Addition formulae for Abelian functions associated with specialized curves

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We discuss a family of multi-term addition formulae for Weierstrass functions on specialized curves of low genus with many automorphisms, concentrating mostly on the case of genus 1 and 2. In the genus 1 case, we give addition formulae for the equianharmonic and lemniscate cases, and in genus 2 we find some new addition formulae for a number of curves.

Keywords: Abelian functions; sigma functions; Weierstrass functions; addition formulae

1. Introduction

The aim of this paper is to introduce some new addition formulae for the Weierstrass $\sigma$ and $\varphi$ functions in genus 1, and some generalizations to some higher genus cases. These formulae are found in the special case when some of the coefficients (moduli) of the associated algebraic curves are chosen to be zero, and as a result the curves have additional automorphisms (extra symmetries).

Although elliptic functions, including the Weierstrass elliptic functions, have been extensively used (or perhaps overused) to enumerate travelling wave solutions of nonlinear wave equations, relatively little has been written about the correspondingly higher genus generalizations. This is partly because no general handbooks exist which play the same role as the familiar treatises on elliptic functions. This paper is part of a project to provide the material for such a compendium.

Those coming to this paper because of possible applications to number theory may prefer to see it as extending the classical theory of complex multiplication for elliptic functions to higher genus functions. In this paper, the complex multiplications are of ‘cyclotomic type’, i.e. involving complex roots of unity. These generalize the more well-known addition formulae involving results for $f(u + v)f(u - v)$, where we think of the $(\pm 1)v$ as involving the two real roots of unity.

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We begin by summarizing well-documented existing results for the genus 1 case. In this case, we start with an elliptic curve reduced to the standard Weierstrass form

\[ y^2 = 4x^3 - g_2x - g_3. \]  

(1.1)

The function \( \wp(u) \) is the inverse function \( u \mapsto x \) determined by

\[ u = \int_{\infty}^{(x,y)} \frac{dx}{y}, \]  

(1.2)

and \( \sigma(u) \) is an entire function satisfying

\[ \wp(u) = -\frac{d^2}{du^2} \log \sigma(u). \]  

(1.3)

The function \( \wp(u) \) satisfies the well-known formula

\[ (\wp')^2 = 4\wp^3 - g_2\wp - g_3, \]  

(1.4)

which is isomorphic to the genus 1 curve (but note that this result does not hold for the higher genus cases).

The following two-variable addition formula plays an important role in the theory of the Weierstrass \( \sigma \) and \( \wp \) functions, and its generalizations are central to this paper:

\[ -\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v). \]  

(1.5)

Taking the second logarithmic derivative of equation (1.5) gives the well-known addition formula involving just \( \wp \) and \( \wp' \), which is also an addition formula on the curve (1.1).

A three-variable addition formula is also known from the work of Frobenius and Stickelburger [1] (see also [2])

\[ \frac{\sigma(u-w)\sigma(v-w)\sigma(u-v)\sigma(u+v+w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3} = -\frac{1}{2} \begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix}. \]  

(1.6)

In the genus 2 case, starting with the hyperelliptic curve

\[ y^2 = x^5 + \mu_2x^4 + \mu_4x^3 + \mu_6x^2 + \mu_8x + \mu_{10}, \]  

(1.7)

one can define generalized \( \sigma \) and \( \wp \) functions (see the classical book by Baker [3] or Buchstaber et al. [4] for a modern treatment). The main difference is that \( \sigma \) and \( \wp \) are now functions of \( g = 2 \) variables, \( u = \{u_1, u_2\} \), and there are now three possible versions of the \( \wp \) function, owing to the different possible logarithmic differentials of the \( \sigma \) function,

\[ \wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad 1 \leq i \leq j \leq 2. \]  

(1.8)
Addition formulae for Abelian functions

(Note that in this notation the \( \varphi \) of the genus 1 theory would be written as \( \varphi_{11} \).) The functions \( \varphi_{ij} \) and \( \varphi_{ijk} = (\partial/\partial u_k)\varphi_{ij}(u) \) satisfy equations analogous to equation (1.4). The genus 2 \( \sigma \) and \( \varphi \) functions satisfy an analogue of the elliptic addition formula (1.5)

\[
- \frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = \varphi_{11}(u) - \varphi_{11}(v) + \varphi_{12}(u)\varphi_{22}(v) - \varphi_{22}(u)\varphi_{12}(v),
\]

as described in Baker [3,5]. One possible naive generalization of equation (1.6) in the genus 2 case has been derived [6], but this is a rather complicated formula. Other generalizations are possible.

Our main aim in this paper is to point out that Abelian functions associated with a curve with many automorphisms, namely with extra symmetries relative to the genus 1 theory would be written as \( \varphi_{ij} \), and Abelian functions on its Jacobian variety have an addition formula different from equation (1.5). For example, proposition 6.2 expresses the following formula (taken from theorem 5.1):

\[
- \frac{\sigma(u \pm v)\sigma(u \pm \zeta v)\sigma(u \pm \zeta^2 v)}{\sigma(u)^3\sigma(v)^3} = \pm \frac{1}{2} (\varphi'(u) \pm \varphi'(v)).
\]

The formula is novel in the sense that, although it can be derived from equation (1.5), it is only valid in the case \( \varphi_2 = 0 \).

As another example, while the curve (1.7) with generic parameters has only two automorphisms \((x, y) \mapsto (x, \pm y)\), the special curve

\[
y^2 = x^3 - g_3 \quad \text{with } g_3 \neq 0
\]

has six automorphisms \((x, y) \mapsto (\zeta^j x, \pm y)\) with \( j = 0, 1, 2 \) and \( \zeta = \exp(2\pi i/3) \). This has other addition formulae different from equation (1.5); for example, the following formula (taken from theorem 5.1):

\[
- \frac{\sigma(u \pm v)\sigma(u \pm \zeta v)\sigma(u \pm \zeta^2 v)}{\sigma(u)^3\sigma(v)^3} = \pm \frac{1}{2} (\varphi'(u) \pm \varphi'(v)).
\]

As another example, while the curve (1.7) with generic parameters has only two automorphisms \((x, y) \mapsto (x, \pm y)\), the special curve

\[
y^2 = x^5 + \mu_{10} \quad \text{with } \mu_{10} \neq 0
\]

has 10 automorphisms, \((x, y) \mapsto (\zeta^j x, \pm y), j = 0, 1, \ldots, 4\), with \( \zeta = \exp(2\pi i/5) \), and Abelian functions on its Jacobian variety have an addition formula different from equation (1.9). For example, proposition 6.2 expresses

\[
- \frac{\sigma(u + v)\sigma(u + [\zeta]v)\sigma(u + [\zeta^2]v)\sigma(u + [\zeta^3]v)\sigma(u + [\zeta^4]v)}{\sigma(u)^5\sigma(v)^5} = \pm \frac{1}{2} (\varphi'(u) \pm \varphi'(v)).
\]

as a polynomial in \( \varphi_{ij}(u), \varphi_{ij}(v), \varphi_{ijk}(u) \) and \( \varphi_{ijk}(v) \) (for a slightly more general form of the curve). Here \( \zeta = \exp(2\pi i/5) \) and \([\zeta^j]v = [\zeta^j](v_1, v_2) = (\zeta^j v_1, \zeta^{2j} v_2)\).

In the genus 1 case, we obtain also three-term and four-term addition formulae by using equation (1.6). While formulae of a similar type exist in the higher genus cases, we do not mention them here because of their complexity.

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We have two ways to prove these formulae. One is by simplifying the expression in terms of \( \wp \) and its derivative given by taking the product of modified formulae from equation (1.5) or (1.6), or their higher genus generalizations. The other is similar to the method used in Eilbeck et al. [7], that is, by balancing an expression such as equation (1.11) with a linear combination of suitable \( \wp \)-functions, in which the derivations of the correct coefficients are aided by algebraic computing software.

The paper is laid out as follows. We first cover some basic theory mainly needed for genus 2 and higher. After introducing basic notations in §2, we define the \( \sigma(u) \) and \( \wp \) functions in §3. In §4, we define the types of curves we shall be considering and their symmetries. We consider the genus 1 case in §5, giving some detail to provide a pedagogical background for the general methods. The genus 2 case is discussed in §6. In §7, we discuss briefly further generalizations to higher genus cases, a topic which will be covered in more detail elsewhere.

In this paper, we as usual denote by \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \) the ring of integers, the fields of rational numbers, real numbers and complex numbers, respectively. We denote the imaginary unit by \( i \).

2. Differential forms, etc.

We recall fundamentals in this section on elliptic and hyperelliptic curves. Let
\[
f(x, y) = y^2 + (\mu_1 x^g + \mu_3 x^{g-1} + \cdots + \mu_{2g+1}) y - (x^{2g+1} + \mu_2 x^{2g} + \cdots + \mu_{4g+2}).
\]
Let \( C \) be the smooth projective curve defined by \( f(x, y) = 0 \) which has point \( \infty \) at infinity. The space of differential forms on \( C \) is spanned by
\[
\omega_1 = \frac{dx}{f_y(x, y)}, \quad \omega_2 = \frac{x \, dx}{f_y(x, y)}, \ldots, \quad \omega_g = \frac{x^{g-1} \, dx}{f_y(x, y)},
\]
where \( f_y(x, y) = (\partial/\partial y)f(x, y) \). For variable \( g \) points \((x_1, y_1), (x_2, y_2), \ldots, (x_g, y_g)\) on \( C \), we consider the integrals
\[
u = (u_1, u_2, \ldots, u_g)
\]
\[
= \int_{\infty}^{(x_1, y_1)} \omega + \int_{\infty}^{(x_2, y_2)} \omega + \cdots + \int_{\infty}^{(x_g, y_g)} \omega,
\]
where
\[
\omega = (\omega_1, \omega_2, \ldots, \omega_g).
\]
Let \( t = x^g/y \). This is a local parameter at \( \infty \). We see easily that the \( x \) and \( y \), and the \( \omega_j \)'s, can be expanded as power series with respect to \( t \) with coefficients in \( \mathbb{Z}([\mu_j]) \). We shall choose differential forms \( \{\eta_1, \ldots, \eta_g\} \) such that \( \eta_j \) has poles of order \( 2j \) at \( \infty \) and no poles elsewhere by requiring that the 2-form, which is called Klein’s fundamental 2-form,
\[
\xi(x, y; z, w) = \omega_1(x, y) \frac{d}{dz} \left( \frac{1}{x - z} \right) \left| \frac{f(Z, y) - f(Z, w)}{y - w} \right|_{Z = z} \, dz
\]
\[- \sum_{j=1}^{g} \omega_j(x, y) \eta_j(z, w),
\]

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where \((x, y)\) and \((z, w)\) are variable coordinates on \(C\), has the following two properties:

1. it is symmetric, i.e. \(\xi(x, y; z, w) = \xi(z, w; x, y)\), and
2. it has an expansion of the form \(\xi(x, y; z, w) \in 1/(t_2 - t_1)^2 + \mathbb{Z}[\mu][[t_1, t_2]]\),

where \(t_1\) and \(t_2\) are local parameters defined above of \((x, y)\) and \((z, w)\), respectively.

These conditions determine \(\eta_1, \ldots, \eta_g\) up to the addition of linear combinations of \(\omega_j\)'s. We shall choose the simplest one in a certain sense. Since in this paper we treat mainly curves of genus 1 and genus 2, here we only give explicit versions of those forms in these cases. If \(g = 1\), we have

\[ \eta_1 = \frac{xdx}{fy(x, y)}. \]  (2.6)

If \(g = 2\), we have

\[ \eta_1 = \frac{x^2dx}{fy(x, y)}, \quad \eta_2 = \frac{(3x^3 + (\mu_1^2 + 2\mu_2)x^2 + (\mu_3\mu_1 + \mu_4)x + \mu_1y)dx}{fy(x, y)}. \]  (2.7)

Let \(L\) be the lattice in \(\mathbb{C}^g\) generated by the loop integrals of \(\omega\),

\[ L = \left\{ \oint \omega \right\}. \]  (2.8)

Then the Jacobian variety \(J\) of \(C\) is given by \(\mathbb{C}^g/L\). For \(k = 1, 2, \ldots, g\), the map

\[ \iota : \mathrm{Sym}^k(C) \to J \]  (2.9)

and

\[ (P_1, \ldots, P_k) \mapsto \left( \int_{\infty}^{P_1} \omega + \cdots + \int_{\infty}^{P_k} \omega \right) \mod L \]  (2.10)

is an injection outside a certain small dimensional (relative to \(k\)) subset. If \(k = g\), the map is surjective. We denote the image \(\iota(\mathrm{Sym}^k(C))\) by \(\Theta^{[k]}\). Let

\[ R_1 = \mathrm{rslt}_z(\mathrm{rslt}_y(f(x, y), f_x(x, y)), \mathrm{rslt}_y(f(x, y), f_y(x, y))), \]
\[ R_2 = \mathrm{rslt}_y(\mathrm{rslt}_x(f(x, y), f_x(x, y)), \mathrm{rslt}_x(f(x, y), f_y(x, y))) \]  (2.11)

and

\[ R_3 = \gcd(R_1, R_2), \]

where \(\mathrm{rslt}_z\) represents the resultant, namely the determinant of the Sylvester matrix with respect to the variable \(z\). Then \(R_3\) is a perfect square in the ring \(\mathbb{Z}[\mu_j]\). So, we define

\[ D = R_3^{1/2}, \]  (2.12)

where the square root is chosen as explained in equation (3.5) later.
3. The sigma function

(a) The definition of \( \sigma(u) \)

We define here an entire function \( \sigma(u) = \sigma(u_1, \ldots, u_g) \) on \( \mathbb{C}^g \) associated with \( C \), which we call the \( \sigma \)-function. As usual, let

\[
\alpha_i, \beta_j \quad (1 \leq i, j \leq g)
\]

be closed paths on \( C \) which generate \( H_1(C, \mathbb{Z}) \) such that their intersection numbers are \( \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0 \) and \( \alpha_i \cdot \beta_j = \delta_{ij} \).

Define the period matrices by

\[
\begin{bmatrix}
\omega' \omega''
\end{bmatrix} = \left[ \int_{\alpha_i} \omega_j \int_{\beta_i} \omega_j \right]_{i,j=1,\ldots,g} \quad \text{and} \quad
\begin{bmatrix}
\eta' \eta''
\end{bmatrix} = \left[ \int_{\alpha_i} \eta_j \int_{\beta_i} \eta_j \right]_{i,j=1,\ldots,g}.
\]

From equation (2.2), we see that the canonical divisor class of \( C \) is given by \( 4\infty \), and we are taking \( \infty \) as the base point of the Abel map (2.9) for \( k = g \). Hence the Riemann constant is an element of \( (\frac{1}{2}\mathbb{Z})^{2g} \) (see [8, corollary 3.11, p. 166]). Let

\[
\delta = \left[ \begin{array}{c}
\delta' \\
\delta''
\end{array} \right] \in \left( \frac{1}{2}\mathbb{Z} \right)^{2g},
\]

be the theta characteristic which gives the Riemann constant with respect to the base point \( \infty \) and the period matrix \( [\omega' \omega''] \). Note that we use \( \delta' \) and \( \delta'' \) as well as \( n \) in equation (3.4) as columns, to keep the notation simple. We define

\[
\sigma(u) = \sigma(u_1, \ldots, u_g)
\]

\[
= c \exp \left( -\frac{1}{2} u \eta' \omega'^{-1} u \right) \Theta[\delta](\omega'^{-1} u; \omega'^{-1} \omega'')
\]

\[
= c \exp \left( -\frac{1}{2} u \eta' \omega'^{-1} u \right) \sum_{n \in \mathbb{Z}^g} \exp \left[ 2\pi i \left\{ \frac{1}{2} t(n + \delta') \omega'^{-1} \omega''(n + \delta') + t(n + \delta')(\omega'^{-1} u + \delta'') \right\} \right],
\]

where

\[
c = \frac{1}{\sqrt{D}} \left( \frac{\pi^g}{|\omega'|} \right)^{1/2}
\]

with \( D \) from equation (2.12). Here the sign of a root of equation (3.5) is chosen so that the leading term of \( \sigma(u) \) is just the Schur–Weierstrass polynomial as defined in Buchstaber et al. [9]. However, we use the modified version from Ônishi [10, p. 711]. The series (3.4) converges because the imaginary part of \( \omega'^{-1} \omega'' \) is positive-definite.
In what follows, for a given \(u \in \mathbb{C}^g\), we denote by \(u'\) and \(u''\) the unique elements in \(\mathbb{R}^g\) such that
\[
u = u'\omega' + u''\omega''. \tag{3.6}
\]
Then for \(u, v \in \mathbb{C}^g\), and \(\ell (= \ell'\omega' + \ell''\omega'') \in A\), we define
\[
L(u, v) := u(\eta'\nu' + \eta''\nu'') \quad \text{and} \quad \chi(\ell) := \exp[\pi i(2(\ell'\delta' - \ell''\delta') + \ell''\ell'')] (\in \{1, -1\}). \tag{3.7}
\]
In this situation, the most important properties of \(s(u; M)\) are as follows.

**Lemma 3.1.** The function \(s(u)\) is an entire function which is independent of choice of the paths \(\alpha_j\) and \(\beta_j\) of equation (3.1). Let \(\kappa : \mathbb{C}^g \to J = \mathbb{C}^g/\Lambda\) be the natural map. For all \(u \in \mathbb{C}^g\), \(\ell \in \Lambda\) and we have
\[
s(u + \ell) = \chi(\ell)s(u) \exp L(u + \frac{1}{2}\ell, \ell) \tag{3.8}
\]
and
\[
\{u \in \mathbb{C}^g | s(u) = 0\} = \kappa^{-1}(\Theta^{[g-1]}), \tag{3.9}
\]
where \(\Theta^{[g-1]}\) is as defined following equation (2.9). Moreover, any entire function \(G\) on \(\mathbb{C}^g\) having the same property \(\{u \in \mathbb{C}^g | G(u) = 0\} = \kappa^{-1}(\Theta^{[g-1]})\) is \(s(u)\) times an entire function.

**Proof.** These are essentially classical results, and can be proved as in [7, lemma 4.1]. So we omit the proof. \(\blacksquare\)

**Lemma 3.2.** The coefficients in the expansion of the function \(s(u)\) at the origin are polynomials of \(\mu_j\)'s in (4.1) over the rationals \(\mathbb{Q}\).

**Proof.** See corollary 1 of Buchstaber et al. [11] and also [12]. \(\blacksquare\)

**Lemma 3.3.** Let \(\chi\) and \(L\) be defined as above. The space of entire functions \(\varphi(u)\) on \(\mathbb{C}^g\) satisfying
\[
\varphi(u + \ell) = \chi(\ell)\varphi(u) \exp L(u + \frac{1}{2}\ell, \ell)
\]
is one dimensional.

**Proof.** This is shown by the fact that the Pfaffian of the Riemann form attached to \(L(,)\) is 1 (see [13, p. 93, theorem 3.1]). \(\blacksquare\)

**Definition 3.4.** We introduce a weight by fixing the weight of \(\mu_j\) to be \(-j\), that of \(u_j\) to be \(2(g - j) + 1\), that of \(x\) to be \(-2\), and that of \(y\) to be \(-2g - 1\).

We can easily show that any formula on \(J\) is a sum of terms homogeneous in this weight. In many cases, the computation will be easier if the weight is taken into consideration. We can further simplify by subdividing the calculations according to the separate weights of the \(\mu_i\) and the \(u_j\) terms.
4. Elliptic and hyperelliptic curves of cyclotomic type

In this section, we describe clearly the curves which we shall consider. Let \( a \geq 2 \) and \( m \) be positive integers. We consider two type of curves according to whether \( am \) is odd or even. Namely, let

\[
\begin{align*}
 f(x, y) &= \begin{cases} 
 y^2 + \mu_{am} y - (x^{am} + \mu_{2a} x^{a(m-1)} + \mu_{4a} x^{a(m-2)} + \cdots + \mu_{2a(m-1)} x^a + \mu_{2am}) 
 & \quad \text{if } am \text{ is odd,} \\
 y^2 - (x^{am+1} + \mu_{2a} x^{a(m-1)+1} + \mu_{4a} x^{a(m-2)+1} + \cdots + \mu_{2a(m-1)} x^{a+1} + \mu_{2am} x) 
 & \quad \text{if } am \text{ is even.}
\end{cases}
\end{align*}
\]  

We consider the projective curve \( C \) defined by the affine equation

\[ f(x, y) = 0 \]

by adding the unique point \( \infty \) at infinity. The genus of \( C \) is

\[ g = \left\lfloor \frac{am}{2} \right\rfloor \]

if it is non-singular. We refer to the curve that is defined by the former equation as the \((2, a[m])\)-curve, and to that defined by the latter equation as the \((2, a[m]+1)\)-curve. Here, in the first entry, the number ‘2’ indicates that these curves are either elliptic or hyperelliptic curves, namely the power of \( y \) in the defining equation. We are aiming to treat any algebraic curves with a unique point at infinity, but we restrict ourselves here to elliptic and hyperelliptic curves and to present our idea simply.

Both the \((2, a[m])\)-curve and the \((2, a[m]+1)\)-curve are acted on by the group \( W_{2a} \) of \( 2a \)-th roots of 1 as automorphisms,

\[
[\xi]: (x, y) \mapsto (\xi^2 x, -y - \mu_{am}) \quad \text{for a } (2, a[m])\text{-curve}
\]

and

\[
[\xi]: (x, y) \mapsto (\xi^2 x, \xi y) \quad \text{for a } (2, a[m]+1)\text{-curve},
\]

where \( \xi = \exp(2\pi i/(2a)) \).

**Examples.** We give some examples here:

- The general \((2, 3[1])\)-curve is defined by \( y^2 + \mu_3 y = x^3 + \mu_6 \). This is acted on by \( W_6 \).
- The general \((2, 2[1]+1)\)-curve is defined by \( y^2 = x^3 + \mu_4 x \). This is acted on by \( W_4 \).
- The general \((2, 2[2]+1)\)-curve is defined by \( y^2 = x^5 + \mu_4 x^3 + \mu_8 x \), which is the famous Burnside curve. This is acted on by \( W_4 \).
- The general \((2, 5[1])\)-curve is defined by \( y^2 + \mu_5 y = x^5 + \mu_{10} \). This is acted on by \( W_{10} \).
- The general \((2, 4[1]+1)\)-curve is defined by \( y^2 = x^5 + \mu_8 x \). This is acted on by \( W_8 \).
- The general \((2, 3[2]+1)\)-curve is defined by \( y^2 = x^7 + \mu_6 x^4 + \mu_{12} x \). This is acted on by \( W_6 \).

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— The general \((2, 2[3]+1)\)-curve is defined by \(y^2 = x^7 + \mu_4 x^5 + \mu_8 x^3 + \mu_{12} x\). This is acted on by \(W_4\).
— The general \((2, 7[1])\)-curve is defined by \(y^2 + \mu_7 y = x^7 + \mu_{12} x\). This is acted on by \(W_{14}\).

In this paper, we suppose that \(m_j\) is 0, if it does not appear in the equation of \(C, f(x, y) = 0\).

(a) Complex multiplication of \(\sigma(u)\)

By the map (3.3), the action (4.2) of the group \(W_{2a}\) of \(2a\)-th roots of 1 on the curve \(C\) induce naturally an action on the space \(\mathbb{C}^g\). This is described explicitly as follows:

\[
\begin{align*}
&\left[\zeta\right](u_1, u_2, \ldots, u_g) = (-\zeta^2 u_1, -\zeta^4 u_2, \ldots, -\zeta^{2g} u_g) \quad \text{for a } (2, a[m])\text{-curve} \\
&\left[\zeta\right](u_1, u_2, \ldots, u_g) = (\zeta u_1, \zeta^3 u_2, \ldots, \zeta^{2g-1} u_g) \quad \text{for a } (2, a[m] + 1)\text{-curve}.
\end{align*}
\]

(4.3)

**Lemma 4.1.** Let \(C\) be a \((2, a[m])\)- or \((2, a[m]+1)\)-curve. Let \(\sigma(u)\) be the sigma function associated with \(C\) as above. Let

\[
w = \begin{cases} 
\frac{(am)^2 - 1}{8} & \text{if } am \text{ is odd}, \\
\frac{(am + 1)^2 - 1}{8} & \text{if } am \text{ is even}.
\end{cases}
\]

Then we see that

\[
\sigma([\zeta]u) = \begin{cases} 
(-1)^{m+1} \zeta^w \sigma(u) & \text{for a } (2, a[m])\text{-curve}, \\
(-1)^{m} \zeta^w \sigma(u) & \text{for a } (2, a[m] + 1)\text{-curve}.
\end{cases}
\]

**Proof.** Let \(A\) be the lattice in \(\mathbb{C}^g\) as above. Then, we have

\([\zeta]A = A\).

By lemma 3.3, there is a constant \(K\) such that

\(\sigma([\zeta]u) = K \sigma(u)\).

Because \([\zeta]^{2a}\) is the identity on \(\mathbb{C}^g\), we see

\(K^{2a} = 1\).

By looking at the leading terms of \(\sigma(u)\), we have the desired formula. \(\blacksquare\)

To define the \(\wp\)-functions from the sigma functions defined above, we let

\[
\wp_{jk}(u) = -\frac{\partial^2}{\partial u_j \partial u_k} \log \sigma(u), \quad \wp_{jkl}(u) = \frac{\partial}{\partial u_l} \wp_{jk}(u), \ldots.
\]

(4.4)

Then by equation (3.8), these functions are periodic with respect to the \(A\) of equation (2.8). If the genus of \(C\) is \(g = 1\), then, as usual, we write more classically \(\wp_{11}(u) = \wp(u)\) and \(\wp_{111}(u) = \wp'(u)\).
5. Genus 1

(a) Generalities
For completeness we start off with $C$ being the general elliptic curve defined by
\[ y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6. \]
Then the $\wp(u)$ defined by equation (4.4) satisfies
\[ \wp'(u) = 2y + \mu_1 x + \mu_3, \quad \wp(u) = x \]
when
\[ u = \int_{\infty}^{(x,y)} \frac{dx}{2y + \mu_1 x + \mu_3}, \]
and the $s(u), \wp(u)$ satisfy equation (1.5) in §1.

(b) Equianharmonic case
We now specialize $C$ to the curve $y^2 + \mu_3 y = x^3 + \mu_6$. Then we have
\[ (\wp')^2 = 4\wp^3 + 4(\mu_3^2 + \mu_6). \]
As usual, by putting $g_3 = -4(\mu_3^2 + \mu_6)$, we rewrite equation (5.1) as
\[ (\wp')^2 = 4\wp^3 - g_3. \]
This is usually called the equianharmonic case (see [14]). Let $\zeta = \exp(2\pi i/3)$. Then $\zeta^2 = -\zeta - 1$, and
\[ \sigma(\zeta u) = \zeta \sigma(u), \quad \wp(\zeta u) = \zeta \wp(u) \quad \text{and} \quad \wp'(\zeta u) = \wp'(u), \]
by equation (4.1).

The main results for the equianharmonic case are two novel addition formulae, one for two variables and one for three variables, as follows.

Proposition 5.1.
\[ -\frac{\sigma(u \pm v)\sigma(u \pm \zeta v)\sigma(u \pm \zeta^2 v)}{\sigma(u)^3\sigma(v)^3} \pm \frac{1}{2}(\wp'(u) \pm \wp'(v)) \]
and
\[ \frac{\sigma(u + v + w)\sigma(u + \zeta v + \zeta^2 w)\sigma(u + \zeta^2 v + \zeta w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3} \]
\[ = \frac{1}{4}(\wp'(u)\wp'(v) + \wp'(u)\wp'(w) + \wp'(v)\wp'(w)) \]
\[ - \frac{3}{4}(4\wp(u)\wp(v)\wp(w) - g_3). \]

Proof. We give two proofs of these results, the first based on straightforward manipulations of equations (1.5) and (1.6), and the second based on a pole argument. As we use both techniques in the genus 2 case, we give some detail here for completeness, and to aid understanding.
First proof. In equation (1.5), put \( v = \zeta u \) and use equation (5.3) to get
\[
\frac{\sigma((1 - \zeta)u)}{\sigma(u)^3} = (1 - \zeta)\wp(u).
\]

Next put \( w = \zeta u \) in equation (1.6) and use the above result, the fact that \( \sigma \) is an odd function of its argument, and equation (5.3) to give equation (5.4).

Now consider equation (5.5). Firstly, we make use of equation (1.6) by taking \( (u, v) \) as \( (v, w), (\zeta v, \zeta^2 w), (\zeta^2 v, \zeta w) \) in turn. Multiplying all three versions together, we get
\[
\prod_{j=0}^{j=2} \sigma(u - \zeta^{2j}w)\sigma(\zeta^jv - \zeta^{2j}w)\sigma(u - \zeta^jv)\sigma(u + \zeta^jv + \zeta^{2j}w)
\]
\[
= -\frac{1}{8} \begin{vmatrix}
\wp(u) & \wp'(u) \\
\wp(v) & \wp'(v) \\
\wp(w) & \wp'(w)
\end{vmatrix}
\begin{vmatrix}
\wp(u) & \wp'(u) \\
\wp(\zeta v) & \wp'(\zeta v) \\
\wp(\zeta^2 w) & \wp'(\zeta^2 w)
\end{vmatrix}
\begin{vmatrix}
\wp(u) & \wp'(u) \\
\wp(\zeta^2 v) & \wp'(\zeta^2 v) \\
\wp(\zeta w) & \wp'(\zeta w)
\end{vmatrix}.
\]
(5.6)

Now note the denominator of the l.h.s. simplifies using equation (5.3) to
\[
\sigma(u)^9\sigma(v)^9\sigma(w)^9.
\]

Consider now the r.h.s. Multiply this out, simplify using equation (5.3), then replace all occurrences of \( \wp(\cdot)^3 \) with \( \frac{1}{4}(\wp'(\cdot)^2 + g_3) \). Then factor (Maple is useful for this calculation!). The result is
\[
-\frac{1}{32}(\wp'(u) - \wp'(w))(\wp'(v) - \wp'(w))(\wp'(v) - \wp'(u))
\]
\[
\times (\wp'(v)\wp'(u) + \wp'(v)\wp'(w) + \wp'(w)\wp'(u) + 3g_3 - 12\wp(u)\wp(v)\wp(w)).
\]

Now apply equation (5.4) with the minus sign and with \( (u, v) = (u, v), (u, w), (v, w) \) in turn to the numerator of the l.h.s. of equation (5.6). Finally, cancelling common factors, we have equation (5.5).

Second proof. Both sides of equation (5.4) are elliptic functions, by equation (3.8). Fixing \( u \) and regarding both sides of equation (5.4) as a function of \( v \), we see that both sides have the same poles and zeroes with the same order at

\[ v = 0 \text{ of order } -3, \quad u \text{ of order } 1, \quad \zeta u \text{ of order } 1, \quad \zeta^2 u \text{ of order } 1, \]

and with no poles or zeroes elsewhere, because of equation (5.3). Hence the two sides coincide up to a non-zero multiplicative constant. Looking at the coefficients of Laurent expansion with respect to \( v \), we see that the two sides are equal.

Now consider equation (5.5). Recall that, in this case, \( J \) is isomorphic to \( C \), and the space of functions on \( J \) having a pole only at \( 0 \) of order at most \( n \) is given by
\[
\Gamma(J, O(n \cdot \emptyset)) = \begin{cases} 
C & \text{if } n = 0 \text{ or } 1, \\
C \oplus C \wp(u) & \text{if } n = 2, \\
\Gamma(J, O((n - 2) \cdot \emptyset)) \oplus C \wp^{(n-1)}(u) & \text{if } n \geq 3,
\end{cases}
\]

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where we denote $\Theta^{[0]}$ by $\circ$, that is, the origin of $J$. The function $\sigma(u)$ is expanded as

$$\sigma(u) = u - \frac{1}{120} (\mu_3^2 + \mu_6) u^7 + O(u^{13}).$$

The l.h.s. of equation (5.5) is invariant under $u \leftrightarrow \zeta u$, $v \leftrightarrow \zeta v$ and $w \leftrightarrow \zeta w$, and all exchanges of $u$, $v$ and $w$. It is an even function under $u \leftrightarrow -u$, $v \leftrightarrow -v$ and $w \leftrightarrow -w$ simultaneously. Moreover, it is of homogeneous weight $-6$. Hence, it must be of the form

$$a (\wp'(u)\wp'(v) + \wp'(u)\wp'(w) + \wp'(v)\wp'(w)) + b \wp(u)\wp(v)\wp(w) + c \mu_6,$$

where $a$, $b$ and $c$ are constants independent of $g_3$. Then, by using the first few terms of the power-series expansion with respect to $u$ or $v$, and by balancing the two sides, we determine these coefficients to obtain equation (5.4).

**Remark 5.2.** In the ‘rational’ case, $\mu_3 = \mu_6 = 0$, $\sigma(u) = u$, $\wp(u) = 1/u^2$, the formula (5.5) becomes the well-known identity

$$(a + b + c)(a + \zeta b + \zeta^2 c)(a + \zeta^2 b + \zeta c) = a^3 + b^3 + c^3 - 3abc.$$

**Remark 5.3.** Formula (5.5) turns up as a special case in the study of exceptional completely decomposable quasi-linear (CDQL) webs globally defined on compact complex surfaces [15].

(c) **Lemniscate case (the $(2, 2[1] + 1)$-curve)**

For the curve $y^2 = x^3 + \mu_4 x$, we have

$$\begin{align*}
\wp'(u) &= 2y, \\
\wp(u) &= x
\end{align*}$$

if $u = \int_{\infty}^{(x,y)} \frac{dx}{2y + \mu_3},$ (5.7)

and

$$(\wp')^2 = 4\wp^3 + 4\mu_4 \wp.$$ (5.8)

As usual by putting $\mu_4 = -g_2/4$, we rewrite equation (5.8) as

$$(\wp')^2 = 4\wp^3 - g_2 \wp.$$ (5.9)

This is usually called the **lemniscate case** (see [14]). By equation (4.1), we see that, for the $\wp$ satisfying equation (5.9),

$$\sigma(iu) = i\sigma(u), \quad \wp(iu) = -\wp(u), \quad \wp'(iu) = i\wp'(u).$$ (5.10)

In this case, we have from equation (1.5) with $v \to iv$ and equation (5.10) that

$$\frac{-\sigma(u + iv)\sigma(u - iv)}{\sigma(u)^2\sigma(v)^2} = \wp(u) + \wp(v).$$ (5.11)

Generalizing equation (1.5) and this, the main results for the lemniscate case are the following addition formulae.
Proposition 5.4.

\[
\frac{\sigma(u + v + w)\sigma(u + v - w)\sigma(u - v + w)\sigma(u - v - w)}{\sigma(u)^4\sigma(v)^4\sigma(w)^4}
= \frac{1}{16} g_2^2 + \frac{1}{2} g_2 (\varphi(v)\varphi(w) + \varphi(u)\varphi(w) + \varphi(u)\varphi(v))
+ \varphi(u)^2\varphi(v)^2 + \varphi(u)^2\varphi(w)^2 + \varphi(w)^2\varphi(v)^2
- 2\varphi(u)\varphi(v)\varphi(w)(\varphi(u) + \varphi(v) + \varphi(w)) \equiv E_0(u, v, w),
\]
(5.12)

and

\[
\frac{\sigma(u + iv + w)\sigma(u + iv - w)\sigma(u - iv + w)\sigma(u - iv - w)}{\sigma(u)^4\sigma(v)^4\sigma(w)^4}
= \frac{1}{16} g_2^2 + \frac{1}{2} g_2 (\varphi(u)\varphi(w) - \varphi(v)\varphi(w) - \varphi(u)\varphi(v))
+ \varphi(u)^2\varphi(v)^2 + \varphi(u)^2\varphi(w)^2 + \varphi(w)^2\varphi(v)^2
+ 2\varphi(v)\varphi(u)\varphi(w)(\varphi(w) - \varphi(v) + \varphi(u)) \equiv E_1(u, w; v).
\]
(5.13)

By symmetry we have two further formulae under the transformations \(u \rightarrow iu\) and \(w \rightarrow iw\), and finally we have the 16-term formula

\[
\prod_{n,m=0,1,2,3} \sigma(u + i^n v + i^m w) = E_0(u, v, w)E_1(u, v; w)E_1(u, w; v)E_1(v, w; u).
\]

Proof. The four-term formulae are constructed from products of the relation (1.6) in the same way as in §5b. Similar relations for other permutations of terms can also be constructed.

Remark 5.5. The addition formulae in this section can be proved by another method, as in §6b.

6. Genus 2

In this section, we treat curves of genus 2. So, \(am = 4\) or 5.

(a) Basis of spaces of Abelian functions

Using the functions in equation (4.4), we denote

\[
\Delta = \det[\varphi_{ij}] = \varphi_{11}\varphi_{22} - \varphi_{12}^2,
\]
\[
\Delta_j = \frac{\partial}{\partial u_j}\Delta, \quad \Delta_{ij} = \frac{\partial^2}{\partial u_j\partial u_i}\Delta, \ldots.
\]
(6.1)
Lemma 6.1. Let \( n \geq 2 \) be an integer. The space \( \Gamma(J, \mathcal{O}(n\Theta^{[1]})) \) of the functions having no pole outside \( \Theta^{[1]} \) and at most of order \( n \) on \( \Theta^{[1]} \) is given recursively by

\[
\Gamma(J, \mathcal{O}(2\Theta^{[1]})) = \mathbb{C}1 \oplus \mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{12} \oplus \mathbb{C}\wp_{22},
\]

\[
\Gamma(J, \mathcal{O}((n+1)\Theta^{[1]})) = \frac{\partial}{\partial u_1} \Gamma(J, \mathcal{O}(n\Theta^{[1]})) \cup \frac{\partial}{\partial u_2} \Gamma(J, \mathcal{O}(n\Theta^{[1]})) .
\]

In particular,

\[
\Gamma(J, \mathcal{O}(3\Theta^{[1]})) = \Gamma(J, \mathcal{O}(2\Theta^{[2]})) \oplus \mathbb{C}\wp_{111} \oplus \mathbb{C}\wp_{112} \oplus \mathbb{C}\wp_{122} \oplus \mathbb{C}\wp_{222} \oplus \mathbb{C}\Delta,
\]

\[
\Gamma(J, \mathcal{O}(4\Theta^{[1]})) = \Gamma(J, \mathcal{O}(3\Theta^{[1]})) \oplus \mathbb{C}\wp_{1111} \oplus \mathbb{C}\wp_{1112} \oplus \mathbb{C}\wp_{1122} \oplus \mathbb{C}\wp_{1222}
\]

\[
\quad \oplus \mathbb{C}\wp_{2222} \oplus \mathbb{C}\Delta_1 \oplus \mathbb{C}\Delta_2,
\]

and

\[
\Gamma(J, \mathcal{O}(5\Theta^{[1]})) = \Gamma(J, \mathcal{O}(4\Theta^{[1]})) \oplus \mathbb{C}\wp_{11111} \oplus \mathbb{C}\wp_{11112} \oplus \mathbb{C}\wp_{11122} \oplus \mathbb{C}\wp_{11222}
\]

\[
\quad \oplus \mathbb{C}\wp_{12222} \oplus \mathbb{C}\wp_{22222} \oplus \mathbb{C}\Delta_{11} \oplus \mathbb{C}\Delta_{12} \oplus \mathbb{C}\Delta_{22}.
\]

For the convenience of the reader, we list their weights below:

<table>
<thead>
<tr>
<th>function</th>
<th>( \wp_{11} )</th>
<th>( \wp_{12} )</th>
<th>( \wp_{22} )</th>
<th>( \wp_{111} )</th>
<th>( \wp_{112} )</th>
<th>( \wp_{122} )</th>
<th>( \wp_{222} )</th>
<th>( \Delta )</th>
<th>( \Delta_1 )</th>
<th>( \Delta_2 )</th>
<th>( \wp_{1111} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>weight</td>
<td>-6</td>
<td>-4</td>
<td>-2</td>
<td>-9</td>
<td>-7</td>
<td>-5</td>
<td>-3</td>
<td>-8</td>
<td>-11</td>
<td>-9</td>
<td>-12</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccccccccc}
\wp_{1112} & \wp_{1122} & \wp_{1222} & \wp_{2222} & \wp_{11111} & \wp_{11112} & \wp_{11122} & \wp_{11222} & \wp_{12222} & \wp_{22222} \\
\end{array}
\]

\[\text{Proof. This is shown in Cho & Nakayashiki [16]; see, especially, the example for } g = 2 \text{ in } \S 9 \text{ of that paper.}\]

\[\blacklozenge\]  

(b) Equipentamic case

We propose the name ‘equipentamic’ for the \((2,5[1])\)-curve \( C \) defined by

\[ f(x, y) = y^2 + \mu_5 y - (x^5 + \mu_{10}). \]

This is isomorphic to the curve of type II in Bolza [17]. In this case, we have

\[ \wp_{11}([-\zeta]u) = \zeta^3 \wp_{11}(u), \quad \wp_{12}([-\zeta]u) = \zeta^2 \wp_{12}(u), \quad \wp_{22}([-\zeta]u) = \zeta \wp_{22}(u), \]

\[ \wp_{111}([-\zeta]u) = -\zeta^2 \wp_{111}(u), \quad \ldots \]

for \( \zeta = \zeta_5 = \exp(2\pi i/5), \) because of lemma 4.1.
Proposition 6.2. We have
\[
\sigma(u + v)\sigma(u + [\zeta]v)\sigma(u + [\zeta^2]v)\sigma(u + [\zeta^3]v)\sigma(u + [\zeta^4]v)
\]
\[
\frac{\sigma(u)^5\sigma(v)^5}{\sigma(u)\sigma(v)}
\]
\[
= \frac{5}{18}\{\varphi_{122}(u)\varphi_{1112}(v) + \varphi(v)\varphi_{1112}(u)\}
- \frac{5}{144}\{\varphi_{122}(u)\Delta_{22}(v) + \varphi_{122}(v)\Delta_{22}(u)\}
\]
\[
- \frac{1}{144}\{\varphi_{122}(u)\varphi_{22222}(v) + \varphi_{1112}(v)\varphi_{22222}(u)\}
- \frac{1}{576}\{\Delta_{22}(u)\varphi_{22222}(v) + \Delta_{22}(v)\varphi_{22222}(u)\}
\frac{1}{24}(\mu_10 + \frac{1}{3}\mu_5^2)[\varphi_{22222}(u) + \varphi_{22222}(v)],
\]
(6.2)

where \(\zeta = \exp(2\pi i/5)\), and \([\zeta]v = [\zeta](v_1, v_2) = (\zeta v_1, \zeta^2 v_2)\). Alternatively, the r.h.s. of the relation above can be written as
\[
= \frac{1}{4}\varphi_{222}(u)\varphi_{222}(v)(\varphi_{12}(v)\varphi_{11}(v))^2 \varphi_{22}(v)^2 \varphi_{11}(v)^2 \varphi_{112}(v)^2
\]
\[
\frac{1}{2}\varphi_{122}(u)\varphi_{12}(v)\varphi_{11}(v) + \varphi_{222}(v)\varphi_{12}(v)^2 \varphi_{11}(v)^2 \varphi_{112}(v)^2
\frac{1}{2}\varphi_{11}(u)\varphi_{111}(u) + \mu_{10}\varphi_{22}(u)\varphi_{222}(u) + (u \leftrightarrow v),
\]
(6.3)

Proof. The l.h.s. is an Abelian function of weight \(-15\) as a function of \(u\) (resp. \(v\)) because of equation (3.8), and has poles only along \(\Theta^{[1]}\) of order 5 by the last statement in lemma 3.1 and equation (3.9). It is also invariant under \(u \leftrightarrow [\zeta]u\), \(v \leftrightarrow [\zeta]v\) and \(u \leftrightarrow v\). According to lemma 3.2, it must be a linear combination of homogeneous weight \(-15\) of terms of the form
\[
c_j(X_i(u)Y_j(v) + X_j(v)Y_i(u)),
\]
(6.4)

where \(X_i(u)\) and \(Y_j(u)\) are members of the list just below lemma 6.1, with coefficients \(c_j\) being polynomial of \(\mu_{10}\) over the rationals. So, there are six possible terms of equation (6.4), namely those appearing on the r.h.s. of equation (6.2). Its coefficients follow from expanding both sides in power series in \(u\) and \(v\) with the first several terms after multiplying \(\sigma(u)^5\sigma(v)^5\) on both sides, by computer calculation using Maple. To prove equation (6.2), we use the known expansions of the four-index \(\varphi_{ijkl}\) relations in this case
\[
\varphi_{2222} = 6\varphi_{22}^2 + 4\varphi_{12}, \quad \varphi_{1222} = 6\varphi_{22}\varphi_{12} - 2\varphi_{11},
\]
\[
\varphi_{1122} = 4\varphi_{12}^2 + 2\varphi_{11}\varphi_{22}, \quad \varphi_{1112} = 6\varphi_{12}\varphi_{11} - 4\mu_{10},
\]
\[
\varphi_{1111} = 6\varphi_{11}^2 - 12\varphi_{22}\mu_{10},
\]
together with the derivatives of these equations with respect to the \(u_i\). In addition we use the known quadratic three-index relations
\[
\varphi_{222} = 4\varphi_{22}^3 + 4\varphi_{11} + 4\varphi_{22}\varphi_{12},
\]
\[
\varphi_{122}\varphi_{222} = s - 2\varphi_{22}\varphi_{12} + 4\varphi_{22}^2 + 1 + 2\varphi_{12}^2,
\]
\[
\cdots = \cdots
\]
These substitutions lead eventually to equation (6.3).
Remark 6.3. Other types of addition formulae exist; for example, for \( u, v, w \in \mathbb{C}^2 \),

\[
\sigma(u + v + w)\sigma(u + \zeta v + \zeta^2 w)\sigma(u + \zeta^2 v + \zeta^4 w) \\
\times \sigma(u + \zeta^3 v + \zeta w)\sigma(u + \zeta^4 v + \zeta^3 w) / (\sigma(u)^5 \sigma(v)^5 \sigma(w)^5)
\]

is expressed in terms of \( \wp \)-functions and their derivatives. But it would require a big calculation to get the explicit expression.

(c) The \((2, 2[2] + 1)\)-curve

Here we treat the \((2, 2[2] + 1)\)-curve \( C \) given by

\[
f(x, y) = y^2 - (x^5 + \mu_4 x^3 + \mu_8 x).
\]

This is isomorphic to the curve of type VI in Bolza [17]. The result here is not so interesting because it is essentially a product of two of equation (1.9) in §1. For completeness, we describe the result here in compressed form. This curve \( C \) has the automorphism

\[
[i] : (x, y) \mapsto (-x, iy), \quad \left( \frac{dx}{2y}, \frac{x dx}{2y} \right) \mapsto \left( \frac{dx}{2y}, -i \frac{x dx}{2y} \right).
\]

We see that

\[
[i]^2 = [-1] : (x, y) \mapsto (x, -y),
\]

\[
[i] : (u_1, u_2) \mapsto (iu_1, -iu_2), \quad [i]^2(u_1, u_2) \mapsto (-u_1, -u_2),
\]

\[
\wp_{11}([i]u) = -\wp(u), \quad \wp_{12}([i]u) = \wp_{12}(u),
\]

and

\[
\wp_{22}([i]u) = -\wp(u), \quad \wp_{111}([i]u) = i \wp_{12}(u), \ldots
\]

We trivially have the following formula:

\[
\frac{\sigma(u + v)\sigma(u + [i]v)\sigma(u + [i]^2 v)\sigma(u + [i]^3 v)}{\sigma(u)^4 \sigma(v)^4}
\]

\[
= (\wp_{11}(u) - \wp_{11}(v) + \wp_{12}(u)\wp_{22}(v) - \wp_{22}(u)\wp_{12}(v))
\]

\[
\times (\wp_{11}(u) + \wp_{11}(v) - \wp_{12}(u)\wp_{22}(v) - \wp_{12}(v)\wp_{22}(u)).
\]

This is not new because \([i]^2\) is no other than the standard involution \( u \mapsto -u \). We remark further on this in §7.
Before closing this section we remark that, if the genus 2 curve under consideration covers a genus 1 curve, it is possible to write the corresponding genus 2 \( \theta \) functions as products of genus 1 \( \theta \) functions \([18,19]\). It should be possible to expand the \( \wp \) formulae in such cases in a similar way.

7. Higher genus and non-hyperelliptic curves

These new results described above for genus 1 and 2 were inspired by results for the trigonal genus 3 case \([7]\). In this paper, we derived a three-term two-variable addition formula which generalizes equation (5.4) to the purely trigonal case of genus 3

\[
f(x, y) = y^3 - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}).
\] (7.1)

We have also recently proved the existence of a similar three-term two-variable addition formula for the purely trigonal case of genus 4 \([20]\)

\[
f(x, y) = y^3 - (x^5 + \mu_3 x^4 + \mu_6 x^3 + \cdots + \mu_{15}).
\] (7.2)

In addition we showed in that paper the existence of a three-term three-variable addition formula for equations (7.1) and (7.2) which generalize equation (5.5).

(a) The three-term three-variable addition formula for the \((3,4)\) purely trigonal curve

The formula in this case generalizes the three-variable formula given in equation (5.5) for the genus 1 case. This type of formula is expected to be quite complicated, as the family of members in the corresponding natural basis in each case is large.

For example, in the case of equation (7.1), we have an addition formula of the type

\[
\sigma(u + v + w)\sigma(u + [\zeta]v + [\zeta^2]w)\sigma(u + [\zeta^2]v + [\zeta]w)
\]

\[
\sigma(u)^3\sigma(v)^3\sigma(w)^3
\]

\[
= \sum_{i=1}^{27} \sum_{j=1}^{27} \sum_{k=1}^{27} c_{ijk} U_i(u) V_j(v) W_k(w),
\] (7.3)

where the functions \( U_i, V_j \) and \( W_k \) are all basis functions for the space \( \Gamma(J, O(3\Theta^{12})) \). These are enumerated in \([7]\). We can write the r.h.s. as

\[
C_{30} + C_{27} + \cdots + C_0,
\]

where \( C_n \) has weight \(-n\) in \( u, v \) and \( w \) combined, and weight \( n - 30 \) in the \( \lambda_i, i = 3, \ldots, 0 \). Both the l.h.s. and the r.h.s. are symmetric under all permutations in \((u, v, w)\). So far only the terms from \( C_{30} \) up to \( C_{18} \) have been calculated, and a full
description of this formula will be given elsewhere. To illustrate the complexity 
we give here the formula for \( C_{30} \)
\[
C_{30} = \frac{1}{6} \varphi_{13}(u) \partial_3 Q_{1333}(v) \varphi_{111}(w) + \frac{1}{18} \partial_3 Q_{13333}(u) \partial_3 Q_{13333}(v) \varphi^{[22]}(w) \\
- \frac{3}{2} \varphi_{12}(u) \varphi_{11}(v) \varphi^{[23]}(w) - \varphi_{13}(u) \varphi^{[22]}(v) \varphi^{[22]}(w) \\
+ \frac{1}{8} \varphi_{[12]}(u) \partial_3 Q_{13333}(v) \varphi_{112}(w) - \frac{1}{2} \varphi_{11}(u) \varphi_{11}(v) \varphi_{11}(w) \\
- \frac{1}{16} \varphi_{222}(u) \varphi^{[22]}(v) \varphi_{112}(w) - \frac{3}{8} \varphi_{222}(u) \varphi^{[23]}(v) \varphi^{[23]}(w) \\
- \frac{1}{8} \varphi_{[11]}(u) \varphi^{[22]}(v) \varphi^{[22]}(w) + \frac{3}{16} \varphi_{122}(u) \varphi_{122}(v) \varphi^{[22]}(w) \\
- \frac{1}{8} \varphi^{[13]}(u) \varphi^{[13]}(v) \varphi^{[13]}(w) + \frac{3}{8} \varphi_{123}(u) \varphi_{123}(v) \varphi^{[33]}(w) \\
- \frac{1}{8} Q_{13333}(u) Q_{13333}(v) \varphi^{[33]}(w) - \frac{3}{8} \varphi_{333}(u) \varphi^{[33]}(v) \varphi^{[33]}(w) \\
- \frac{1}{8} \varphi_{1333}(u) \partial_2 Q_{13333}(v) \varphi^{[23]}(w) + \frac{1}{4} \varphi_{1333}(u) \varphi_{111}(v) \partial_1 Q_{13333}(w) \\
- \frac{1}{8} \varphi_{1333}(u) \varphi^{[13]}(v) \partial_1 Q_{13333}(w) - \frac{3}{8} \varphi_{223}(u) \varphi_{113}(v) \varphi^{[33]}(w) \\
+ \frac{1}{4} \varphi_{[12]}(u) \varphi^{[12]}(v) \varphi^{[22]}(w) - \frac{1}{4} \varphi_{13} \varphi_{122}(v) \varphi_{111}(w) + \frac{1}{8} \varphi_{111}(u) \varphi_{111}(v) \\
+ \frac{3}{8} Q_{1333}(u) \varphi_{113}(u) \varphi_{113}(w) + \text{all permutations of } (u, v, w).
\]

(b) Other formulae for higher genus curves

There are a number of other addition formulae waiting to be computed in 
explicit form for special cases of \( g > 2 \) hyperelliptic curves and for other special 
cases of trigonal curves and curves with higher gonal numbers. For instance, let 
\[
\begin{align*}
\varphi(x, y) &= \left\{ \begin{array}{ll}
y^3 - (x^am + \mu_3x^{a(m-1)} + \mu_6x^{a(m-2)}) \\
&\quad + \cdots + \mu_3x^{a(m+1)}x^a + \mu_3am \\
y^3 - (x^am+1 + \mu_3x^{a(m-1)+1} + \mu_6x^{a(m-2)+1}) \\
&\quad + \cdots + \mu_3x^{a(m+2)}x^a + \mu_3amx \\
\end{array} \right. \\
\text{if }\gcd(am, 3) = 1, \\
\text{if }\gcd(am, 3) = 3;
\end{align*}
\]
and \( C \) be the curve defined by \( f(x, y) = 0 \). This should be called the \((3, a[m])\)-curve or \((3, a[m] + 1)\)-curve, respectively. Let \( \zeta = \exp(2\pi i/(3a)) \). Then \( C \) has automorphisms 
\[
[\zeta] : \left\{ \begin{array}{ll}
(x, y) \mapsto (\zeta^3x, \zeta^ay) & \text{if }\gcd(am, 3) = 1, \\
(x, y) \mapsto (\zeta^3x, \zeta^y) & \text{if }\gcd(am, 3) = 3.
\end{array} \right. \tag{7.4}
\]
Namely, it is acted on by \( W_{3a} \), the group of \( 3a \)-th roots of 1. Associated with 
such a curve \( C \), we will have various multi-term addition formulae.

We shall give a remark by an explicit example. For the \((3, 4[1])\)-curve \( y^3 = x^4 + \mu_{12} \) and the automorphism 
\[
[i] : (x, y) \mapsto (-ix, y), \tag{7.5}
\]
the action \([i]^2\) is different from the standard involution \( u \mapsto -u \) on the variable 
space of associated Abelian functions. The formula that expresses 
\[
\frac{\sigma(u + v)\sigma(u + [i]v)\sigma(u + [i]^2v)\sigma(u + [i]^3v)}{\sigma(u)^4\sigma(v)^4} \tag{7.6}
\]
in terms of \( \varphi \)-functions seems to be interesting.
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References