2.2)) is covariant with respect to this transformation provided that, for \( 0 \leq n \leq N \), the following DB ‘dressing’ transformations are satisfied:

\[
H_I[N] \Xi_{N-n}(\Delta) = \Xi_{N-n}(\Delta) H_I + i \partial_t \Xi_{N-n}(\Delta)
\]

\[
- \frac{1}{2} \left[ D, (\Xi_{N-n+1}(\Delta) - \Delta \Xi_{N-n}(\Delta)) \right] \quad (2.4)
\]

and

\[
\rho[N] \Xi_{N-n}(\Delta) = \Xi_{N-n}(\Delta) \rho + \frac{2i}{\nu_0} \partial_z \Xi_{N-n+1}(\Delta), \quad (2.5)
\]

together with the convention \( \Xi_{N+1}(\Delta) = \Xi_0(\Delta) = I \). The meaning of equations (2.4) and (2.5) is that they connect the ‘seed’ solutions \( H_I, \rho \) of the nonlinear system with the dressed \( (N\text{-soliton}) \) solutions \( H_I[N], \rho[N] \). So as to derive the matrices \( \{\Xi_k\}_{k=1}^{N} \), we specify a set of solutions \( \{\Psi_k\}_{k=1}^{N} \) corresponding to certain fixed values of the matrix spectral parameter \( L \), i.e. \( \{\mathcal{L}_k\}_{k=1}^{N} \), where

\[
\mathcal{L}_k = \begin{pmatrix}
\lambda_{k-1}^* & 0 & 0 \\
0 & \lambda_k^* & 0 \\
0 & 0 & \lambda_{k-1}^*
\end{pmatrix} \quad \text{and} \quad \mathcal{P}_k^{-1} = \mathcal{L}_k - \Delta \cdot I.
\]
We then demand
\[
\sum_{n=0}^{N} (-1)^{n+1} \Xi_{N-n}(\Delta) \Psi_k P_k^{-n} = 0, \quad k = 1, \ldots, N. \tag{2.6}
\]
This linear system allows the ‘dressing matrices’ \( \{ \Xi_k(\Delta) \}_{k=1}^N \) to be obtained via Cramer’s rule. It can be shown that the solutions of equation (2.6) satisfy the relations (2.4) and (2.5).

Since in what follows we only discuss the case \( N = 1 \) for convenience, we change now the notation in the following way:
\[
\tilde{H}_1 = H_1 - \frac{1}{2} [D, \Xi(0)], \quad \tilde{\rho} = \Xi(\Delta) \rho \Xi^{-1}(\Delta)
\]
and
\[
\tilde{\Psi} = \Psi \rho^{-1} - \Xi(\Delta) \Psi,
\]
while from the linear system (2.6), we obtain
\[
\Xi(\Delta) = \Psi_1 (L_1 - \Delta \cdot I) \Psi_1^{-1}. \tag{2.9}
\]
As explained above, matrix \( \Psi_1 \) is found by expressing \( \Psi \) at a specific value of the matrix spectral parameter,
\[
L_1 = \begin{pmatrix}
\lambda_0^* & 0 & 0 \\
0 & \lambda_0^* & 0 \\
0 & 0 & \lambda_0
\end{pmatrix}.
\]
We denote by \( \Phi_0 \) the fundamental matrix of solutions for the linear system equations (1.7) and (1.8) with \( \lambda = \lambda_0^* \). It can be shown that, for \( \lambda = \lambda_0^* \), the fundamental matrix is given by \( \Phi_0 = (\Phi_0^{-1})^\dagger \). Since the subspace of solutions related to \( \lambda_0^* \) is two dimensional, matrix \( \Psi_1 \) is constructed in the following way.

The vector \( \Psi_1^{(3)} = c_1 \Phi_0^{(1)} + c_2 \Phi_0^{(2)} + c_3 \Phi_0^{(3)} \) is the general solution to the linear problem with \( \lambda = \lambda_0 \). Here, the superscript in the brackets \( i = 1, 2, 3 \) denotes the column of the respective vector. In order to satisfy the structure of the operator \( \Xi \) of equation (2.8), we require that the scalar product (in a three-dimensional complex vector space) \( (\Psi_1^{(3)}, \Psi_1^{(1,2)}) = 0 \), and the vectors \( \Psi_1^{(1,2)} \) are related to \( \lambda = \lambda_0^* \). It is evident that \( (\Phi_0^{(i)}, \Phi_0^{(j)}) = \delta_{i,j} \).

We can easily find two appropriate orthogonal vectors \( \Psi_1^{(1,2)} \) such that
\[
\Psi_1^{(1)} = (c_2^* + c_3^*) \Phi_0^{(1)} - c_1^* (\Phi_0^{(2)} + \Phi_0^{(3)})
\]
and
\[
\Psi_1^{(2)} = c_3^* \Phi_0^{(2)} - c_2^* \Phi_0^{(3)}.
\]

An algorithm for finding new solutions to the nonlinear system (equations (1.5) and (1.6)) can fairly easily be formulated: find a solution \( \Phi_0 \) of the associated linear-system equations (1.7) and (1.8), corresponding to a certain seed solution of the nonlinear-system equations (1.5) and (1.6), construct then \( \Psi_1 \) and \( \Xi(\Delta) \), and finally use the dressing transformation (equation (2.7)). It is straightforward
to show that, for the state equation (1.9) of the atom–field system, a general solution of linear systems (1.7) and (1.8) can be expressed in the form

$$\Phi_0 = \begin{pmatrix} e^{(i/2)(\lambda \tau + (v_0 \xi / (\lambda - A))} & 0 \\ 0 & T(\tau, \lambda) \end{pmatrix},$$  

(2.10)
in which the $2 \times 2$ matrix $T(\tau, \lambda)$ is defined in terms of two complex functions $w(\tau, \lambda)$ and $z(\tau, \lambda)$, such that

$$T(\tau, \lambda) = (I + W(\tau, \lambda))e^{Z(\tau, \lambda)},$$

$$W(\tau, \lambda) = \begin{pmatrix} 0 & -w^*(\tau, \lambda) \\ w(\tau, \lambda) & 0 \end{pmatrix},$$

and

$$Z(\tau, \lambda) = \begin{pmatrix} \lambda/2 + z(\tau, \lambda) & 0 \\ 0 & -\lambda/2 + z^*(\tau, \lambda) \end{pmatrix}.$$  

(2.11)

Here, $I$ is a $2 \times 2$ identity matrix. The function $w(\tau, \lambda)$ satisfies the Riccati equation

$$-i\partial_\tau w(\tau, \lambda) = -\lambda w(\tau, \lambda) + \frac{1}{2}Q(\tau) - \frac{1}{2}Q^*(\tau) w^2(\tau, \lambda),$$  

(2.12)

and the function $z(\tau, \lambda)$ is defined via $w(\tau, \lambda)$,

$$-i\partial_\tau z(\tau, \lambda) = \frac{1}{2}Q^*(\tau) w(\tau, \lambda).$$  

(2.13)

It is easy to check that $\bar{\Phi}_0$ has the same form as $\Phi_0$, with $\lambda$ replaced by $\lambda^*$, and $T$ is replaced by

$$\bar{T}(\tau, \lambda) = \frac{(I + W(\tau, \lambda^*))e^{-Z^*(\tau, \lambda^*)}}{1 + w(\tau, \lambda^*) w^*(\tau, \lambda^*)}.$$  

(2.14)

Applying the procedure described above, we find for the fields

$$\bar{Q}_a = -2\Xi(0)_{3,1} = -2(\lambda_0 - \lambda^*_0) e^{-i\phi_1} \frac{(w(\tau, \lambda)e^{\phi_2} + e^{\phi_3})}{N}$$  

(2.15)

and

$$\bar{Q}_b = \Omega(\tau) - 2\Xi(0)_{3,2} = \Omega(\tau) - 2(\lambda_0 - \lambda^*_0)$$

$$\times \frac{(e^{\phi_2} - w(\tau, \lambda^*) e^{\phi_3})(w(\tau, \lambda)e^{\phi_2} + e^{\phi_3})}{N}.$$  

(2.16)

The corresponding density matrix $\bar{\rho} = |\bar{\psi}_{at}\rangle \langle \bar{\psi}_{at}|$ is given by

$$|\bar{\psi}_{at}\rangle = \Xi(\Delta)_{1,1}|1\rangle + \Xi(\Delta)_{2,1}|2\rangle + \Xi(\Delta)_{3,1}|3\rangle = \left( \frac{\lambda^*_0 - \Delta}{|\lambda_0 - \Delta|} + \frac{\lambda_0 - \lambda^*_0}{|\lambda_0 - \Delta|N} \right) |1\rangle + e^{-i\phi_1} (\frac{\lambda_0 - \lambda^*_0}{|\lambda_0 - \Delta|}) \left( \frac{(e^{\phi_2} - w^*(\tau, \lambda)e^{\phi_3})}{N} |2\rangle + \frac{(w(\tau, \lambda)e^{\phi_2} + e^{\phi_3})}{N} |3\rangle \right).$$  

(2.17)

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Here, $\varphi_1$ is the phase of coefficient $c_1$. The absolute value of this coefficient can be set to unity without loss of generality. For making notions shorter, we have also defined two phases (cf. equations (2.10) and (2.11)),

$$\varphi_2 = Z(\tau, \lambda)_{1,1} + \log(c_2) - \frac{i}{2} \left( \lambda \tau + \frac{v_0 \zeta}{\lambda - \Delta} \right)$$

$$= z(\tau, \lambda) + \log(c_2) - \frac{i v_0 \zeta}{2(\lambda - \Delta)}$$

and

$$\varphi_3 = Z(\tau, \lambda)_{2,2} + \log(c_3) - \frac{i}{2} \left( \lambda \tau + \frac{v_0 \zeta}{\lambda - \Delta} \right)$$

$$= -i \lambda \tau + z^*(\tau, \lambda) + \log(c_3) - \frac{i v_0 \zeta}{2(\lambda - \Delta)},$$

and a normalization function

$$N = 1 + \text{Re}[(w(\tau, \lambda) - w(\tau, \lambda^*)) e^{i \varphi_2 + \varphi_3^*}] + (1 + |w(\tau, \lambda)|^2)(e^{i \varphi_2 + \varphi_3^*} + e^{i \varphi_3 + \varphi_3^*}).$$

We conclude this section by noting an important difference between $\varphi_2$ and $\varphi_3$. It will be shown below that, for a constant or slowly varying background field, the magnitude of function $z(\tau, \lambda)$ is of the same order as that of the control field $|\Omega(\tau)|^2$. Therefore, for small intensities, phase $\varphi_2$ is slowly varying in time and describes the slow-light soliton, while phase $\varphi_3$ varies with a speed close to that of light in vacuum owing to the term $\lambda \tau$.

### 3. The transparency gate

In this section, we introduce a concept of fast and slow-light solitons in the $\Lambda$ model, and explain how the nonlinear interplay between such solitons creates a possibility to control the transparency of the dielectric medium. We discuss first the way to control the transparency of the medium to the slow-light soliton. We explain how a fast soliton propagating in the $a$ channel hops to the $b$ channel, where the slow-light soliton is propagating. The fast soliton then destroys the slow-light solitons stopping thereby its propagation, and then disappears itself as a result of the strong relaxation of the system.

As indicated above, we consider here exact solutions of the Maxwell–Bloch system (equations (1.5) and (1.6)) with a finite background field. This background field plays a similar role as the controlling field in the conventional linear theory of EIT, but it appears in the exact solutions in a very nonlinear fashion. We begin with the case of a time-independent field, such that

$$\Omega_a = 0 \quad \text{and} \quad \Omega_b = \Omega_0 e^{i k z}.$$  

Here, $k \ll k_{a,b}$ is introduced so as to take into account small spatial variations in the phase of the field. The intensity of the field $\Omega_0$ is an experimentally adjustable parameter that provides control over the transparency of optical gates.
and determines the speed of the slow-light soliton. The Maxwell–Bloch system is satisfied for the following initial state of atoms:

\[
\rho_0 = \begin{pmatrix}
1 - \frac{k}{v_0} x & 0 & 0 \\
0 & \frac{k}{v_0} \left( \frac{x}{2} + \Delta \right) & \frac{k}{v_0} \Omega_0 e^{-ik\zeta} \\
0 & \frac{k}{v_0} \Omega_0 e^{ik\zeta} & \frac{k}{v_0} \left( \frac{x}{2} - \Delta \right)
\end{pmatrix}.
\] (3.2)

Parameter \( x \) determines the population of the excited state and has to be larger than \( 2\Delta \). It is important to notice that, for a time-independent background field, atoms can be prepared in a mixture of dark and polarized states only for a non-vanishing parameter \( k \), and this allows access to a wide range of physically interesting situations.

For the time-independent background field above, we immediately find solutions to equations (2.12) and (2.13), namely

\[
w(\tau, \lambda) = w_0 = \frac{\Omega_0 e^{ik\zeta}}{\lambda + \sqrt{\lambda^2 + \Omega_0^2}}
\] (3.3)

and

\[
z(\tau, \lambda) = z_0 \tau = \frac{i}{2} \Omega_0 e^{-ik\zeta} w_0 \tau = \frac{i\Omega_0^2 \tau}{2(\lambda + \sqrt{\lambda^2 + \Omega_0^2})}.
\] (3.4)

As already noted in the previous section, when \( \Omega_b \) depends on \( \zeta \), we find that

\[\left( \Phi_0 \right)_{1,1} = e^{(i/2)(\lambda \tau + ((\nu_0 - kx)\zeta/(\lambda - \Delta)))},\]

and the structure of the solution \( T(\tau, \lambda) \) of equations (2.11) is slightly modified, i.e. we have to replace \( Z(\tau, \lambda) \) with

\[
Z_1(\tau, \lambda) = Z \left( \tau + \frac{k\zeta}{\lambda - \Delta}, \lambda \right) + \frac{i k x \zeta}{4(\lambda - \Delta)} I - \frac{i k \zeta}{2} \sigma_3,
\] (3.5)

where \( \sigma_3 \) is a Pauli matrix. Hence, the phases of equations (2.18) and (2.19) can be expressed in the form

\[
\varphi_2 = \log(c_2) + \frac{i\Omega_0^2 \tau}{2(\lambda + \sqrt{\lambda^2 + \Omega_0^2})} - \frac{ik\zeta}{2} + \frac{i(3kx + 2k\sqrt{\lambda^2 + \Omega_0^2} - 2\nu_0)\zeta}{4(\lambda - \Delta)}
\] (3.6)

and

\[
\varphi_3 = \log(c_3) - \frac{i\lambda\tau}{2} - \frac{i\sqrt{\lambda^2 + \Omega_0^2} \tau}{2} + \frac{ik\zeta}{2} + \frac{i(3kx - 2k\sqrt{\lambda^2 + \Omega_0^2} - 2\nu_0)\zeta}{4(\lambda - \Delta)}.
\] (3.7)

Using the general solutions (2.15) and (2.16), we can determine the dynamics that describe the formation of a transparency gate for the initial conditions specified in equations (3.1) and (3.2). For simplicity, in this section, we assume the spectral parameter to be purely imaginary, \( \lambda_0 = i\epsilon_0 \), and for a soliton type of solution \( \epsilon_0 > \Omega_0 \). The solution corresponding to the phases equations (3.6) and (3.7) describes nonlinear interactions of fast and slow-light solitons. This solution
is parametrized by constants $c_{2,3}$, which define the position and phase of the two solitons. As we have already indicated above, the phase $\phi_2$ determines the position of the slow-light soliton, whereas $\phi_3$ determines the position of the fast soliton. In practice, these constants are defined by the initial condition that specifies the actual pulse of light that enters the medium at point $z = 0$. To understand the structure of the slow-light soliton, we can set $c_3 = 0$. This choice corresponds to having the fast soliton at $\tau = -\infty$, i.e. it is in effect removed and the slow-light soliton is singled out. The latter can then be expressed in the form

$$\tilde{Q}_a = \frac{(\lambda^*_0 - \lambda) w_0 e^{i(\text{Im} \phi_2 - \phi_1)}}{\sqrt{1 + |w_0|^2}} \text{sech}(\phi_s)$$  \hspace{1cm} (3.8)$$

and

$$\tilde{Q}_b = -\Omega_0 e^{ik_0} \tanh(\phi_s),$$  \hspace{1cm} (3.9)$$

in which

$$\phi_s = \text{Re} \phi_2 + \frac{1}{2} \log(1 + |w_0|^2)$$  \hspace{1cm} (3.10)$$

is the phase of the slow-light soliton. For simplicity, we let $k = 0$ in the following. From the expression above with the simplifying approximation $\Omega_0^2/\varepsilon_0^2 \ll 1$, $\Delta = 0$, the group velocity of the slow-light soliton can easily be derived,

$$v_s \approx c \frac{\Omega_0^2}{v_0}.$$  \hspace{1cm} (3.11)$$

The pure state of the atomic subsystem that corresponds to the slow-light soliton solution is given by

$$|\tilde{\psi}_{at}\rangle = \frac{\text{Re} \lambda - \Delta - i \text{Im} \lambda \tanh \phi_s}{|\lambda - \Delta|} |1\rangle - \frac{\tilde{Q}_a}{2|\lambda - \Delta|} |2\rangle - \frac{\tilde{Q}_a}{2|\lambda - \Delta|} |3\rangle.$$  \hspace{1cm} (3.12)$$

Notice that the population of the upper level $|3\rangle$ is proportional to the intensity of the background field. The speed of the slow-light soliton is also proportional to $\Omega_0^2$. This means that the slower the soliton, the smaller the population of the $|3\rangle$ level and, therefore, the dynamics of the nonlinear system as a whole is less affected by relaxation processes.

So as to understand the structure of the fast soliton, we can choose $c_2 = 0$. We can then arrive at an expression that describes a signal moving with the speed of light on a constant background (fast soliton),

$$\tilde{Q}_a = \frac{(\lambda^*_0 - \lambda) e^{i(\text{Im} \phi_3 - \phi_1)}}{\sqrt{1 + |w_0|^2}} \text{sech}(\phi_f)$$ \hspace{1cm} (3.13)$$

and

$$\tilde{Q}_b = -\Omega_0 e^{ik_0} \tanh(\phi_f),$$

where the phase of the fast soliton is given by

$$\phi_f = \text{Re} \phi_3 + \frac{1}{2} \log(1 + |w_0|^2).$$

The atomic state of the system is described by

$$|\tilde{\psi}_{af}\rangle = \frac{\text{Re} \lambda - \Delta - i \text{Im} \lambda \tanh \phi_f}{|\lambda - \Delta|} |1\rangle + \frac{w_0^* \tilde{Q}_a}{2|\lambda - \Delta|} |2\rangle - \frac{\tilde{Q}_a}{2|\lambda - \Delta|} |3\rangle.$$  \hspace{1cm} (3.14)$$

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We must emphasize the principal difference between the fast and slow-light soliton. The latter vanishes together with the controlling field owing to the factor $w_0$ in equation (3.8). Another important feature is that, for the slow-light soliton, the population of the upper level $|3\rangle$ is proportional to $\Omega_0$, and is thus small for a small background field and, therefore, stable against optical relaxation. In contrast with this, the amplitude of the fast soliton is not limited by $\Omega_0$ and is determined by the spectral parameter $\varepsilon_0$. The population of the $|3\rangle$ level is also determined by $\varepsilon_0$, which means that for large spectral parameters, the fast signal will be attenuated as a result of relaxation. As we discuss below, in the absence of a background field, the fast signal behaves as a conventional SIT soliton in the two-level system $|1\rangle \leftrightarrow |3\rangle$.

Figure 2 illustrates the propagation and collision of a fast and slow-light soliton as given by equations (2.15) and (2.16). $I_a$ describes the intensities of the signals in channel $a$. We see that before the collision, only the slow-light soliton exists in channel $a$, while after the collision, the slow-light soliton disappears and a fast intensive signal appears, whose speed is slightly below that of light. In the intensity $I_b$, the slow-light soliton appears as a groove in the background field $\Omega_0$. It is evident that after the collision, the slow-light soliton ceases to propagate in channel $b$, while a small trace of the fast soliton can still be noticed there. The process described above can be interpreted as a destruction of the slow-light soliton by the fast soliton. The notion of a transparency gate relies on the

\[ \text{Figure 2. Knocking out of the slow-light soliton. Plots } (a,b) \text{ show the intensities of fields } \Omega_a \text{ and } \Omega_b, \text{ while plots } (c,d) \text{ show the populations of levels } |2\rangle \text{ and } |3\rangle. \text{ The parameters of the plotted solutions are } c_2 = c_3 = 1, \lambda_0 = 4.1i, \mathcal{J} = 0. \]
existence of two distinctly different regimes, a transparent (open-gate) and opaque (closed-gate) regime. In the absence of a fast soliton, the gate is open for the slow-light soliton. When a fast soliton is present, the slow-light soliton is destroyed, while the fast intensive signal created after the collision in channel $a$ is attenuated as a result of strong relaxation in the atomic subsystem. The gate thus closes in the course of the dynamics owing to relaxation. To further explain the process, we show, in figure 2, the populations $P_2, P_3$ of levels $|2\rangle$ and $|3\rangle$, respectively.

Notice that before the collision, the population of the upper atomic level $|3\rangle$ is negligible, and is approximately given by the slow-light soliton solution (3.12) (see the bottom right plot of $P_3$). The populations $P_1, P_2$ of the lower levels $|1, 2\rangle$ are determined by the slow-light soliton (see the bottom left plot of $P_2$).

Indeed, the fast signal that exists in channel $b$ does not interact with the atoms because, at the onset of the dynamics, their state coincides with the dark state $|1\rangle$. Figure 2 shows that, after the collision, the atoms of the dielectric medium are highly excited and, therefore, level $|3\rangle$ is strongly populated. This leads to the fast attenuation of the rapid intensive signal in channel $a$ as a result of relaxation. The optical gate closes.

Until now, we have described a control mechanism for the transparency of the medium to a particular type of slowly moving signals (slow-light soliton). We now discuss the possibility to read information thus stored in the atomic subsystem. Let us assume that the background field vanishes, i.e. $\Omega_0 = 0$. As explained above, the speed of the slow-light soliton then also vanishes. Information about its polarization is, however, stored in the atomic subsystem. This effect can be interpreted in terms of a polariton, which is a collective excitation of the atom–field system. The notion of polariton has been used before in connection with the $\Delta$ model. In the linear system, the dark-state polariton was discussed in Fleischhauer & Lukin [34]. In the strongly nonlinear regime, we consider here a similar excitation is also possible. Indeed, the field component of a slow-light soliton solution can be interpreted as a contribution to a slow-light polariton. When the controlling field $\Omega_0$ vanishes, this contribution also vanishes, along with the speed of the polariton. The resulting immobile polariton then contains only excitation of the atomic subsystem. The general solution (equations (2.15) and (2.16)) is then reduced so that it can be expressed in the form

$$
\tilde{\Omega}_a = \frac{2\epsilon_0 \exp[i(\Delta n_0 z/2(\epsilon_0^2 + \Delta^2)) + i \log(c_3/|c_2|) - i \varphi_1]}{\cosh(\phi_{s0}) + (1/2) \exp[2\phi_{f0} - \phi_{s0}]} \tilde{\Omega}_a,
$$

and

$$
\tilde{\Omega}_b = e^{i\varphi_1 + \log(c_2) - (i\nu_0 z/2(\Delta + \epsilon_0^2))} \tilde{\Omega}_a,
$$

in which $\phi_{s0} = \epsilon_0 n_0 z/2(\Delta^2 + \epsilon_0^2) + \log(|c_2|)$ is the phase of the slow-light soliton, and $\phi_{f0} = \epsilon_0 \tau + (\epsilon_0 n_0 z/2(\epsilon_0^2 + \Delta^2)) + \log(|c_3|)$ is the phase of the fast soliton for a vanishing background field $\Omega_0$. The form of the above fields resembles a superposition of a fast and slow-light soliton of equations (2.15) and (2.16), with a vanishing velocity of the latter. The last exponential term in the denominator above is the contribution of the fast signal. The term that contains a hyperbolic cosine describes the information about the slow-light soliton, which is stored in the medium when that soliton was stopped. The dynamics of the system then

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includes scattering of the fast soliton by the frozen pattern of atomic polarization. The atomic state that describes this scattering is given by

$$|\psi\rangle = -\left(\frac{i\epsilon_0}{\sqrt{\Delta^2 + \epsilon_0^2}} e^{2\phi_0} - 1 + e^{2\phi_0} + \frac{\Delta}{\sqrt{\Delta^2 + \epsilon_0^2}}\right) |1\rangle + \frac{2i\epsilon_0 \exp[i(\Delta v_0 \zeta/2(\epsilon_0^2 + \Delta^2)) + \text{arg}(c_2) - \phi_1 + \phi_0]}{\sqrt{\Delta^2 + \epsilon_0^2}} \left(\frac{2}{\sqrt{\Delta^2 + \epsilon_0^2}}\right) |2\rangle - \frac{\tilde{Q}_a}{\sqrt{\Delta^2 + \epsilon_0^2}} |3\rangle.$$ (3.16)

For $c_3 = 0$, the fields vanish, while the atomic state is reduced to a form that corresponds to a frozen polariton described by equation (3.12) with $Q_0 = 0$. In other words, when the slow-light soliton is completely stopped, the soliton-borne information is stored as a localized spin-polarization of the atoms. As long as the upper state $|3\rangle$ is not populated, the state of the atomic subsystem is not sensitive to the destructive influence of the optical relaxation processes, and the stored information remains intact.

The conventional way [6] to read the information stored in atoms is to increase the intensity of the background field. The method we suggest for reading the information is different. We propose to send a fast soliton into the dielectric medium, where the information is stored. The polarization of the frozen polariton is flipped by the fast signal, which facilitates reading of the information stored. This way of reading optical information is advantageous because it involves fast and easily detectable processes. Figure 3 illustrates the mechanisms involved.

Notice that the act of reading, based on induced flipping of a localized polarization by the fast signal, can be realized at a very short time scale in comparison with typical relaxation times.

4. Non-adiabatic manipulation of slow-light solitons

In this section, we discuss an exactly solvable, but still physical, case of controlled preparation, manipulation and readout of slow-light solitons in cold atomic vapours and Bose–Einstein condensates. The group velocity of a slow-light soliton depends explicitly on the field $\Omega_0$, i.e. $v_g \approx c(\Omega_0^2/2v_0)$ (cf. equation (3.11)). This expression immediately suggests a plausible conjecture that, when the controlling field is switched off, the soliton stops propagating while the information borne by it remains as a spatially localized polarization pattern in the medium. It thus includes optical memory that can be recovered. For brevity, we refer to the localized polariton as a ‘memory bit’.

Consider now the following scenario for the dynamics of the system (figure 4). Before $\tau = 0$, we create in the medium a slow-light soliton, and assume it is propagating on a constant background field $\Omega_0$. We then slow down the soliton by switching off the background field. Assume, for simplicity, an exponential decay thereafter of the background field with a decay constant of $\alpha$, i.e. $\Omega_0 e^{-\alpha \tau}$. At a certain moment of time, say $T_1 = 4/\alpha$, the field becomes negligibly small. Therefore, we then cut off the exponential tail and approximate it by zero. At this point, the soliton is completely stopped. The position where the soliton stops
Figure 3. Reading of optical information by a fast soliton. Plots (a,b) show the dynamics of fields $\Omega_a$ and $\Omega_b$. Plots (c,d) show the populations of levels $|2\rangle$ and $|3\rangle$. The stationary peak in $P_2$ corresponds to information stored in the form of a localized polarization. The rapidly moving localized excitation of atoms evident in $P_3$ represents the act of reading. The background field is $\Omega_0 = 0$, and the coupling constant is the same as before, $\nu_0 = 4.5$.

depends on the time of switching off the field and the decay constant $\alpha$. As described above, the soliton-borne information is stored as a frozen polarization pattern. Such formation can be preserved for a relatively long time in cold atomic vapours or BECs [15]. At time $T$, we restore the slow-light soliton by abruptly switching on the laser. The whole dynamics is divided into four time intervals $\bigcup_{i=0}^3 D_i = (-\infty,0] \cup [0,T_1] \cup (T_1,T] \cup (T,\infty)$. The time dependence of the intensity of the background field at the entrance into the medium is shown in figure 4.

Before the pulse of light enters the medium, the physical system is assumed to be prepared in the state of equation (1.9). The function $\Omega(\tau)$ now includes switching off and back on of the (controlling) background field, and is given by (cf. figure 4)

$$\Omega(\tau) = \Omega_0 [\Theta(-\tau) + e^{-\alpha \tau} (\Theta(\tau) - \Theta(\tau - T_1)) + \Theta(\tau - T)].$$

Here, $\Theta(\cdot)$ is the Heaviside step function with $\Theta(0) = 1/2$. For the state of equations (1.9) and (4.1), we exactly solve the nonlinear system of equations (1.5) and (1.6), as well as the auxiliary scattering problem of equations (1.7) and (1.8) underlying its complete integrability. Solution of the latter problem is the cornerstone behind the progress achieved in analytical solutions. It allows
mounting of a soliton on a background field, as given in equations (1.9) and (4.1), using the Darboux–Bäcklund transformation described in §2 (cf. [28]). According to the results of §2, the one-soliton solution related to the time-dependent background of equation (1.9) is given by

\[
\begin{align*}
\tilde{Q}_a &= \frac{(\lambda^* - \lambda) w(\tau, \lambda)}{\sqrt{1 + |w(\tau, \lambda)|^2}} e^{i\phi_s} \text{sech} \phi_s \\
\tilde{Q}_b &= \frac{\lambda - \lambda^*}{1 + |w(\tau, \lambda)|^2} e^{\phi_s} \text{sech} \phi_s - \Omega(\tau),
\end{align*}
\]

(4.2)

with the atomic state \(\tilde{\rho} = |\tilde{\psi}_{at}\rangle \langle \tilde{\psi}_{at}|\), where

\[
|\tilde{\psi}_{at}\rangle = \frac{\text{Re} \lambda - \Delta - i \text{Im} \lambda \tanh \phi_s |1\rangle + \frac{\tilde{Q}_a}{2|\lambda - \Delta| w(\tau, \lambda)} |2\rangle - \frac{\tilde{Q}_a}{2|\lambda - \Delta|} |3\rangle. \tag{4.3}
\]

Here,

\[
\begin{align*}
\phi_s &= \phi_0 + \frac{\nu_0 \zeta}{2} \text{Im} \frac{1}{\lambda - \Delta} + \text{Re}(z(\tau, \lambda)) + \ln \sqrt{1 + |w(\tau, \lambda)|^2} \\
\theta_s &= \theta_0 - \frac{\nu_0 \zeta}{2} \text{Re} \frac{1}{\lambda - \Delta} + \text{Im}(z(\tau, \lambda)),
\end{align*}
\]

(4.4)

where \(\lambda\) is an arbitrary complex parameter. Functions \(w(\tau, \lambda)\) and \(z(\tau, \lambda)\) are piecewise continuous, being smooth functions within each of the time regions \(D_i\).

For clarity, we show, in table 1, all the components of the solution separately for the four time regions. Function \(u_0\) is defined as in equation (3.3), but with \(k = 0\), and the index of the Bessel functions is defined such that \(\gamma = (\alpha + i\lambda)/(2\alpha)\).
Table 1. Parameters for the solution equations (4.2) and (4.3).

<table>
<thead>
<tr>
<th>$D$</th>
<th>$\Omega(\tau)$</th>
<th>$w(\tau, \lambda)$</th>
<th>$z(\tau, \lambda)$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0$</td>
<td>$\Omega_0$</td>
<td>$w_0$</td>
<td>$\frac{i}{2} \Omega_0 w_0 \tau$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>$\Omega_0 e^{-\alpha \tau}$</td>
<td>$\frac{C J_1 - \gamma (-\Omega(\tau)/2\alpha) - J_{\gamma-1} (-\Omega(\tau)/2\alpha)}{C J_{-\gamma} (-\Omega(\tau)/2\alpha) + J_{\gamma} (-\Omega(\tau)/2\alpha)}$</td>
<td>$-\alpha \gamma \tau + \ln \frac{C J_{-\gamma} (-\Omega(\tau)/2\alpha) + J_{\gamma} (-\Omega(\tau)/2\alpha)}{C J_{-\gamma} (-\Omega_0/2\alpha) + J_{\gamma} (-\Omega_0/2\alpha)}$</td>
<td>$-i w_0 J_{\gamma} (-\Omega_0/2\alpha) + J_{\gamma-1} (-\Omega_0/2\alpha) - i w_0 J_{-\gamma} (-\Omega_0/2\alpha)$</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\ln \frac{C (-\Omega_0/4\alpha)^{-\gamma} / (1 - \gamma)}{C J_{-\gamma} (-\Omega_0/2\alpha) + J_{\gamma} (-\Omega_0/2\alpha)}$</td>
<td>$C_2 = C_1$</td>
</tr>
<tr>
<td>$D_3$</td>
<td>$\Omega_0$</td>
<td>$\frac{\Omega_0 \tan((1/2) \sqrt{\lambda^2 + \Omega_0^2} (\tau - T))}{\lambda \tan((1/2) \sqrt{\lambda^2 + \Omega_0^2} (\tau - T)) - i \sqrt{\lambda^2 + \Omega_0^2}}$</td>
<td>$\ln \frac{C e^{-i(\lambda + \sqrt{\lambda^2 + \Omega_0^2}) (\tau - T)/2} + e^{-i(\lambda - \sqrt{\lambda^2 + \Omega_0^2}) (\tau - T)/2}}{C + 1 + z_2}$</td>
<td>$\frac{\Omega_0^2 + 2\lambda (\lambda - \sqrt{\lambda^2 + \Omega_0^2})}{\Omega_0^2}$</td>
</tr>
</tbody>
</table>
The values $C_i$, $i = 0, 1, 2, 3$, of $C$ that are constant in each of the time regions $D_i$, $i = 0, 1, 2, 3$, form the right-most column of the table, and time $T$ is chosen as in figure 4, i.e. $T = 4/\alpha + 3$. Notice that in this table, $w_2 = w_1(\infty, \lambda)$ and $z_2 = z_1(\infty, \lambda)$. The solution in region $D_2$ is thus parametrized by the asymptotic values of the data for region $D_1$, in which there is no cut-off in the exponentially vanishing tail. Region $D_2$ describes the phase when the slow-light soliton is stopped: the fields vanish and the soliton-borne information is stored as a frozen polarization pattern in the medium. At time $T$, the laser is instantly turned on again. The frozen polarization pattern then generates a moving slow-light soliton. This process is described by the solution in the region $D_3$. Except for time $T_1$, functions $w$ and $z$ are, in fact, continuous in $\tau$. This ensures that the physical variables such as the wave function and field amplitudes are always continuous.

Typical dynamics of the field intensities $I_b$ and $I_a$ are illustrated in figures 4 and 5, respectively. In figure 4, the decaying shock wave, whose front has an exponential profile, propagates with the speed of light, reaches the slow-light soliton and stops it. An intense and narrow peak is developed in the background field once the shock wave has hit the soliton (dotted curve). This peak describes the transfer of energy from the soliton to the background field. After the laser field has been switched on again, another step-like shock wave reaches the polariton localized in the medium, and retrieves the stored information in the form of a new slow-light soliton. The narrow deep depression in the right-most part of the dotted curve means that now energy is transferred back from the background field to the restored soliton. The intensity $T_a$ of field $\tilde{Q}_a$ is plotted in figure 5 (white means non-zero intensity). There is a sharp end in the intensity when the background
field is switched off and the propagating soliton disappears. After restoration of a soliton by switching back on the field, characteristics of the soliton, i.e. the width and group velocity, are very similar to those of the original soliton.

We now calculate the half width of the polarization pattern frozen in the medium when the soliton is stopped. The result is

$$W_s = 4c \ln(2 + \sqrt{3}) \frac{|\Delta - \lambda|^2}{v_0|\text{Im}(\lambda)|}. \quad (4.5)$$

Notice that the width equation (4.5) of the optical memory bit does not depend on rate $\alpha$. It means that the memory bit is not sensitive to how rapidly, i.e. non-adiabatically, the controlling field is switched off. This leads to an important conclusion. Using the externally adjustable parameter $\alpha$, it is possible to control the location of the memory bit, while its characteristic size remains the same. Dutton & Hau [15] have already reported that, when the switching of the field is quick in comparison with the natural lifetime of the upper level, the adiabatic assumptions are no longer valid, but, remarkably enough, the quality of the storage is not reduced. Our analytical results are in excellent agreement with this observation.

The group velocity of the slow-light soliton is given by

$$v_g = \frac{|w(\tau, \lambda)|^2}{c, (v_0(1 + |w(\tau, \lambda)|^2)/2|\Delta - \lambda|^2 + |w(\tau, \lambda)|^2}. \quad (4.6)$$

Notice that in the case of a constant background field, i.e. in the case $\alpha = 0$, the conventional expressions for the slow-light soliton equations (3.8) and (3.9), along with the expression for the group-velocity equation (3.11)—one of the main results reported here—can be readily recovered from equations (4.2) and (4.6).

The distance $L_s(\alpha)$ that the slow-light soliton will propagate after switching off the laser before its full stop is given by

$$L_s(\alpha) = \frac{2c|\Delta - \lambda|^2}{v_0 \text{Im}(\lambda)} \frac{\phi_s(\tau = \infty)}{\phi_s(\tau = 0)} = \frac{2c|\Delta - \lambda|^2}{v_0|\text{Im}(\lambda)|} \left[ \ln \left(1 + \frac{|w_0|^2}{2} - \text{Re}(z(\infty, \lambda)) \right) \right]. \quad (4.7)$$

It is evident from the result equation (4.7) that the soliton has inertia. Indeed, even if the controlling field were switched off instantly (notice that $\lim_{\alpha \to \infty} \text{Re}(z(\infty, \lambda)) = 0$), the soliton would still propagate over a finite distance after the shock wave related to vanishing of the field and propagating with the speed of light has reached the soliton.

In figure 6, we show the dynamics related to a localized polariton, and to the soliton dynamics shown in figure 5. The population-level dynamics determined analytically here is remarkably similar to that shown in Bajcsy et al. [8] as a result of an experiment. The central part of the plot in figure 6 shows the localized memory bit imprinted in the medium by the slow-light soliton. Notice that, in the presence of a soliton, the flip of population from level $|1\rangle$ to $|2\rangle$ is almost complete at the centre of the peak. This property of the solution manifests a major difference between the strongly nonlinear regime considered here and the linear EIT theory. As has already been pointed out before [15,21,24], when the two fields are comparable in magnitude, new avenues are opened up for an effective control over the superposition of two lower states of the atoms. Changing the parameters of the slow-light soliton, we can coherently drive the system to access
any point on the Bloch sphere that describes the lower levels. For zero detuning, the solution discussed here shows that, when the field vanishes, the maximum population of the second level reaches unity. Using equation (4.3), it is also not difficult to estimate the maximum population of level |2⟩ after the soliton has stopped completely; also for a non-zero detuning, |λ|^2/|Δ − λ|^2. Notice that only a small fraction of the total population is located on the upper level |3⟩ and gives a contribution to the atom–field interaction. This population is proportional to |Ω₀|^2. Numerical studies of the Maxwell–Bloch equations with relaxation terms included [15,21] show that, for experimentally feasible group velocities of the slow-light pulses, i.e. when the maximum intensity of the controlling field is not very high, the pulses are stable against relaxation from level |3⟩. We considered above the same range of parameters, and found that the destructive influence of relaxation on the solution equation (4.2) is indeed negligible.

5. The case of an arbitrary controlling field

In this section, we construct a single-soliton solution on the background state of the overall atom–field system described by equation (1.9) for quite a general (complex) controlling field \( \Omega(\tau) \). The single-soliton solution that corresponds to a background field \( \Omega(\tau) \) is given by equations (4.2) and (4.3), with the functions \( w(\tau, \lambda) \) and \( z(\tau, \lambda) \) defined below.

We envisage dynamics similar to those dealt with in §4. We assume now that the slow-light soliton propagates in a nonlinear superposition with the background field, which is constant at \( \tau \to -\infty \) and vanishes at \( \tau \to +\infty \). The speed of the slow-light soliton is controlled by the intensity of the background field. Therefore, when the field decreases, the slow-light soliton slows down and eventually stops, thereby disappearing and leaving behind a localized polariton, i.e. an optical memory bit. Should the background field increase again, the soliton will be reborn and will then accelerate accordingly.
To be specific, we define the asymptotic behaviour of field $Q(\tau)$ in the form
\[ Q(\tau \to -\infty) = Q_0 \quad \text{and} \quad Q(\tau \to +\infty) = 0. \quad (5.1) \]
The asymptotic boundary conditions (equation (5.1)) dictate the following asymptotic behaviours of functions $w(\tau, \lambda)$ and $z(\tau, \lambda)$ defined by equations (2.12) and (2.13):
\[ w(-\infty, \lambda) = w_0 = \frac{Q_0}{2k(\lambda)}, \quad w(+\infty, \lambda) = 0 \quad (5.2) \]
and
\[ z(-\infty, \lambda) = z_0 = i\frac{|Q_0|^2}{4k(\lambda)}, \quad (5.3) \]
in which $k(\lambda) = (\lambda + \sqrt{\lambda^2 + |Q_0|^2})/2$. The function $z(\tau, \lambda)$ that satisfies the asymptotical conditions equation (5.3) is given by
\[ z(\tau, \lambda) = z_0 + \int_{-\infty}^{\infty} \left( \frac{i}{2} Q^*(\tau') w(\tau', \lambda) - z_0 \right) \Theta(\tau - \tau') d\tau'. \quad (5.4) \]
Similarly, function $w(\tau, \lambda)$ is given by
\[ w(\tau, \lambda) = i \int_{-\infty}^{\infty} e^{-ik(\tau - s)} \Theta(\tau - s) \tilde{w}(s, \lambda) ds \quad (5.5) \]
and
\[ \tilde{w}(\tau, \lambda) = \frac{Q(\tau)}{2} + \frac{1}{k^2} \left( \frac{|Q_0|^2}{4} k w - \frac{Q^*(\tau)}{2} (k w)^2 \right). \quad (5.6) \]
Here, $\Theta(\tau)$ is the Heaviside step function. We rewrite the relations (equations (5.5) and (5.6)) in the form of a nonlinear integral equation,
\[ \tilde{w}(\tau, \lambda) = \frac{Q(\tau)}{2} + \int_{-\infty}^{\infty} e^{-ik(\tau - s)} \Theta(\tau - s) \tilde{w}(s, \lambda) ds \times \int_{-\infty}^{\infty} e^{-ik(\tau - s)} \Theta(\tau - s) \left( \frac{Q^*(\tau)}{2} \tilde{w}(s, \lambda) - \frac{|Q_0|^2}{4} \right) ds. \quad (5.7) \]
Hence, we can construct a solution $\tilde{w}(\tau, \lambda)$ by iterating equation (5.7), and starting the iterations with $\tilde{w}_0(\tau, \lambda) = (1/2)Q(\tau)$.
Notice that the last term in equation (5.6) provides a correction of the order of $k^{-2}$, because function $w(\tau, \lambda)$ asymptotically behaves as $1/k$. In the adiabatic regime when the background field varies slowly, i.e. when all the derivatives of $Q(\tau)$ are much smaller than $k$, we can integrate equation (5.5) by parts. Preserving only the lowest order term with respect to $k$, we find that
\[ w(\tau, \lambda) \approx \frac{Q(\tau)}{2k}. \quad (5.8) \]
Therefore, to the lowest order in \( k \), we find that
\[
z(\tau, \lambda) \approx z_0 \tau + \int_{-\infty}^{\infty} \left( \frac{i}{4k} |Q(\tau')|^2 - z_0 \right) \Theta(\tau - \tau') d\tau'.
\] (5.9)

It is evident that this expression is in agreement with the asymptotic condition equation (5.3).

For an arbitrary dependence of the background field on the retarded time \( \tau \), the speed of the slow-light soliton can be expressed in the form
\[
v_g = \frac{\partial \phi_s}{c - \partial \phi_s - \partial \phi_s}.
\] (5.10)

It follows easily that
\[
\frac{\partial \phi_s}{\partial \tau} = \frac{\text{Im}(\lambda)|w(\tau, \lambda)|^2}{1 + |w(\tau, \lambda)|^2} \quad \text{and} \quad \frac{\partial \phi_s}{\partial \zeta} = \nu_0 \frac{1}{2} \frac{\text{Im} 1}{\lambda - \Delta}.
\] (5.11)

We have thus found a general solution for the velocity \( v_g \) of the slow-light soliton propagating on an arbitrary time-dependent background field, in terms of the function \( w(\tau, \lambda) \) given by equation (5.7). This result provides a new way to study the dynamics of localized optical signals in nonlinear EIT systems. It allows an easy construction of many different schemes to slow down light, stop it and reaccelerate it in the form of a slow-light soliton contribution to the probing pulse. With such techniques, one can, for example, probe different regions of the medium by changing the time that the soliton dwells at a particular location. The dwelling time is important in situations where interaction between light and impurities in the EIT medium is weak, and requires slowing down the signal in the vicinity of impurities in order to gain more information about the structure of the medium.

We also introduce above the notion of the distance \( \mathcal{L}[\Omega] \) that the slow-light soliton propagates before stopping fully. This quantity is important because it describes the writing and location of the imprinted memory bit. The brackets \([\cdot]\) indicate a functional dependence of the distance on the controlling field \( \Omega(\tau) \). To begin with, we consider the case when the field is instantly switched off at time \( \tau = 0 \), i.e. \( \Omega(\tau) = \Omega_0 \Theta(-\tau) \). We now easily find the solutions for \( w \) and \( z \),
\[
w(\tau, \lambda) = w_0 (\Theta(-\tau) + \Theta(\tau)e^{-i\omega_0 \tau}) \quad \text{and} \quad z(\tau, \lambda) = z_0 \Theta(-\tau) \tau.
\]

From these, we can determine the distance \( \mathcal{L}_0 \) that the soliton propagates between times \( \tau = 0 \) and \( \tau \to \infty \) (the stopping time),
\[
\mathcal{L}_0 = \frac{c|\Delta - \lambda|^2}{\nu_0 |\text{Im}(\lambda)|} \ln(1 + |w_0|^2).
\]

We made use above of the assumption \( \text{Im}(\lambda) < 0 \).

\( \text{Phil. Trans. R. Soc. A} \ (2011) \)
We can now determine the stopping distance $L[\mathcal{U}]$ for a generic field $\mathcal{U}(\tau)$ that satisfies the conditions equation (5.1). It is convenient to determine it as a relative distance, namely as the difference between the location of the maximum of the stopped signal and the distance $L_0$. This relative distance is given by

$$
L[\mathcal{U}] = \frac{2c|\Delta - \lambda|^2}{\nu_0 \text{ Im}(\lambda)} \int_{-\infty}^{\infty} \text{ Re} \left( \frac{i}{2} \mathcal{U}^*(\tau) \mathcal{U}(\tau, \lambda) - z_0 \Theta(-\tau) \right) d\tau. 
$$

Using the representation equation (5.6), we further find that

$$
L[\mathcal{U}] = \frac{2c|\Delta - \lambda|^2}{\nu_0 \text{ Im}(\lambda)} \text{ Re} \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-ik(\tau-s)} \Theta(\tau-s) \left( \frac{|\mathcal{U}_0|^2}{4} \Theta(-\tau) - \frac{\mathcal{U}^*(\tau)}{2} \tilde{w}(s, \lambda) \right) ds d\tau \right). 
$$

(5.12)

If we assume that $\mathcal{U}(\tau)$ is a smooth function and substitute in the above expression the solution for $\tilde{w}(\tau, \lambda)$, we find the result in the form of a series,

$$
L[\mathcal{U}] = \frac{2c|\Delta - \lambda|^2}{2\nu_0 \text{ Im}(\lambda)} \text{ Im} \left( \sum_{n=1}^{\infty} \frac{I_n}{k^n} \right),
$$

in which $I_n[\mathcal{U}]$ are regularized Zakharov–Shabat functionals [25]. The first two functionals are $I_1[\mathcal{U}] = -\int_{-\infty}^{\infty} (|\mathcal{U}(\tau)|^2 - |\mathcal{U}_0|^2 \Theta(-\tau)) d\tau$ and $I_2[\mathcal{U}] = (1/2i) \int_{-\infty}^{\infty} (\mathcal{U}^*(s) \partial_s \mathcal{U}(s) - \mathcal{U}(s) \partial_s \mathcal{U}^*(s)) ds$. The other functionals can be obtained through the iteration procedure described above. As is usual for boundary conditions of the type of finite density, $I_1$ is actually not a proper functional on the complex manifold of physical observables, in the sense described in Faddeev & Takhtadjan [25]. In that same sense, all the other functionals in the expansion with respect to $k$ are proper. It is a plausible conjecture that the minimum of the functional for the stopping length, equation (5.13), i.e. $\delta L[\mathcal{U}]/\delta \mathcal{U} = 0$ with $\delta^2 L[\mathcal{U}]/\delta \mathcal{U}^2 > 0$, is achieved when the controlling field is switched off instantly. It thus seems intuitive that the minimum is given by the function $\mathcal{U}_0 \Theta(-\tau)$ discussed above. This conjecture is also supported by the case discussed in §4, when the controlling field vanishes exponentially, i.e. $\mathcal{U}(\tau) = \mathcal{U}_0 (\Theta(-\tau) + \Theta(\tau)e^{-\alpha t})$. In that case, the minimum of the stopping length is given by the singular limit $\alpha \to \infty$, i.e. that of instant switching off of the controlling field.

Another important characteristic of the system is the shape of the imprinted memory bit. It is easy to show that the width $W_0$ of the imprint is not sensitive to the functional form of $\mathcal{U}(\tau)$ and is the same as that given by equation (4.5). It means that this exact result is valid regardless of how rapidly we switch off the background field. The actual functional form of $\mathcal{U}(\tau)$ only influences the location of the stored signal and not its shape. This result is strongly supported by recent experiments [15]. It is emphasized in this reference that, phenomenologically, the quality of the storage is not sensitive to how the control laser is switched off. Our exact result equation (4.5) provides a rigorous basis for this observation.

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Figure 7. Comparison of exact and effective time solutions for the soliton of §4 with $\epsilon_0 = 4.1$, $\Delta = 0$ and $\Omega(\tau) = 0.5e^{-4\tau}$. It is evident that, when the background field changes relatively fast, the exact solution denoted by the solid and dotted lines are clearly different from the approximative solution denoted by the dashed and dot-dashed lines, respectively.

To conclude this section, we discuss briefly the applicability of the concept of effective time in the regime of non-adiabatic variation of the controlling field. This concept has been used before in Grobe et al. [18], Bajcsy et al. [8] and Matsko et al. [24] for an approximative description of pulse propagation on a background of time-dependent controlling field. In these references, an effective time variable (the ‘scaled time’ of [8] and the function $Z(\tau)$ of [18]) is introduced, and it shows that effective (scaled) time is a very useful concept in the regime of linear EIT, while in Bajcsy et al. [8], the applicability of this concept is demonstrated in the strongly nonlinear adiabatic regime. In the present formalism, the effective time approximation is provided by equations (5.8) and (5.9) when employed in the slow-light soliton solution equations (4.2) and (4.3). In figure 7, we compare the resulting approximate solution with the exact solution discussed in §4. We can conclude that the effective time has a rather limited applicability in the strongly nonlinear non-adiabatic regime, which is the most relevant one in the current experiments [15]. Indeed, figure 7 demonstrates that, when effective time is used for the slow-light soliton, large errors appear in the non-adiabatic case. Therefore, heuristic attempts [35] to substitute an effective time into the slow-light soliton solution (equations (3.8) and (3.9)) have not been very useful.

6. Slow-light solitons and dilatonic gravity

For convenience, we begin this section by expressing the wave equations for the Rabi frequencies of the two fields, equation (1.5), in the form

$$\partial_\zeta \Omega_a = i\nu_0 \psi_3 \psi_1^*$$ \quad and \quad \partial_\zeta \Omega_b = i\nu_0 \psi_3 \psi_2^*. \quad (6.1)$$

Here, $\zeta = z/c$ and $\tau = t - z/c$, while in the sharp line limit (no detuning, $\Delta = 0$) equation (1.6) can be reformulated as a Schrödinger equation for the...
amplitudes $\psi_{1,2,3}$ of the atomic wave function,

$$
\begin{align*}
\partial_\tau \psi_1 &= \frac{i}{2} \Omega_a^* \psi_3, \\
\partial_\tau \psi_2 &= \frac{i}{2} \Omega_b^* \psi_3 \\
\partial_\tau \psi_3 &= -i\gamma \psi_3 + \frac{i}{2}(\Omega_a \psi_1 + \Omega_b \psi_2).
\end{align*}
$$

(6.2)

Here, $\gamma$ describes the relaxation rate, and we have set $\hbar = 1$. We intend to study the dynamics of coupled atom–field modulations in a $\Lambda$ type model that can, to a large extent, preserve their spatial shape during their propagation in the medium. We consider such modulation solutions as a generalization of the dark-state polariton [34].

In the linear theory, the probe field only appears in $\Omega_a$, whereas in the nonlinear theory, it also forms a nonlinear superposition with the controlling field in channel $b$. Nevertheless, in both cases, the Rabi frequency $\Omega_a$ describes probe-field modulations. Indeed, the field in channel $a$ induces atomic transitions from state $|1\rangle$ to the excited state $|3\rangle$. On the other hand, the amplitude of the excited state drives the field $\Omega_a$. From the results of linear and nonlinear theories of EIT [15, 16, 18, 21, 28, 36], we can infer a physically plausible assumption that the population of the upper level is proportional to the intensity of the field in the probe channel $a$, i.e. that $|\Omega_a|^2 \approx |\psi_3|^2$. We assume here that this relation is characteristic of slow-light phenomena. Based on this, we postulate that

$$
|\psi_3|^2 = \frac{2}{\nu_0} k |\Omega_a|^2,
$$

(6.3)

where $k$ is an arbitrary parameter.

We emphasize that the imposed constraint equation (6.3) reduces equations (6.1) to a simplified nonlinear system that, however, provides an adequate description for the propagation of slow light. In this sense, the relation equation (6.3) is essential for this section. As we show below, this condition is sufficient and necessary for the slow-light solitons to exist.

For convenience, we introduce two new notations: $|\Omega_a| \equiv e^{-\rho}$ and $\Omega_b = \eta$, and find from equation (6.1), together with equation (6.3), that the $\rho$ field satisfies the Liouville equation

$$
\partial_{\xi \tau} \rho = -ke^{-2\rho},
$$

(6.4)

with the constraint

$$
\partial_{\xi \tau} \eta + \partial_\tau \rho \partial_{\xi} \eta = ke^{-2\rho} \eta
$$

(6.5)

and the auxiliary equation

$$
(4k(\partial_\tau + \gamma) + \partial_{\xi})e^{-2\rho} + \partial_{\xi} \eta^2 = 0.
$$

(6.6)

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For $\gamma = 0$, equations (6.4)–(6.6) can be easily solved, and we find that

$$\rho = -\frac{1}{2} \log \left[ \frac{\partial_\zeta A_+(\zeta) \partial_\tau A_-(\tau)}{(1 - kA_+A_-)^2} \right],$$

(6.7)

$$A_+(\zeta) = -\frac{1}{k} \exp[-8\epsilon_0 k\zeta],$$

(6.8)

$$A_-(\tau) = \exp \left[ 2\epsilon_0 \int \frac{d\tau}{m^2(\tau) + 1} \right],$$

(6.9)

and

$$\eta = 2(\partial_\tau m - m \partial_\tau \rho).$$

(6.10)

Here, $m(\tau)$ is an arbitrary function. The original fields $\Omega_{a,b}$ are now given by

$$\Omega_a = \frac{2\epsilon_0}{\sqrt{m^2(\tau) + 1}} \sech(\varphi),$$

(6.11)

$$\Omega_b = \frac{2\epsilon_0 m(\tau)}{m^2(\tau) + 1} \tanh(\varphi) + \frac{1}{2} \frac{\partial_\tau m(\tau)}{m^2(\tau) + 1},$$

(6.12)

and

$$\varphi = -4k\epsilon_0 \zeta + \int \frac{\epsilon_0 d\tau}{m^2(\tau) + 1},$$

(6.13)

while the background field $\Omega(\tau)$ is

$$\Omega(\tau) = \frac{4\epsilon_0 m(\tau) + \partial_\tau m(\tau)}{2(m^2(\tau) + 1)}.$$  

(6.14)

Function $\Omega(\tau)$ models the controlling field that governs the dynamics of the system. The time dependence of this function can result, for example, from modulation of the intensity of the auxiliary laser. For a constant controlling field, $\Omega(\tau) = \Omega_0$, and making in addition the simplifying approximation $\Omega_0^2/\epsilon_0^2 \ll 1$, for $\Delta = 0$, the group velocity of the slow-light soliton can easily be derived, and is given by equation (3.11). This expression for the group velocity immediately suggests that the signal stops when $\Omega_0 = 0$. It was, therefore, one of the main motivations for the previous analytical studies of the slow-light soliton (see [28,36] and references therein). We can envisage the following dynamics scenario. Assume that a slow-light soliton is created in the system before the time $\tau = 0$, and propagates then on the background of a constant controlling field $\Omega_0$. At time $\tau = 0$, the laser source of the controlling field is switched off, and it decays thereafter reasonably rapidly, as described by the ‘switch-off’ function $f(\tau)$. The front of the decaying controlling field described by $f(\tau)$ propagates into the medium, starting at point $\zeta = 0$ where the laser is placed. The state of the quantum system equation (1.9) is dark for the controlling field. The medium is thus transparent for the spreading front of the decaying field that propagates with the speed of light such that it eventually overtakes the slow-light soliton and stops it. To realize this scenario, we assume the controlling field $\Omega(\tau)$ to be a constant $\Omega_0$ for negative $\tau$, and a switch-off time ($\tau$) dependent function $f(\tau)$ for positive $\tau$, i.e. $\Omega(\tau) = \Omega_0 \Theta(-\tau) + f(\tau) \Theta(\tau)$. Here, $\Theta(\tau)$ is the Heaviside

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step function, while \( f(0) = Q_0 \). The velocity of the slow-light soliton can easily be found, and we find that

\[
v_g = \frac{1}{4k} \frac{1}{m^2(\tau) + 1},
\]

while the stopping distance is given by

\[
L = \frac{1}{4k} \int_0^\infty \frac{d\tau}{m^2(\tau) + 1}.
\]

This distance designates the stretch in the medium, where the slow-light soliton creates a localized polaron in the process of its own disappearance.

From equation (6.14), a number of exactly solvable regimes for stopping of the slow-light soliton can easily be identified. For example, for \( m(\tau) = e^{a(\tau - \tau_0)} \), \( \alpha > 0 \), we find that \( f(\tau) = (e_0 + (\alpha/4)) \text{sech} \alpha(\tau - \tau_0) \) and \( L = (1/8\alpha k) \ln(1 + e^{2\alpha\tau_0}) \). Another physically interesting case, \( f(\tau) = Q_0 e^{-\alpha\tau} \), was discussed in §4 above.

Notice also that the Liouville equation (6.4) appears in the theory of two-dimensional gravity. The corresponding action is given by

\[
I = \frac{1}{2} \int \sqrt{-g} \left( \frac{\eta}{2} R[g] + V(\eta) \right) d^2x,
\]

in which \( R[g] \) is the two-dimensional curvature scalar that describes a gravitational field defined by the metric \( g_{ab} = e^{-2\eta} \gamma_{ab} \), with \( \gamma_{ab} \) being the two-dimensional Minkowski metric. In this context, \( V(\eta) = -4k\eta \) plays the role of the dilatonic potential.

### 7. Slow-light solitons: influence of relaxation

In this section, we consider equations (6.1) and (6.2) with a relaxation constant \( \gamma \neq 0 \). From the results of linear and nonlinear theories of EIT [15,16,18,21,28,37], we inferred above a physically plausible assumption that the population of the excited state \( |3\rangle \) is proportional to the amplitude of the field in the probe channel, \( Q_a \). We have previously demonstrated that a slow-light soliton without relaxation can be obtained in the system by taking \( \psi_3 = -(1/2|\lambda - A|)Q_a \). Here, \( \lambda \) is an arbitrary parameter limited from below by the condition \( |\psi_3| \leq 1 \). In order to take into account relaxation, we propose a more general relation between the amplitude of level \( |3\rangle \) and \( Q_a \). The population of the upper level, although very small, should remain non-zero in order to preserve a necessary coupling between the probe and controlling field. At the same time, relaxation should manifest itself in the form of an effective damping of the probe field. In this way, we find, for the relation between the amplitudes,

\[
e^{a(\tau)} \psi_3 = -\frac{1}{2|\lambda - A|} Q_a,
\]
in which \( \alpha(\tau) \) is a first-order correction to the exact slow-light soliton solution. It is shown below that \( \alpha(\tau) \) is always negative. Let us introduce for convenience the notations

\[
|\Omega_a| \equiv e^{-\rho}, \quad \Omega_b \equiv \eta, \quad \tilde{\rho} \equiv \rho + \alpha \quad \text{and} \quad k = \frac{\nu_0}{8|\lambda - \Delta|^2}.
\]

The first three equations of the system equations (6.1) and (6.4) can then be expressed in the form

\[
\partial_\tau \alpha \partial_\zeta \tilde{\rho} + \partial_\zeta \tilde{\rho} = -ke^{-2\tilde{\rho}}, \quad (7.2)
\]

\[
\partial_\tau \eta + \partial_\zeta \tilde{\rho} \partial_\zeta \eta = ke^{-2\tilde{\rho}} \eta \quad (7.3)
\]

and

\[
4k(\partial_\tau + \gamma)e^{-2\tilde{\rho}} = -\partial_\zeta (\eta^2 + e^{-2\rho}). \quad (7.4)
\]

The last of these equations (for phase \( \varphi_3 \)) can easily be integrated once the first three equations have been solved.

Assuming now that the effective relaxation described by \( \alpha \) varies slowly in time \( \tau \) and neglecting, therefore, the first term in equation (7.2), we can find a solution to the system of equations (7.2)–(7.4). We provide a rigorous justification for this approximation after first constructing the solution.

If we neglect the first term on the left-hand side in equation (7.2), assuming that \( \partial_\tau \alpha \) is small in comparison with the term on the right-hand side, this equation is transformed to the Liouville equation

\[
\partial_\zeta \tilde{\rho} = -ke^{-2\tilde{\rho}}. \quad (7.5)
\]

The Liouville equation has an exact solution, which is given by

\[
\rho = -\frac{1}{2} \log \left[ \frac{(1/k)e^{2\alpha} \partial_\zeta A_+(\zeta) \partial_\tau A_- (\tau)}{(1 - A_+ A_-)^2} \right], \quad (7.6)
\]

in which \( A_+(\zeta) \) and \( A_- (\tau) \) are arbitrary functions. We can now find the solution to the whole system including equations (7.3) and (7.4) if we choose these functions such that

\[
A_+(\zeta) = -\exp[-8\varepsilon_0 k \zeta], \quad (7.7)
\]

\[
A_- (\tau) = \exp \left[ 2\varepsilon_0 \int \frac{e^{2\alpha(\tau)}}{p(\tau)^2 + 1} \, d\tau \right], \quad (7.8)
\]

\[
\eta = -2p \partial_\tau \rho + 2\partial_\tau p - 2p \partial_\tau \alpha \quad (7.9)
\]

and

\[
\partial_\tau \alpha (\tau) = -\frac{\gamma/2}{p(\tau)^2 + 1}, \quad (7.10)
\]

with \( p(\tau) \) an arbitrary function that describes the controlling field \( \Omega(\tau) \), and \( \lambda = i\varepsilon_0 \). From equation (7.9), we can conclude that correction \( \alpha \) to the exact solution without relaxation obtained above vanishes for \( \gamma = 0 \), as expected.

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The fields can now be found, and they are given by

\[
\mathcal{U}_a(t, z) = \frac{2\varepsilon_0 e^{2\alpha}}{\sqrt{p(t)^2 + 1}} \text{sech}(\varphi)
\]

and

\[
\mathcal{U}_b(t, z) = -\frac{2\varepsilon_0 p(t) e^{2\alpha}}{p(t)^2 + 1} \tanh(\varphi) + \frac{2\partial_\tau p(t) - \gamma p(t)}{p(t)^2 + 1},
\]

with the phase

\[
\varphi = -4k\varepsilon_0(\zeta - \zeta_0) + \varepsilon_0 \frac{1 - e^{2\alpha(\tau)}}{\gamma},
\]

in which we have chosen \(\alpha(0) = 0\). From this solution for the phase of the slow-light soliton, we find its velocity in the form

\[
v_g = \frac{1}{4k} \frac{e^{2\alpha}}{p(t)^2 + 1}.
\]

We can now return to our approximation of small \(\partial_\tau \alpha\) and analyse its validity, i.e. that the first term on the left-hand side of equation (7.2) is much smaller than \(k e^{-2\beta}\). This requirement leads to the following condition on \(\alpha\):

\[
|\partial_\tau \alpha| \ll \left| \frac{8\varepsilon_0 e^{2\alpha}}{(p^2 + 1) \sinh(2\varphi)} \right|.
\]

Equation (7.14) can be simplified with the help of equation (7.9), and we find that

\[
\frac{\gamma}{16\varepsilon_0} \ll \left| \frac{e^{2\alpha}}{\sinh(2\varphi)} \right|.
\]

This condition is always fulfilled at the maximum of the slow-light soliton as \(\varphi = 0\) there. The position of the maximum is given by the function

\[
\zeta_c(\tau) = 1 - \frac{e^{2\alpha(\tau)}}{4k\gamma} + \zeta_0,
\]

in which we have chosen \(\zeta_c(0) = \zeta_0\). Hence, the phase of the soliton can be expressed in the form

\[
\varphi = -4k\varepsilon_0(\zeta - \zeta_c(\tau)).
\]

From the condition equation (7.15), we can now determine the spatial range around the centre of the slow-light soliton, \(\Delta\zeta(\tau) = \zeta - \zeta_c(\tau)\), where our approximation is valid and where the slow-light soliton thus gives a correct description of the pulse shape,

\[
|\sinh(-8k\varepsilon_0\Delta\zeta)| \ll \frac{16\varepsilon_0}{\gamma} e^{2\alpha}.
\]

At the initial time, for \(\gamma < 16\varepsilon_0\), we find that \(k\varepsilon_0 \Delta\zeta_0 \approx \ln(2\varepsilon_0/\gamma)\). Hence, the full width at half maximum (FWHM) of the slow-light soliton solution is \(w_s \approx 0.66/(k|\varepsilon_0|)\). As the FWHM can also be expressed in the form \(w_s = 2\Delta\zeta_0\), in order
to have the solution valid at least within the FWHM around the soliton peak, parameter $\varepsilon_0$ should obey the condition

$$\varepsilon_0 \geq 0.7 \gamma. \tag{7.19}$$

Notice that $\alpha$ is a negative monotonically decreasing function of $\tau$. Therefore, the spatial range of validity of the approximation closes down with increasing retarded time $\tau$. However, the group velocity of the slow-light soliton tends to zero as $\tau$ increases, and the soliton slows down and is eventually fully stopped (see equation (7.13)). The stopping distance is given by

$$\mathcal{L} \equiv \zeta_c(\infty) - \zeta_0 \leq \frac{1}{4k\gamma}. \tag{7.20}$$

It is evident from this expression that the maximum distance that our soliton can travel in the medium is limited by the magnitude of the relaxation constant. The stronger the relaxation, the shorter is the distance that the soliton traverses.

We discussed above how the controlling field generated by an auxiliary laser at the entrance into the medium supports the propagation of the slow-light soliton. An appropriately chosen modulation of this field ensures a nonlinear coupling between the soliton and atoms, while this coupling preserves in turn the secant shape of the soliton pulse. However, in the ideal case when there is no relaxation, the temporal scale of modulation of the field induces a similar temporal scale in the signal in channel $a$. After the soliton has been generated, the auxiliary laser continues to emit a constant beam of light, which supports the propagation at constant velocity of the soliton. We demonstrated above how this ideal situation, in which the soliton amplitude remains constant, corresponds to a constant function $p(\tau) = p_0$. If the relaxation is not vanishingly small, the amplitude and velocity of the soliton will decay in time $\tau$ according to equation (7.11). The asymptotic value of the background field, beyond the modulation that supports the soliton, can be found from $\Omega_b(\tau, \zeta)$ by taking the limit $\zeta \to \infty$ at the initial time,

$$\Omega_0 = \frac{(2|\varepsilon_0| - \gamma)p_0}{p_0^2 + 1}. \tag{7.21}$$

Assuming $\gamma \gg \Omega_0$, we now find that

$$p_0 = \frac{2|\varepsilon_0| - \gamma}{2\Omega_0} + \sqrt{\left(\frac{2|\varepsilon_0| - \gamma}{2\Omega_0}\right)^2 - 1} \simeq \frac{2|\varepsilon_0| - \gamma}{\Omega_0}. \tag{7.22}$$

In the absence of relaxation, the temporal length of the soliton pulse at the entrance into the medium is determined by the phase $\varphi_0(t) = \varepsilon_0 t/(p_0^2 + 1)$. Hence, we obtain a second condition on the two, so far, arbitrary parameters $p_0$ and $\varepsilon_0$ in terms of realistic experimental parameters $\Omega_0$ and $t_p$,

$$\text{sech}\left(\frac{|\varepsilon_0|}{p_0^2 + 1} \frac{t_p}{2}\right) = 0.5. \tag{7.23}$$

Choosing $|\varepsilon_0| = \gamma$, we can verify that the conditions (7.19), (7.21) and (7.23) are well satisfied for experimentally feasible values of $t_p \sim 1\mu s$ and $\Omega_0 \sim 10\text{MHz}$. 

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It is instructive to compare our solutions with experimental results reported for sodium atoms [5]. From these experiments, we get the following values for the relevant parameters: $\gamma = 6.3 \times 10^7$ rad s$^{-1}$, $t_p = 2.5 \mu$s and $\Omega_0 = 0.56\gamma$. Using these values in the solution of equations (7.21) and (7.23), we find that $p_0 \simeq 18.4$ and $\varepsilon_0 \simeq 5.7\gamma$. We can also determine the reduction in the strength of optical relaxation in the dynamics of the slow-light soliton. According to our solution equation (7.11), soliton amplitude decays with the rate $2a$. We can thus introduce an effective relaxation constant for the soliton (see equation (7.9)),

$$\gamma^* = \frac{\gamma}{p_0^2 + 1} .$$  

(7.24)

For the above parameters, $\gamma^* \simeq \gamma/340$. The effective relaxation constant of the soliton is thus significantly lower than the decay coefficient of an arbitrary pulse. This means that spontaneous emission of the soliton pulses is markedly suppressed as a result of its nonlinear interaction with the medium. The effective optical relaxation time $\tau_{rel}^*$ is longer than the duration of the pulse, and is about $2.2 t_p$.

For a pulse delay of $\Delta t = 7.05 \mu$s reported by Hau [5], we calculated the decay of the maximum amplitude of the soliton, and compared it with the amplitude of a reference pulse $\Omega^*_a$ propagating in a similar system, but with no relaxation. We found that

$$\frac{\Omega_a(t_p/2, \zeta_c)}{\Omega^*_a} \simeq 0.2 .$$

We added a half of the pulse duration to the time at which the soliton amplitude was evaluated because it started to interact with the medium before, by about $t_p/2$, the entrance in the medium of the maximum amplitude. This result agrees very well with those reported by Hau. Notice also that the distance traversed by the slow-light soliton during the time $\Delta t + t_p$ is about $200 \mu$m, which is of the order of the size of the atomic cloud in the experiment [5]. We emphasize that, in the presence of relaxation, the velocity of the soliton does not remain constant any more (see equation (7.13). We estimated that, for the above parameters, the average value of the soliton velocity is about $22 \text{ m s}^{-1}$, while in the experiments, the velocity of a Gaussian pulse was found to be $32 \text{ m s}^{-1}$.

An important feature of the dynamics of the system is that the spatial form of the soliton does not depend on its relaxation. Its shape remains unchanged during its propagation in the medium (figure 8), which is of crucial importance in the storage and retrieval of optical information there.

Notice that the present results remain valid beyond the constraints set by the transparency window of the linear theory. We also demonstrated above that, as a result of strong nonlinear interactions between the probe and controlling fields, it is possible to preserve the shape of optical (solitonic) signals, even in the presence of strong optical relaxation. Present theoretical results and experimental results of Hau et al. [5] and Liu et al. [6] were found to be in very good agreement. It means that the Λ model applied here captures well the relevant properties of at least cold sodium atoms. In addition, we provided rigorous analytical estimates for quantities that have not, so far, lent themselves to experimental analysis, such as the largest distance that a slow-light soliton can propagate in a lossy medium and the dependence of soliton velocity on the relaxation constant $\gamma$.  

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8. Conclusions and discussion

In this article, we analyse an exactly solvable model for an optically active medium, which is shown to facilitate controlled creation and readout of optical memory bits in the strongly nonlinear, and which is even more important, non-adiabatic regime of the system. We show how a transparency gate can be formed and controlled for a slow-light soliton in the medium, how this soliton creates a localized immobile polariton when stopped, and how this polariton can be turned again into a moving slow-light soliton. We explain the mechanisms of a dynamic control of the propagation and stopping of the slow-light soliton with two electromagnetic fields, the background field and an auxiliary (probe) field. We also determine the location and shape of the localized polariton, i.e. the memory bit. Its shape (width) does not depend on the form of the controlling fields and is determined by the parameters of the slow-light soliton alone.

At this point, it is perhaps worth discussing how to actually create slow-light solitons in a $\Lambda$-type medium. The physical feature that underlies the mathematical property of complete integrability needed for true solitons to appear is a delicate balance between dispersion and nonlinearity in the medium. In such a case, almost any sufficiently intensive initial pulse creates solitons. The nonlinear evolution of this pulse leads to formation of a number of solitons and a decaying tail that eventually vanishes. In an ideal lossless medium, the solitons thus formed survive forever. The solution (equations (3.8) and (3.9)) is a slow-light soliton solution of this kind to the nonlinear system (equations (1.5) and (1.6)). Weak dissipation does not typically change the nature of soliton solutions, which only begin to suffer then from energy losses when moving. It is therefore very plausible that such slow-light solitons can indeed be created. This belief is strengthened by the approximative treatment above of an active medium with optical relaxation. The slow-light solitons of that system were found to be rather immune to relaxation, i.e. were much less affected by it than arbitrary optical pulses as a result of strong nonlinear interactions between the two fields, and could thus

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Figure 8. Effect of relaxation on the decay of the slow-light soliton. The solid line denotes a soliton that has propagated a time of $\Delta t + t_p$ in a system with no relaxation, and the dashed line denotes an initially similar soliton that has propagated the same time in a system with non-zero relaxation.
easily remain mobile in systems of realistic size so that they can be manipulated in the manner described for solitons in a lossless medium. Also, comparison of theoretical and experimental results [5,6] showed very good agreement.

We would like to emphasize that in the present analysis, an important role is played by the background field $\Omega(\tau)$: it is a nonlinear analogue of the control field that appears in the conventional (linear) theory of EIT. In the nonlinear case, the control field and the slow-light soliton solution are, however, present in the same channel as a nonlinear superposition. Also, we derived above the slow-light soliton using the sufficient condition equation (6.3). The IS analysis of equations (1.5) and (1.6) in Rybin et al. [36] shows, however, that for slow-light solitons, even a stronger condition holds, namely

$$\psi_3 = -\frac{1}{2|\lambda - \Delta|} \Omega_a,$$

(8.1)

where $\lambda$ is a complex number that parametrizes the soliton solution (in the present case, $\lambda = i\varepsilon_0$). This means that the condition (6.3) is also a necessary condition for the slow-light soliton to exist with $\kappa = \nu_0/(8|\lambda - \Delta|^2)$.

Finally, the Liouville equation (6.4) that appears in the present problem as a result of a few simplifying assumptions, describes also a two-dimensional gravitational field defined by the metric $g_{ab} = e^{-2\phi} \gamma_{ab}$, with $\gamma_{ab}$ the two-dimensional Minkowski metric. Within this interpretation, $V(\eta) = -4\kappa \eta$ plays the role of the dilatonic potential. We did not work out here the ramifications of this connection, but only remark that there is a fascinating analogy between stopping of a slow-light soliton and formation of a black hole in two-dimensional gravity.

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References


Review. Nonlinear theory of slow light


