Exactly and quasi-exactly solvable ‘discrete’ quantum mechanics

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A brief introduction to discrete quantum mechanics is given together with the main results on various exactly solvable systems. Namely, the intertwining relations, shape invariance, Heisenberg operator solutions, annihilation/creation operators and dynamical symmetry algebras, including the \( q \)-oscillator algebra and the Askey–Wilson algebra. A simple recipe to construct exactly and quasi-exactly solvable (QES) Hamiltonians in one-dimensional ‘discrete’ quantum mechanics is presented. It reproduces all the known Hamiltonians whose eigenfunctions consist of the Askey scheme of hypergeometric orthogonal polynomials of a continuous or a discrete variable. Several new exactly and QES Hamiltonians are constructed. The sinusoidal coordinate plays an essential role.

Keywords: difference Schrödinger equation; exact solvability; quasi-exact solvability; shape invariance; Heisenberg operator solutions; Askey–Wilson algebra

1. Introduction

This is an expanded version of my talk at the Robin Bullough Memorial Symposium, 11 June 2009, Manchester. I met Robin Bullough in the Niels Bohr Institute, Copenhagen in 1979, after finishing two papers on a unified geometric theory of soliton equations [1,2]. Robin was my mentor and navigator when I started cruising into an unknown sea of integrable systems and soliton theory. Thirty years ago, the subject was young and rough but it was very fertile and full of good promises. He gave me good suggestions and comments on my third and fourth papers on geometric approach to soliton equations [3,4]. We produced joint papers, one on the geometry of the AKNS-ZS (Ablowitz–Kaup–Newell–Segur–Zhakharov–Shabata) scheme [5] and the other on non-local conservation laws of the sine-Gordon equation [6]. Collaboration was a fruitfull experience and I learned a lot from him and got to know many friends and experts through him. As a result, I am still working in the general area of integrable systems, and in particular, the general mechanism of integrability and solvability as a means to understand nonlinear interactions beyond perturbation, rather than some specific properties of certain equations.

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One contribution of 13 to a Theme Issue ‘Nonlinear phenomena, optical and quantum solitons’.
Here, I present an overview of the problems I have been working on over the past twelve years: exactly and quasi-exactly solvable (QES) quantum particle dynamics. Owing to the lack of space, I have to concentrate on the most basic but most interesting part of the problems: the quantum mechanics (QM) of one degree of freedom. The subjects to be covered in this article include: general setting of discrete quantum mechanics (dQM) with pure imaginary shifts and real shifts, shape invariance, exact solvability in the Heisenberg picture as well as in the Schrödinger picture, dynamical symmetry algebras, unified theory of exactly and QES dQM. Let me emphasize that the basic structure of integrability and methods of solutions are shared by ordinary quantum mechanics (oQM) and dQM. For each specific subject, I will present basic formulas common to oQM and dQM, and then provide explicit forms of various quantities which are different in oQM or the two types of dQM.

One underlying motivation of this series of works is to inaugurate a new theory for the Askey scheme of hypergeometric orthogonal polynomials [7–10] within the framework of QM. The abundant concepts and methods of QM accumulated over 80 years after the construction of QM would be available for the research of orthogonal polynomials. These include Crum’s [11] theory of the associated Hamiltonians together with the determinant expressions of the eigenfunctions, the factorization method [12] or the method of supersymmetric QM [13] combined with the intertwining method, shape invariance [14] for complete solvability in the Schrödinger picture, solvability in the Heisenberg picture in terms of the annihilation/creation operators leading to Rodrigues type formulas for the eigenfunctions, the concepts of dynamical symmetry algebras and coherent states. So far this project has been extremely successful. Every Hamiltonian system describing the hypergeometric orthogonal polynomial is shown to be shape invariant thus exactly solvable in the Schrödinger picture [15–17]. As a by-product, a universal Rodrigues type formula (2.49) for all the hypergeometric orthogonal polynomials is obtained. It also turned out that all these Hamiltonian systems describing the hypergeometric orthogonal polynomials are exactly solvable in the Heisenberg picture [18,19]. Namely, the Heisenberg equation for the sinusoidal coordinate \( \eta(x) \) is exactly solvable. The positive/negative energy parts of the exact Heisenberg solution for \( \eta(x) \) provide the annihilation/creation operators, which together with the Hamiltonian constitute the dynamical symmetry algebra of the exactly solvable system. All the eigenfunctions are obtained algebraically by multiple applications of the creation operator on the groundstate eigenfunction. The q-oscillator algebra is obtained as a dynamical symmetry algebra of the system for the q-Hermite polynomials [20]. The relationship dictated by the annihilation/creation operators can be interpreted as quantum mechanical disguise of the three term recurrence relations of the orthogonal polynomials. Various coherent states as the eigenvectors of the annihilation operators are explicitly obtained [17–19]. This project also opens a rich frontier for various deformed polynomials in terms of modification of Crum’s theorem in oQM [21,22] and for dQM [23]. The problem of determining the deformed weight functions can be solved without recourse to the cumbersome moment problems. One recent highlight of the project is the discovery of infinitely many exceptional \((X_\ell)\) orthogonal polynomials related to the Laguerre, Jacobi, continuous Hahn, Wilson and Askey–Wilson polynomials [23–29]. These polynomials are exceptional in the sense that they start at degree \( \ell (\ell \geq 1) \) instead of a degree zero constant term.
They form a complete set but they are not constrained by any generalization of Bochner’s theorem [30], which states that the orthogonal polynomials satisfying a second-order differential equation must be one of the classical polynomials, the Hermite, Laguerre, Jacobi and Bessel.

This paper is organized as follows: In §2, I explain various concepts and methods in dQM together with three explicit examples for each of oQM, pure imaginary shifts QM and real shifts QM, as the basic structure is common in all the three cases. These are: (i) the Hamiltonians (2.7)–(2.10) and the ‘Schrödinger equations’ for the eigenpolynomials (2.24) and (2.25) together with the groundstate eigenfunctions (2.29)–(2.31) as the orthogonality weight functions (2.27) and (2.28). (ii) The intertwining relations connecting various Hamiltonians (2.33), (2.39) and (2.40), and eigenfunctions (2.36), (2.42) and (2.44). This is a brief summary of Crum’s theory [11] in our language. Its deformation à la Krein–Adler [21,22] is also mentioned. (iii) Shape invariance (2.47) and the full energy spectrum (2.48) and the unified Rodrigues type formula (2.49) for all the eigenfunctions. (iv) Solvability in the Heisenberg picture (2.67) and the closure relation (2.65). The formulas for the creation/annihilation operators (2.70) and (2.71) are given. (v) Dual closure relation (2.81) together with its connection with the Askey–Wilson algebra. In §3, I provide the essence of the unified theory of exact and quasi-exact solvability in dQM. All the known examples of the shape invariant and thus exactly solvable dQM can be reproduced in this way. It also generates several new exactly solvable Hamiltonians as well as QES ones.

2. ‘Discrete’ quantum mechanics

Throughout this paper, we consider one-dimensional QM. The dynamical variables are the coordinate \( x \) and its conjugate momentum \( p \), which is realized as a differential operator \( p = -i(d/dx) \equiv -i\partial_x \). The dQM is a generalization of QM in which the Schrödinger equation is a difference equation instead of differential in the oQM [16,17]. In other words, the Hamiltonian contains the momentum operator in exponentiated forms \( e^{\pm i\beta p} \), which work as shift operators on the wave function

\[
e^{\pm i\beta p} \psi(x) = \psi(x \mp i) . \tag{2.1}
\]

According to the two choices of the parameter \( \beta \), either real or pure imaginary, we have two types of dQM, with (i) pure imaginary shifts \( \beta = \gamma \in \mathbb{R}_{\neq 0} \) (pdQM), or (ii) real shifts \( \beta = i \) (rdQM), respectively. In the case of pdQM, \( \psi(x \mp i) \), we require the wave function to be an analytic function of \( x \) with its domain including the real axis or a part of it on which the dynamical variable \( x \) is defined. For the rdQM, the difference equation gives constraints on wave functions only on equally spaced lattice points. Then we choose, after proper rescaling, the variable \( x \) to be a non-negative integer, with the total number either finite \((N + 1)\) or infinite.

To sum up, the dynamical variable \( x \) of the one-dimensional dQM takes continuous or discrete values:

\[
\text{imaginary shifts: } x \in \mathbb{R}, \quad x \in (x_1, x_2), \tag{2.2}
\]

and

\[
\text{real shifts: } x \in \mathbb{Z}, \quad x \in [0, N] \text{ or } [0, \infty) . \tag{2.3}
\]
Here \(x_1, x_2\) may be finite, \(-\infty\) or \(+\infty\). Correspondingly, the inner product of the wave functions has the following form:

**imaginary shifts**: \( (f, g) = \int_{x_1}^{x_2} f^*(x)g(x)dx, \) (2.4)

and

**real shifts**: \( (f, g) = \sum_{x=0}^{N} f(x)^*g(x) \) or \( \sum_{x=0}^{\infty} f(x)^*g(x), \) (2.5)

and the norm of \(f(x)\) is \( ||f|| = \sqrt{(f, f)}. \)

We will consider the Hamiltonians having a finite (rdQM with finite \(N\)) or semi-infinite number of discrete energy levels only:

\[ 0 = \mathcal{E}(0) < \mathcal{E}(1) < \mathcal{E}(2) < \cdots. \] (2.6)

The additive constant of the Hamiltonian is so chosen that the groundstate energy vanishes. That is, the Hamiltonian is *positive semi-definite*. It is a well-known theorem in linear algebra that any positive semi-definite hermitian matrix can be factorized as a product of a certain matrix, say \(A\), and its hermitian conjugate \(A^\dagger\).

As we will see shortly, the Hamiltonians of one-dimensional QM have the same property, either the oQM or the dQM both with the imaginary and real shifts.

**(a) Hamiltonian**

The Hamiltonian of one-dimensional QM has a simple form

\[ \mathcal{H} = A^\dagger A, \] (2.7)

in which the operators \(A\) and \(A^\dagger\) are

**oQM** \( A = \frac{d}{dx} - \frac{dw(x)}{dx}, \quad A^\dagger = -\frac{d}{dx} - \frac{dw(x)}{dx}, \quad w(x) \in \mathbb{R}, \) (2.8)

**pdQM** \( A = i \left( e^{\gamma p/2} \sqrt{V^*(x)} - e^{-\gamma p/2} \sqrt{V(x)} \right), \quad V(x), \ V^*(x) \in \mathbb{C}, \)

\[ A^\dagger = -i \left( \sqrt{V(x)} e^{\gamma p/2} - \sqrt{V^*(x)} e^{-\gamma p/2} \right) \] (2.9)

and **rdQM** \( A = \sqrt{B(x)} - e^\delta \sqrt{D(x)}, \quad B(x) \geq 0, \ D(x) \geq 0, \)

\[ A^\dagger = \sqrt{B(x)} - e^{-\delta} \sqrt{D(x)}, \quad D(0) = 0, \ B(N) = 0. \] (2.10)

Here \(V^*(x)\) is an analytic function of \(x\) obtained from \(V(x)\) by the *-operation, which is defined as follows. If \(f(x) = \sum_n a_n x^n, \ a_n \in \mathbb{C}\), then \(f^*(x) = \sum_n a_n^* x^n\), in which \(a_n^*\) is the complex conjugation of \(a_n\). Obviously \(f^{**}(x) = f(x)\) and \(f(x)^* = f^*(x^*)\). If a function satisfies \(f^* = f\), then it takes real values on the real line.
As mentioned above, \( e^{\pm \beta p} \) are shift operators \( e^{\pm \beta p} f(x) = f(x \mp i\beta) \), and the Schrödinger equation

\[
\mathcal{H} \phi_n(x) = \mathcal{E}(n) \phi_n(x), \quad n = 0, 1, 2, \ldots
\]  

(2.11)
is a second-order differential (oQM) or difference equation (dQM).

Here are some explicit examples, from the oQM. The prepotential \( w(x) \) parametrizes the groundstate wave function \( \phi_0(x) \), which has no node and can be chosen real and positive \( \phi_0(x) = e^{i w(x)} \). Then the above Hamiltonian takes the form \( \mathcal{H} = p^2 + U(x) \), \( U(x) \) is a second-order differential (oQM) or difference equation (dQM):

\[
(i) \text{ Hermite} : \quad w(x) = -\frac{1}{2} x^2, \quad -\infty < x < \infty, \quad \begin{cases} U(x) = x^2 - 1, & \eta(x) = x, \end{cases}
\]  

(2.12)

\[
(ii) \text{ Laguerre} : \quad w(x) = -\frac{1}{2} x^2 + g \log x, \quad g > 1, \quad 0 < x < \infty, \quad \begin{cases} U(x) = x^2 + \frac{g(g - 1)}{x^2} - 1 - 2g, & \eta(x) = x^2, \end{cases}
\]  

(2.13)

\[
(iii) \text{ Jacobi} : \quad w(x) = g \log \sin x + h \log \cos x, \quad g > 1, \quad h > 1, \quad 0 < x < \frac{\pi}{2}, \quad \begin{cases} U(x) = \frac{g(g - 1)}{\sin^2 x} + \frac{h(h - 1)}{\cos^2 x} - (g + h)^2, & \eta(x) = \cos^2 x. \end{cases}
\]  

(2.14)

From the dQM with pure imaginary shifts (\( \beta = \gamma, 0 < q < 1 \)):

\[
(iv) \text{ cont. Hahn} : \quad V(x) = (a_1 + ix)(a_2 + ix), \quad \text{Re}(a_j) > 0, \quad -\infty < x < \infty, \quad \begin{cases} \gamma = 1, & \eta(x) = x, \end{cases}
\]  

(2.15)

\[
(v) \text{ Wilson} : \quad V(x) = \prod_{j=1}^{4} \frac{(a_j + ix)}{2ix(2ix + 1)}, \quad \text{Re}(a_j) > 0, \quad 0 < x < \infty, \quad \begin{cases} \gamma = 1, & \{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\} \text{ (as a set), } \eta(x) = x^2. \end{cases}
\]  

(2.16)

\[
(vi) \text{ Askey–Wilson} : \quad V(x) = \frac{\prod_{j=1}^{4} (1 - a_j e^{ix})}{(1 - e^{2ix})(1 - q e^{2ix})}, \quad |a_j| < 1, \quad 0 < x < \pi, \quad \begin{cases} \gamma = \log q, & \{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\} \text{ (as a set), } \eta(x) = \cos x. \end{cases}
\]  

(2.17)

From the dQM with real shifts (\( \beta = i \)):

\[
(vii) \text{ Hahn} : \quad B(x) = (x + a)(N - x), \quad D(x) = x(b + N - x), \quad a > 0, \quad b > 0, \quad \eta(x) = x,
\]  

(2.18)
(viii) Racah:
\[
B(x) = -\frac{(x + a)(x + b)(x + c)(x + d)}{(2x + d)(2x + 1 + d)},
\]
\[
D(x) = -\frac{(x + d - a)(x + d - b)(x + d - c)x}{(2x - 1 + d)(2x + d)},
\]
\[
c = -N, \quad a \geq b, \quad 0 < b < 1 + d, \quad d > 0, \quad a > N + d,
\]
\[
\tilde{d} \overset{\text{def}}{=} a + b + c - d - 1, \quad \eta(x) = x(x + d),
\]
\[
(2.19)
\]

(ix) \(q\)-Racah:
\[
B(x) = -\frac{(1 - aq^x)(1 - bq^x)(1 - cq^x)(1 - dq^x)}{(1 - dq^{2x})(1 - dq^{2x+1})},
\]
\[
D(x) = -\frac{\tilde{d}(1 - a^{-1}dq^x)(1 - b^{-1}dq^x)(1 - c^{-1}dq^x)(1 - q^x)}{(1 - dq^{2x-1})(1 - dq^{2x})},
\]
\[
c = q^{-N}, \quad a \leq b, \quad 0 < d < 1, \quad 0 < a < q^N d, \quad q d < b < 1,
\]
\[
\tilde{d} < q^{-1}, \quad \tilde{d} \overset{\text{def}}{=} abcd^{-1}q^{-1}, \quad \eta(x) = (q^{-x} - 1)(1 - dq^x).
\]
\[
(2.21)
\]

In the above list, the names like the Hermite, ..., \(q\)-Racah are the names of the eigenpolynomials of the corresponding Hamiltonian in the form
\[
\phi_n(x) = \phi_0(x)P_n(\eta(x)), \quad (2.22)
\]

in which \(P_n(\eta(x))\) is a degree \(n\) polynomial in the sinusoidal coordinate \(\eta(x)\). It is also given in the same list. For the definitions and various properties of the hypergeometric orthogonal polynomials and their \(q\)-versions in general, see [10,16,17] in connection with dQM.

The groundstate wave function \(\phi_0(x)\) is determined as a zero mode of \(A\),
\[
A\phi_0(x) = 0. \quad (2.23)
\]

It is a first-order differential (oQM) or difference (dQM) equation and, therefore, easily solved explicitly. The similarity transformed Hamiltonian \(\tilde{H}\) in terms of the groundstate wave function \(\phi_0(x)\) has a much simpler form than the original Hamiltonian \(H\):
\[
\tilde{H} \overset{\text{def}}{=} \phi_0(x)^{-1} \circ H \circ \phi_0(x) = \varepsilon (V_+(x)(e^{\beta p} - 1) + V_-(x)(e^{-\beta p} - 1)) \quad (2.24)
\]
\[
= \begin{cases} 
-\frac{d^2}{dx^2} - 2\frac{dw(x)}{dx} \frac{d}{dx} & \text{oQM} \\
(V(x)(e^{\gamma p} - 1) + V^*(x)(e^{-\gamma p} - 1)) & \text{pdQM} \\
(B(x)(1 - e^\delta) + D(x)(1 - e^{-\delta})) & \text{rdQM}.
\end{cases} \quad (2.25)
\]
It should be stressed that the square roots in the original expression of \( H \) in dQM (2.9) and (2.10) have disappeared. The ‘Hamiltonian’ \( \tilde{H} \) governs the differential (difference) equation of the polynomial eigenfunctions:

\[
\tilde{H} P_n(\eta(x)) = E(n) P_n(\eta(x)).
\]

(2.26)

Obviously, the square of the groundstate wave function \( \phi_0(x)^2 \) provides the orthogonality weight function for the polynomials:

\[
oQM \text{ and } pdQM \int_{x_1}^{x_2} \phi_0(x)^2 P_n(\eta(x)) P_n(\eta(x)) \, dx \propto \delta_{nm}
\]

(2.27)

and

\[
rQM \sum_{x=0}^{\infty} \phi_0(x)^2 P_n(\eta(x)) P_n(\eta(x)) \propto \delta_{nm}.
\]

(2.28)

Let me emphasize that the weight function, or \( \phi_0(x) \), is determined as a solution of a first-order differential (difference) equation (2.23), without recourse to a moment problem. This situation becomes crucially important when various deformations of orthogonal polynomials are considered. The explicit forms of the squared groundstate wave function (weight function) \( \phi_0(x)^2 \) for the above examples in pure imaginary shifts dQM are

\[
\phi_0(x)^2 = \begin{cases} 
\Gamma(a_1 + ix)\Gamma(a_2 + ix)\Gamma(a_1^* - ix)\Gamma(a_2^* - ix) & : \text{(iv) cont. Hahn} \\
(\Gamma(2ix)\Gamma(-2ix))^{-1}\prod_{j=1}^{4} \Gamma(a_j + ix)\Gamma(a_j - ix) & : \text{(v) Wilson} \\
(e^{2ix}; q)_\infty(e^{-2ix}; q)_\infty\prod_{j=1}^{\infty} (a_je^{ix}; q)_{\infty}^{-1}(a_je^{-ix}; q)_{\infty}^{-1} & : \text{(vi) Askey–Wilson}.
\end{cases}
\]

(2.29)

For the real shifts dQM, the zero-mode equation \( A\phi_0(x) = 0 \) (equation (2.23)) is a two-term recurrence relation, which can be solved elementarily by using the boundary condition (2.10):

\[
\phi_0(x)^2 = \prod_{y=0}^{x-1} \frac{B(y)}{D(y+1)}
\]

(2.30)

\[
= \begin{cases} 
\frac{N!}{x!(N-x)!} \frac{(a)_x(b)_{N-x}}{(a, b, c, d)_x} (1 + d - a, 1 + d - b, 1 + d - c, 1)_x & : \text{(vii) Hahn} \\
\frac{2x + d}{d} \frac{(a, b, c, d; q)_x (a^{-1} dq, b^{-1} dq, c^{-1} dq, q; q)_x d^x}{1 - d} & : \text{(viii) Racah} \\
\frac{1 - dq^2x}{1 - d} & : \text{(ix) } q\text{-Racah}.
\end{cases}
\]

(2.31)

For the oQM, the weight function is simply given by the prepotential \( \phi_0(x)^2 = e^{2w(x)} \).

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(b) Intertwining relations

Let us denote by $\mathcal{H}^{[0]}$ the original factorized Hamiltonian and by $\mathcal{H}^{[1]}$ its partner (associated) Hamiltonian obtained by changing the order of $A^\dagger$ and $A$:

$$\mathcal{H}^{[0]} \overset{\text{def}}{=} A^\dagger A \quad \text{and} \quad \mathcal{H}^{[1]} \overset{\text{def}}{=} AA^\dagger. \quad (2.32)$$

One simple and most important consequence of the factorization is the intertwining relations:

$$A\mathcal{H}^{[0]} = A^\dagger A = \mathcal{H}^{[1]} A \quad \text{and} \quad A^\dagger \mathcal{H}^{[1]} = AA^\dagger = \mathcal{H}^{[0]} A^\dagger, \quad (2.33)$$

which are equally valid in oQM and dQM. The pair of Hamiltonians $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$ are essentially iso-spectral and their eigenfunctions $\{\phi_n^{[0]}(x)\}$ and $\{\phi_n^{[1]}(x)\}$ are related by the Darboux–Crum transformations [11,31]:

$$\mathcal{H}^{[0]} \phi_n^{[0]}(x) = \mathcal{E}(n) \phi_n^{[0]}(x), \quad n = 0, 1, \ldots, \quad (2.34)$$

$$\mathcal{H}^{[1]} \phi_n^{[1]}(x) = \mathcal{E}(n) \phi_n^{[1]}(x), \quad n = 1, 2, \ldots \quad (2.35)$$

and

$$\phi_n^{[1]}(x) = A \phi_n^{[0]}(x), \quad \phi_n^{[0]}(x) = \frac{A^\dagger}{\mathcal{E}(n)} \phi_n^{[1]}(x), \quad n = 1, 2, \ldots. \quad (2.36)$$

The partner Hamiltonian $\mathcal{H}^{[1]}$ has the lowest eigenvalue $\mathcal{E}(1)$. If the groundstate energy $\mathcal{E}(1)$ is subtracted from the partner Hamiltonian $\mathcal{H}^{[1]}$, it is again positive semi-definite and can be factorized in terms of new operators $A^{[1]}$ and $A^{[1]}$:

$$\mathcal{H}^{[1]} \overset{\text{def}}{=} A^{[1]\dagger} A^{[1]} + \mathcal{E}(1). \quad (2.37)$$

By changing the orders of $A^{[1]\dagger}$ and $A^{[1]}$, a new Hamiltonian $\mathcal{H}^{[2]}$ is defined:

$$\mathcal{H}^{[2]} \overset{\text{def}}{=} A^{[1]} A^{[1]\dagger} + \mathcal{E}(1). \quad (2.38)$$

These two Hamiltonians are intertwined by $A^{[1]}$ and $A^{[1]}$:

$$A^{[1]} (\mathcal{H}^{[1]} - \mathcal{E}(1)) = A^{[1]} A^{[1]\dagger} A^{[1]} = (\mathcal{H}^{[2]} - \mathcal{E}(1)) A^{[1]} \quad (2.39)$$

and

$$A^{[1]\dagger} (\mathcal{H}^{[2]} - \mathcal{E}(1)) = A^{[1]\dagger} A^{[1]} A^{[1]\dagger} = (\mathcal{H}^{[1]} - \mathcal{E}(1)) A^{[1]\dagger}. \quad (2.40)$$

The iso-spectrality of the two Hamiltonians $\mathcal{H}^{[1]}$ and $\mathcal{H}^{[2]}$ and the relationship among their eigenfunctions follow as before:

$$\mathcal{H}^{[2]} \phi_n^{[2]}(x) = \mathcal{E}(n) \phi_n^{[2]}(x), \quad n = 2, 3, \ldots \quad (2.41)$$

and

$$\phi_n^{[2]}(x) = A^{[1]} \phi_n^{[1]}(x) \quad \text{and} \quad \phi_n^{[1]}(x) = \frac{A^{[1]\dagger}}{\mathcal{E}(n) - \mathcal{E}(1)} \phi_n^{[2]}(x), \quad n = 2, 3, \ldots. \quad (2.42)$$
This process can go on indefinitely by successively deleting the lowest lying energy level:

$$\mathcal{H}^{[s]} \phi_n^{[s]}(x) = \mathcal{E}(n) \phi_n^{[s]}(x), \quad n = s, s + 1, \ldots \tag{2.43}$$

and

$$\phi_n^{[s]}(x) = A^{[s-1]} \phi_n^{[s-1]}(x) \quad \text{and} \quad \phi_n^{[s-1]}(x) = \frac{A^{[s-1] \dagger}}{\mathcal{E}(n) - \mathcal{E}(s-1)} \phi_n^{[s]}(x), \tag{2.44}$$

for oQM [11] and for dQM as well [32,33]. The determinant expressions of the eigenfunctions $\phi_n^{[s]}(x)$ are also known for the oQM (the Wronskian) and for dQM as well (the Casoratian).

By deleting a finite number of energy levels from the original Hamiltonian systems $\mathcal{H}^{[0]}$ and $\{\phi_n^{[0]}(x)\}$, instead of the successive lowest lying levels, modification of Crum’s theorem provides the essentially iso-spectral modified Hamiltonian $\tilde{\mathcal{H}}$ and its eigenfunctions $\{\tilde{\phi}_n(x)\}$. The set of deleted energy levels $\mathcal{D} = \{d_1, \ldots, d_\ell\}$ must satisfy the conditions

$$\prod_{j=1}^{\ell} (m - d_j) \geq 0, \quad m \in \mathbb{Z}_+ \tag{2.45}$$

in order to guarantee the hermiticity (self-adjointness) of the modified Hamiltonian $\tilde{\mathcal{H}}$ and the non-singularity of the eigenfunctions

$$\tilde{\mathcal{H}} \tilde{\phi}_n(x) = \mathcal{E}(n) \tilde{\phi}_n(x), \quad n \in \mathbb{Z}_+ \setminus \mathcal{D}. \tag{2.46}$$

Again the Wronskian (oQM) and Casoratian (pdQM) expressions of the eigenfunctions are known. For oQM, these results have been known for some time [21,22]. For dQM with pure imaginary shifts, the structure of the modified Crum’s theory was clarified only very recently [34]. Starting from an exactly solvable Hamiltonian, one can construct infinitely many variants of exactly solvable Hamiltonians and their eigenfunctions by the methods of Adler [22] and García-Gutiérrez et al. [34]. The resulting systems are, however, not shape invariant, even if the starting system is. For dQM with real shifts, some partial results are reported in the study of Yermolayeva & Zhedanov [35].

(c) **Shape invariance**

Shape invariance [14] is a sufficient condition for the exact solvability in the Schrödinger picture. Combined with Crum’s theorem [11], or the factorization method [12] or the so-called supersymmetric QM [13], the totality of the discrete eigenvalues and the corresponding eigenfunctions can be easily obtained. It was shown by Odake & Sasaki that the concept of shape invariance works equally well in dQM with pure imaginary shifts [15,17] as well as real shifts [16], providing quantum mechanical explanation of the solvability of the Askey scheme of hypergeometric orthogonal polynomials in general.
In many cases the Hamiltonian contains some parameter(s), \( \lambda = (\lambda_1, \lambda_2, \ldots) \). Here we write parameter-dependence explicitly, \( \mathcal{H}(\lambda), \mathcal{A}(\lambda), \mathcal{E}(n; \lambda), \phi_n(x; \lambda), \) etc., as it is the central issue. The shape invariance condition is

\[
\mathcal{A}(\lambda)\mathcal{A}(\lambda)^\dagger = \kappa \mathcal{A}(\lambda + \delta)^\dagger \mathcal{A}(\lambda + \delta) + \mathcal{E}(1; \lambda),
\]

where \( \kappa \) is a real positive parameter and \( \delta \) is the shift of the parameters. In other words \( \mathcal{H}^{[0]} \) and \( \mathcal{H}^{[1]} \) have the same shape, only the parameters are shifted by \( \delta \). The energy spectrum and the excited state wave function are determined by the data of the groundstate wave function \( \phi_0(x; \lambda) \) and the energy of the first excited state \( \mathcal{E}(1; \lambda) \) as follows:

\[
\mathcal{E}(n; \lambda) = \sum_{s=0}^{n-1} \kappa^s \mathcal{E}(1; \lambda^{[s]}) \quad \text{and} \quad \lambda^{[s]} = \lambda + s\delta,
\]

and

\[
\phi_n(x; \lambda) \propto \mathcal{A}(\lambda^{[0]})^\dagger \mathcal{A}(\lambda^{[1]})^\dagger \mathcal{A}(\lambda^{[2]})^\dagger \cdots \mathcal{A}(\lambda^{[n-1]})^\dagger \phi_0(x; \lambda^{[n]}).
\]

The above formula for the eigenfunctions \( \phi_n(x; \lambda) \) can be considered as the universal Rodrigues type formula for the Askey scheme of hypergeometric polynomials and their \( q \)-analogues. For the explicit forms of the Rodrigues type formula for each polynomial, one only has to substitute the explicit forms of the operator \( \mathcal{A}(\lambda) \) and the groundstate wave function \( \phi_0(x; \lambda) \). For the nine explicit examples given in equations (2.12)–(2.21), it is straightforward to verify the shape invariance conditions (2.47) and the energy (2.48) and the eigenfunction (2.49) formulas. For dQM, the above shape invariance condition (2.47) becomes a relation among the prepotentials (\( \kappa = 1 \)):

\[
\left( \frac{d^2w(x; \lambda)}{dx^2} \right)^2 - \frac{d^2w(x; \lambda)}{dx^2} = \left( \frac{d^2w(x; \lambda + \delta)}{dx^2} \right)^2 + \frac{d^2w(x; \lambda + \delta)}{dx^2} + \mathcal{E}(1; \lambda),
\]

and the data for the three cases (i)–(iii) are

(i) Hermite : \( \lambda = \phi \) (null), \( \mathcal{E}(n) = 2n \),
(ii) Laguerre : \( \lambda = g \), \( \delta = 1 \), \( \mathcal{E}(n; \lambda) = 4n \),
(iii) Jacobi : \( \lambda = (g, h) \), \( \delta = (1, 1) \), \( \mathcal{E}(n; \lambda) = 4n(n + g + h) \).

For dQM with the pure imaginary shifts, the above shape invariance condition (2.47) is rewritten as

\[
V \left( x - i\frac{\gamma}{2}; \lambda \right) V^* \left( x - i\frac{\gamma}{2}; \lambda \right) = \kappa^2 V(x; \lambda + \delta) V^*(x; \lambda + \delta)
\]

and

\[
V \left( x + i\frac{\gamma}{2}; \lambda \right) + V^* \left( x - i\frac{\gamma}{2}; \lambda \right) = \kappa (V(x; \lambda + \delta) + V^*(x; \lambda + \delta)) - \mathcal{E}(1; \lambda).
\]
The data for the three cases in pdQM (iv)–(vi) are [15,17]

(iv) cont. Hahn: \[ \lambda = (a_1, a_2), \quad \delta = \left( \frac{1}{2}, \frac{1}{2} \right), \quad \kappa = 1, \]
\[ E(n; \lambda) = 4n(n + b_1 - 1), \quad b_1 \overset{\text{def}}{=} a_1 + a_2 + a_1^* + a_2^*, \]
\[ (2.56) \]

(v) Wilson: \[ \lambda = (a_1, a_2, a_3, a_4), \quad \delta = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \quad \kappa = 1, \]
\[ E(n; \lambda) = 4n(n + b_1 - 1), \quad b_1 \overset{\text{def}}{=} a_1 + a_2 + a_3 + a_4, \]
\[ (2.57) \]

(vi) Askey–Wilson: \[ q^\lambda = (a_1, a_2, a_3, a_4), \quad \delta = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \quad \kappa = q^{-1}, \]
\[ E(n; \lambda) = (q^{-n} - 1)(1 - b_4 q^{n-1}), \quad b_4 \overset{\text{def}}{=} a_1 a_2 a_3 a_4. \]
\[ (2.58) \]

For dQM with the real shifts, the shape invariance (2.47) is equivalent to the following set of two equations:
\[ B(x + 1; \lambda)D(x + 1; \lambda) = \kappa^2 B(x; \lambda + \delta)D(x + 1; \lambda + \delta) \]
\[ (2.59) \]
and
\[ B(x; \lambda) + D(x + 1; \lambda) = \kappa(B(x; \lambda + \delta) + D(x; \lambda + \delta)) + E(1; \lambda). \]
\[ (2.60) \]

The data for the three cases in rdQM (vii)–(ix) are [16]

(vii) Hahn: \[ \lambda = (a, b, N), \quad \delta = (1, 1, -1), \quad \kappa = 1, \]
\[ E(n; \lambda) = 4n(n + a + b - 1), \]
\[ (2.61) \]

(viii) Racah: \[ \lambda = (a, b, d, N), \quad \delta = (1, 1, 1, -1), \quad \kappa = 1, \]
\[ E(n; \lambda) = 4n(n + d), \]
\[ (2.62) \]

(ix) q-Racah: \[ q^\lambda = (a, b, d, q^{-N}), \quad \delta = (1, 1, 1, 1), \quad \kappa = q^{-1}, \]
\[ E(n; \lambda) = (q^{-n} - 1)(1 - \tilde{d} q^{n-1}). \]
\[ (2.63) \]

It should be stressed that the size of the Hamiltonian (\( N \) in the finite case) decreases by one, since the lowest eigenstate is removed.

(d) Solvability in the Heisenberg picture

All the Hamiltonian systems describing the hypergeometric orthogonal polynomials are exactly solvable in the Heisenberg picture also [18,19]. To be more precise, the Heisenberg operator of the sinusoidal operator \( \eta(x) \)
\[ e^{i\mathcal{H}} \eta(x) e^{-i\mathcal{H}} \]
\[ (2.64) \]
can be evaluated in a closed form. The sufficient condition for that is the closure relation
\[ [\mathcal{H}, [\mathcal{H}, \eta(x)]] = \eta(x) R_0(\mathcal{H}) + [\mathcal{H}, \eta(x)] R_1(\mathcal{H}) + R_{-1}(\mathcal{H}). \]
\[ (2.65) \]
Here, $R_i(y)$ are polynomials in $y$. It is easy to see that the cubic commutator $[\mathcal{H}, [\mathcal{H}, \eta(x)]] \equiv (\text{ad}\mathcal{H})^3\eta(x)$ is reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with $\mathcal{H}$ depending coefficients:

$$(\text{ad}\mathcal{H})^3\eta(x) = [\mathcal{H}, \eta(x)]R_0(\mathcal{H}) + [\mathcal{H}, [\mathcal{H}, \eta(x)]]R_1(\mathcal{H})$$

$$= \eta(x)R_0(\mathcal{H})R_1(\mathcal{H}) + [\mathcal{H}, \eta(x)](R_1(\mathcal{H})^2 + R_0(\mathcal{H}))$$

$$+ R_{-1}(\mathcal{H})R_1(\mathcal{H}), \quad (2.66)$$

in which the definition $(\text{ad}\mathcal{H})X \overset{\text{def}}{=} [\mathcal{H}, X]$ is used. It is trivial to see that all the higher commutators $(\text{ad}\mathcal{H})^n\eta(x)$ can also be reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with $\mathcal{H}$ depending coefficients. Thus, we arrive at

$$e^{i\mathcal{H}}\eta(x)e^{-i\mathcal{H}} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (\text{ad}\mathcal{H})^n\eta(x),$$

$$= [\mathcal{H}, \eta(x)]e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t} - \frac{R_{-1}(\mathcal{H})}{R_0(\mathcal{H})} \left( \alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t} + \alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t} \right) \frac{1}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})}$$

in which the two ‘frequencies’ $\alpha_\pm(\mathcal{H})$ are

$$\alpha_\pm(\mathcal{H}) = \frac{(R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2 + 4R_0(\mathcal{H})})}{2} \quad (2.68)$$

and

$$\alpha_+(\mathcal{H}) + \alpha_-(\mathcal{H}) = R_1(\mathcal{H}) \quad \text{and} \quad \alpha_+(\mathcal{H})\alpha_-(\mathcal{H}) = -R_0(\mathcal{H}). \quad (2.69)$$

The annihilation and creation operators $a^{(\pm)}$ are extracted from this exact Heisenberg operator solution:

$$e^{i\mathcal{H}}\eta(x)e^{-i\mathcal{H}} = a^{(+)}e^{i\alpha_+(\mathcal{H})t} + a^{(-)}e^{i\alpha_-(\mathcal{H})t} - R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1} \quad (2.70)$$

and

$$a^{(\pm)} \overset{\text{def}}{=} \pm ([\mathcal{H}, \eta(x)] - (\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1})\alpha_{\pm}(\mathcal{H}))\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})^{-1}$$

$$= \pm (\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})^{-1})([\mathcal{H}, \eta(x)] + \alpha_+(\mathcal{H})\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}). \quad (2.71)$$

The energy spectrum is determined by the over-determined recursion relations $\mathcal{E}(n + 1) = \mathcal{E}(n) + \alpha_+(\mathcal{E}(n))$ and $\mathcal{E}(n - 1) = \mathcal{E}(n) + \alpha_-(\mathcal{E}(n))$ with $\mathcal{E}(0) = 0$, and the excited state wave functions $\{\phi_n(x)\}$ are obtained by successive action of the creation operator $a^{(+)}$ on the ground state wave function $\phi_0(x)$. This is the exact solvability in the Heisenberg picture.
The data for the three cases in oQM (i)–(iii) are

(i) Hermite: \( R_0(y) = 4, \quad R_1(y) = R_{-1}(y) = 0, \)

(ii) Laguerre: \( R_0(y) = 16, \quad R_1(y) = 0, \quad R_{-1}(y) = -8(y + 2g + 1), \)

(iii) Jacobi: \( \begin{cases} R_0(y) = 16(y + (g + h)^2), & R_1(y) = 0, \\ R_{-1}(y) = 16(g - h)(g + h - 1). \end{cases} \)

The data for the three cases in pdQM (iv)–(vi) are [15,17]

(iv) cont. Hahn: \( \begin{cases} R_0(y) = 4y + 4\text{Re}(a_1 + a_2)(\text{Re}(a_1 + a_2) - 1), \\ R_1(y) = 2, \\ R_{-1}(y) = 2\text{Im}(a_1 + a_2)y + 4(\text{Re}(a_1 + a_2) - 1)\text{Im}(a_1a_2), \end{cases} \)

(v) Wilson: \( \begin{cases} R_0(y) = 4y + b_1(b_1 - 2), & R_1(y) = 2, \\ R_{-1}(y) = -2y^2 + (b_1 - 2b_2)y + (2 - b_1)b_3, \\ b_2 \stackrel{\text{def}}{=} \sum_{1 \leq j < k \leq 4} a_ja_k, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_ja_ka_l, \end{cases} \)

(vi) Askey–Wilson: \( \begin{cases} R_1(y) = (q^{-1/2} - q^{1/2})^2 y', & y' \stackrel{\text{def}}{=} y + 1 + q^{-1}b_4, \\ R_0(y) = (q^{-1/2} - q^{1/2})^2(y^2 - (1 + q^{-1})^2b_4), \\ R_{-1}(y) = -\frac{1}{2}(q^{-1/2} - q^{1/2})^2((b_1 + q^{-1}b_3)y' \\ & - (1 + q^{-1})(b_3 + q^{-1}b_1b_4)), \\ b_1 \stackrel{\text{def}}{=} \sum_{j=1}^4 a_j, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_ja_ka_l. \end{cases} \)

The data for the three cases in rdQM (vii)–(ix) are [16]

(vii) Hahn: \( \begin{cases} R_0(y) = 4y + (a + b - 2)(a + b), & R_1(y) = 2, \\ R_{-1}(y) = -(2N - a + b)y - a(a + b - 2)N, \end{cases} \)

(viii) Racah: \( \begin{cases} R_0(y) = 4y + \tilde{d}^2 - 1, & R_1(y) = 2, \\ R_{-1}(y) = 2y^2 + (2(ab + bc + ca) \\ & - (1 + d)(1 + \tilde{d}))y + abc(\tilde{d} - 1), \end{cases} \)

(ix) q-Racah: \( \begin{cases} R_0(y) = (q^{-1/2} - q^{1/2})^2(y^2 - (q^{-1/2} + q^{1/2})^2\tilde{d}), \\ y' \stackrel{\text{def}}{=} y + 1 + \tilde{d}, \\ R_1(y) = (q^{-1/2} - q^{1/2})^2y', \\ R_{-1}(y) = (q^{-1/2} - q^{1/2})^2((1 + d)y^2 \\ & - (a + b + c + d + \tilde{d} + (ab + bc + ca)q^{-1})y' \\ & + (1 - a)(1 - b)(1 - c)(1 - \tilde{d}q^{-1}) \\
 & + (a + b + c - 1 - d\tilde{d} + (ab + bc + ca)q^{-1})(1 + \tilde{d})). \end{cases} \)
**Dual-closure relation**

The dual-closure relation has the same form as the closure relation (2.65) with the roles of the Hamiltonian $\mathcal{H}$ and the sinusoidal coordinate $\eta(x)$ exchanged:

$$[\eta, [\eta, \mathcal{H}]] = \mathcal{H} R_{\text{dual}}^0 (\eta) + [\eta, \mathcal{H}] R_{\text{dual}}^1 (\eta) + R_{\text{dual}}^{-1} (\eta),$$  
(2.81)

in which

$$R_{\text{dual}}^1 (\eta(x)) = (\eta(x - i\beta) - \eta(x)) + (\eta(x + i\beta) - \eta(x)), \quad (2.82)$$

$$R_{\text{dual}}^0 (\eta(x)) = - (\eta(x - i\beta) - \eta(x)) (\eta(x + i\beta) - \eta(x)), \quad (2.83)$$

and

$$R_{\text{dual}}^{-1} (\eta(x)) = \epsilon (V_+(x) + V_-(x)) R_{\text{dual}}^0 (\eta(x)). \quad (2.84)$$

The dual-closure relation is the characteristic feature shared by all the ‘Hamiltonians’ $\tilde{\mathcal{H}}$, which map a polynomial in $\eta(x)$ into another. Therefore, its dynamical contents are not so constraining as the closure relation, except for the real shifts (the discrete variable) exactly solvable ($L=2$) case, where the closure and the dual-closure relations are on the same footing and they form a dynamical symmetry algebra, which is sometimes called the Askey–Wilson algebra [16,36–39].

## 3. Unified theory of exact and quasi-exact solvability

In §2, various examples of exactly solvable systems are explored and it is shown that the two sufficient conditions for exact solvability, the shape invariance and the closure relation, are satisfied by them. However, these two sufficient conditions do not tell how to build exactly solvable models. In this section, I present a simple theory of constructing exactly solvable Hamiltonians in dQM. It covers all the known examples of exactly solvable dQM with both pure imaginary and real shifts and it predicts several new ones to be explored. The theory is general enough to generate QES Hamiltonians in the same manner. The quasi-exact solvability means, in contrast to the exact solvability, that only a finite number of energy eigenvalues and the corresponding eigenfunctions can be obtained exactly. Many examples are known in oQM [40–43] but only a few are known in dQM [44,45] in spite of the proposal that the $sl(2, R)$ algebra characterization of quasi-exact solvability could be extended to difference Schrödinger equation [46]. This unified theory also incorporates the known examples of QES Hamiltonians in dQM [47–49]. A new type of QES Hamiltonians is constructed. The present approach reveals the common structure underlying the exactly and QES theories. This section is a brief introduction to a recent work by Odake & Sasaki [50].

In the following, I will take the similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ (2.24) instead of $\mathcal{H}$ as the starting point. That is, I reverse the argument and construct directly the ‘Hamiltonian’ $\tilde{\mathcal{H}}$ (2.25) based on the sinusoidal coordinate $\eta(x)$. The general strategy is to construct the ‘Hamiltonian’ $\tilde{\mathcal{H}}$ in such a way that it maps a polynomial in $\eta(x)$ into another:

$$\tilde{\mathcal{H}} \mathcal{V}_n \subseteq \mathcal{V}_{n+L-2} \subset \mathcal{V}_\infty.$$  
(3.1)
Here $\mathcal{V}_n \ (n \in \mathbb{Z}_{\geq 0})$ is defined by

$$
\mathcal{V}_n \equiv \text{Span}[1, \eta(x), \ldots, \eta(x)^n], \quad \mathcal{V}_\infty \equiv \lim_{n \to \infty} \mathcal{V}_n.
$$

(a) Potential functions

The general form of the ‘Hamiltonian’ $\tilde{H}$ mapping a polynomial in $\eta(x)$ into another is achieved by the following form of the potential functions $V_\pm(x)$:

$$
V_\pm(x) = \frac{\tilde{V}_\pm(x)}{(\eta(x + i\beta) - \eta(x))(\eta(x - i\beta) - \eta(x \pm i\beta))}
$$

and

$$
\tilde{V}_\pm(x) = \sum_{k,l=0}^{k+l \leq L} v_{k,l} \eta(x)^k \eta(x \pm i\beta)^l,
$$

where $L$ is a natural number indicating the degree of $\eta(x)$ in $\tilde{V}_\pm(x)$ and $v_{k,l}$ are real constants, with the constraint $\sum_{k+l=L} v_{k,l}^2 \neq 0$. It is important that the same $v_{k,l}$ appears in both $\tilde{V}_\pm(x)$. The ‘Hamiltonian’ $\tilde{H}$ with the above $V_\pm(x)$ maps a degree $n$ polynomial in $\eta(x)$ to a degree $n + L - 2$ polynomial. This can be shown elementarily based on two basic properties of the sinusoidal coordinates called the symmetric shift-addition property:

$$
\eta(x - i\beta) + \eta(x + i\beta) = (2 + r_1^{(1)}) \eta(x) + r_{-1}^{(2)}
$$

and the symmetric shift-multiplication property:

$$
\eta(x - i\beta) \eta(x + i\beta) = (\eta(x) - \eta(-i\beta))(\eta(x) - \eta(i\beta)),
$$

together with $\eta(x) \neq \eta(x - i\beta) \neq \eta(x + i\beta) \neq \eta(x)$. Here $r_1^{(1)}$ and $r_{-1}^{(2)}$ are real parameters. It is straightforward to verify these properties for the explicit forms of the sinusoidal coordinates in dQM listed in the preceding section. There are several more examples of the sinusoidal coordinates satisfying these two properties, see appendix A of Odake & Sasaki [50].

The essential part of the formula (3.3) is the denominators. They have the same form as the generic formula for the coefficients of the three-term recurrence relations of the orthogonal polynomials, eqns (4.52) and (4.53) in Odake & Sasaki [16]. The translation rules are the duality correspondence itself, eqns (3.14)–(3.18) in Odake & Sasaki [16]:

$$
\begin{align*}
&\mathcal{E}(n) \rightarrow \eta(x), \quad -A_n \rightarrow V_+(x), \quad -C_n \rightarrow V_-(x), \\
&\alpha_+(\mathcal{E}(n)) \rightarrow \eta(x - i\beta) - \eta(x) \quad \text{and} \quad \alpha_-(\mathcal{E}(n)) \rightarrow \eta(x + i\beta) - \eta(x).
\end{align*}
$$

Some of the parameters $v_{k,l}$ in equation (3.4) are redundant. It is sufficient to keep $v_{k,l}$ with $l = 0, 1$. The remaining $2L + 1$ parameters $v_{k,l} \ (k + l \leq L, \ l = 0, 1)$ are independent, with one of which corresponding to the overall normalization of the Hamiltonian.

The $L = 2$ case is exactly solvable. Since the Hamiltonian of the polynomial space $\tilde{H}$ is an upper triangular matrix, its eigenvalues and eigenvectors are easily obtained explicitly. For the solutions of a full quantum mechanical problem, however, one needs the square-integrable groundstate wave function $\phi_0(x)$ (2.23),
which is essential for the existence of the Hamiltonian $H$ and the verification of its hermiticity. These conditions would usually restrict the ranges of the parameters $v_{0,0}, \ldots, v_{2,0}$. It is easy to verify that the explicit examples of the potential functions $V(x), V^*(x)$ and $B(x), D(x)$ in equations (2.15)–(2.21) are simply reproduced by proper choices of the parameters $\{v_{k,l}\}$. It should be stressed that the above form of the potential functions (3.3)–(3.4) provides a unified proof of the shape invariance relation (2.47), the closure relation (2.65) and the dual closure relation (2.81) in the $\tilde{H}$ scheme.

The higher $L \geq 3$ cases are obviously non-solvable. Among them, the tame non-solvability of $L = 3$ and 4 can be made QES by adding suitable compensation terms. This is a simple generalization of the method of Sasaki & Takasaki [51] for multi-particle QES in oQM. For a given positive integer $M$, let us try to find a QES ‘Hamiltonian’ $\tilde{H}$, or more precisely its modification $\tilde{H}'$, having an invariant polynomial subspace $V_M$:

$$\tilde{H}'V_M \subseteq V_M.$$  

(3.8)

For $L = 3$, $\tilde{H}'$ is defined by adding one single compensation term of degree one

$$\tilde{H}' \overset{\text{def}}{=} \tilde{H} - e_0(M)\eta(x), \quad (3.9)$$

and we have achieved the quasi-exact solvability $\tilde{H}'V_M \subseteq V_M$. Known discrete QES examples belong to this class [48,49].

For the $L = 4$ case, $\tilde{H}'$ is defined by adding a linear and a quadratic in $\eta(x)$ compensation terms to the Hamiltonian $\tilde{H}$:

$$\tilde{H}' \overset{\text{def}}{=} \tilde{H} - e_0(M)\eta(x)^2 - e_1(M)\eta(x), \quad (3.10)$$

and $\tilde{H}'\eta(x)^M \in V_M$. This type of QES theory is new.

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References

Discrete quantum mechanics


43 Ushveridze, A. G. 1994 *Quasi-exactly solvable models in quantum mechanics*. Bristol, UK: IOP.