Renormalization group for non-relativistic fermions

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A brief introduction is given to the renormalization group for non-relativistic fermions at finite density. It is shown that Landau’s theory of the Fermi liquid arises as a fixed point (with the Landau parameters as marginal couplings) and its instabilities as relevant perturbations. Applications to related areas, nuclear matter, quark matter and quantum dots, are briefly discussed. The focus will be on explaining the main ideas to people in related fields, rather than addressing the experts.

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1. The renormalization group: what, why and how

Imagine that you have a problem that may be cast in the form of a path integral such as the following:

\[ Z(a, b, c, \ldots) = \int dx \int dy e^{-a(x^2+y^2)-b(x+y)^4} = \int dx \int dy \exp[-S(x, y; a, b, c, \ldots)], \]

(1.1)

where \( a, b, c, \ldots \) are the parameters, \( x \) and \( y \) are the field variables being integrated and \( S \) is called the action. The average of any function \( F(x, y) \) is given by

\[ \langle F(x, y) \rangle = \frac{1}{Z} \int dx \int dy F(x, y) e^{-S(x, y; a, b, c, \ldots)}. \]

(1.2)

Suppose that you are just interested in functions of \( x \). Then, you have the option of integrating out \( y \) and working with the new partition function [1–6]

\[ Z(a', b', \ldots) = \int dx e^{-S'(x; a', b', \ldots)}, \quad \text{where} \quad e^{-S'(x; a', b', \ldots)} = \int dy e^{-S(x, y; a, b, c, \ldots)}. \]

(1.3)

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One contribution of 11 to a Theme Issue ‘New applications of the renormalization group in nuclear, particle and condensed matter physics’.
The new parameters $a', b'$, etc. define the effective field theory for $x$. In other words,

$$
\langle F(x) \rangle = \frac{1}{Z'} \int dx F(x) e^{-S(x; a', b', c', \ldots)}, \quad \text{where} \quad Z' = \int dx e^{-S(x; a', b', c', \ldots)}. \quad (1.4)
$$

It is often necessary to follow the elimination of $y$ by a change of scale of the surviving integration variable $x$ and other kinematical entities (like momenta or lengths, which do not appear in this example). It is then understood that $(a', b', \ldots)$ are the ones appearing in the final action $S'$ that arises after all these transformations have been performed.

We see that the new parameters $(a', b', c', \ldots)$ in the theory of just $x$ will reproduce exactly the same averages for $F(x)$ as the original ones $(a, b, c, \ldots)$ did in a theory containing $x$ and $y$. This evolution of parameters with the elimination of uninteresting degrees of freedom is what we mean by renormalization.

The parameters $b$, $c$, etc. are called couplings and the monomials they multiply are called interactions. The quadratic $x^2$ term is called the kinetic or free-field term. We shall see that its coefficient can be often redefined to take a fixed value upon rescaling.

Notice that to get the effective theory, we need to do a non-Gaussian integral. This can only be done perturbatively. At the tree level, we simply drop $y$ and do whatever rescaling is part of the renormalization group (RG). At higher orders, we bring down the non-quadratic terms in the exponential and integrate over $y$ term by term, re-exponentiate and generate effective interactions for $x$. This procedure can be represented by Feynman graphs in which variables in the loop are limited to the ones being eliminated.

Why do we do this? Because certain tendencies of $x$ are not so apparent when $y$ is around, but surface to the top only as we zero in on $x$. For example, we are going to consider a problem in which $x$ stands for low-energy variables and $y$ for high-energy variables. Upon integrating out high-energy variables, a numerically small coupling can grow in size (or an initially impressive one diminishes into oblivion), as we zoom in on the low-energy sector.

This notion can be made more precise as follows. Consider the Gaussian model in which we have just $a \neq 0$. This value does not change as $y$ is eliminated since $x$ and $y$ do not talk to each other. This is called a fixed point of the RG. Now turn on new couplings or ‘interactions’ (corresponding to higher powers of $x$, $y$, etc.) with coefficients $b$, $c$ and so on. Let $a'$, $b'$, etc. be the new couplings after $y$ is eliminated. The mere fact that $b' > b$ does not mean $b$ is more important for the physics of $x$. This is because $a'$ could also be bigger than $a$. So we rescale $x$ so that the kinetic part, $x^2$, has the same coefficient as before. If the quartic term still has a bigger coefficient (still called $b'$), we say it is a relevant interaction. If $b' < b$, we say it is irrelevant. This is because, in reality, $y$ stands for many variables, and as they are eliminated one by one, the coefficient of the quartic term will grow or flow to zero. If a coupling neither grows nor shrinks, it is called marginal.

In Euclidean quantum field theory, where low energy means small momentum, the effective low-energy theory will only contain modes within a tiny ball of radius $A$ centred on the origin of $k$-space. This is also true for non-relativistic fermions in $d = 1$, where the low-energy region is centred around two Fermi points [7,8]. However, when $d > 1$, the low-energy region collapses around the Fermi surface, and the RG exhibits some new features.
2. Non-relativistic fermions at finite density

Consider such fermions in $d = 2$, the dimensionality of interest to high $T_c$ and where the pedagogy is easiest. Ignore interactions and spin and let $\hbar = 1$. The second quantized Hamiltonian is

$$H_0 = \int dK \psi^\dagger (K) \left[ \frac{K^2}{2m} \right] \psi (K). \quad (2.1)$$

This $H_0$ will not have any fermions in the ground state. To ensure a non-zero density of fermions, we subtract a chemical potential $\mu = K_F^2/2m$,

$$H_0 = \int dK \psi^\dagger (K) \left[ \frac{K^2}{2m} - \frac{K_F^2}{2m} \right] \psi (K). \quad (2.2)$$

In the ground state, all momentum states within a circle of radius equal to the Fermi momentum $K_F$ are occupied by one fermion. We choose $K_F$ to ensure the desired density. Excitations on top of this ground state are obtained by knocking electrons out of the filled Fermi sea to empty states. Since such ‘particle–hole’ states can have arbitrarily low-energy cost (if they are arbitrarily close to the Fermi energy), we say the system has no gap.

What happens to the ground state if we add interactions? The Fermi sea will mix with the ‘particle–hole’ states. These excitations will be near the Fermi surface if the interactions are weak. So the ground state and its low-energy excitations will be determined by a band of single-particle states with $|k| < \Lambda$, where $k = K - K_F$, as depicted in figure 1. The effective theory is then obtained by integrating states outside the cut-off $\Lambda$ and eventually sending $\Lambda \to 0$. At this stage, it will become clear if any given interaction is relevant, irrelevant or marginal.
3. The renormalization group analysis: strategy and set-up

To apply our methods, we need to cast the problem in the form of a path integral. Following any number of sources [9–12], we obtain the following expression for the partition function of free fermions:

\[ Z_0 = \int [d\psi \, d\tilde{\psi}] e^{S_0}, \]  

(3.1)

where

\[ S_0 = \int d^2 K \int_{-\infty}^{\infty} d\omega \, \tilde{\psi}(\omega, K) \left( i\omega - \frac{(K^2 - K_F^2)}{2m} \right) \psi(\omega, K), \]  

(3.2)

\( \psi \) and \( \tilde{\psi} \) are Grassmann variables and \([d\psi \, d\tilde{\psi}]\) is the measure for the functional integral.

We now adapt this general expression to the thin annulus to obtain

\[ S_0 = \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega \int_{-A}^{A} dk \tilde{\psi}(i\omega - v k) \psi \]  

(3.3)

upon approximating as follows:

\[ \frac{K^2 - K_F^2}{2m} \approx K_F \cdot \frac{v_F}{m} k, \]  

(3.4)

where \( k = K - K_F \) and \( v_F \) is the Fermi velocity, hereafter set equal to unity. We have also replaced \( K \, dK \) by \( K_F \, dk \) and absorbed \( K_F \) in \( \psi \) and \( \tilde{\psi} \). It will be seen that neglecting \( k \) in relation to \( K_F \) is irrelevant in the technical sense.

We now perform mode elimination and reduce the cut-off by a factor \( s [12–16] \). Since this is a Gaussian integral, mode elimination just leads to a multiplicative constant in which we are not interested. So the result is just the same action as above, but with \( |k| \leq A/s \). Let us now make the following additional transformations:

\( (\omega', k') = s(\omega, k) \)

and

\[ [\psi'(\omega', k'), \tilde{\psi}'(\omega', k')] = s^{-3/2} \left[ \psi \left( \frac{\omega'}{s}, \frac{k'}{s} \right), \tilde{\psi} \left( \frac{\omega'}{s}, \frac{k'}{s} \right) \right]. \]  

(3.5)

When we do this, the action \( S_0 \) returns to its old form in every aspect. So what? Recall that our plan is to evaluate the role of quartic interactions in low-energy physics as we do mode elimination. Now what really matters is not the absolute size of the quartic term, but its size relative to the quadratic term. Keeping the quadratic term identical before and after the RG action makes the comparison easy: if the quartic coupling grows, it is relevant; if it decreases, it is irrelevant; and if it stays the same, it is marginal. The system is clearly gapless if the quartic coupling is irrelevant. Even a marginal coupling implies no gap since any gap will grow under the various rescalings of the RG.
Let us now turn on a generic four-Fermi interaction in path-integral form,

\[ S_4 = \int \bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1)u(4, 3, 2, 1), \quad (3.6) \]

where \( \int \) is a shorthand for

\[ \int \equiv \prod_{i=1}^{3} \int d\theta_i \int_{-A}^{A} dk_i \int_{-\infty}^{\infty} d\omega_i. \quad (3.7) \]

At the tree level, we simply keep the modes within the new cut-off, rescale fields, frequencies and momenta, and read off the new coupling. We find

\[ u'(k', \omega', \theta) = u\left(\frac{k'}{s}, \frac{\omega'}{s}, \theta\right). \quad (3.8) \]

This is the evolution of the coupling function. To deal with coupling constants with which we are more familiar, we expand the functions on both sides in a Taylor series (schematic)

\[ u = u_0 + ku_1 + k^2u_2 + \ldots, \quad (3.9) \]

where \( k \) stands for all the \( k \)'s and \( \omega \)'s. An expansion of this kind is possible as our problem is assumed to have at its microscopic origins only short-range interactions. Given this, mode elimination, which involves integrals that are cut off at both ends, cannot introduce singularities.

Comparing coefficients in equation (3.8), we find that \( u_0 \) is marginal and the rest are irrelevant, as is any coupling of more than four fields. While this is similar to what happens in \( \phi_4^4 \), the scalar field theory in four dimensions with a quartic interaction, there is a big difference. In \( \phi_4^4 \), \( u_0 \) is a single number, the leading term in the Taylor series at the origin in \( k \)-space. Here, since the expansion is not about a point (the origin) but a surface (the Fermi surface), \( u_0 \) will depend on the angles that parametrize it,

\[ u_0 = u(\theta_1, \theta_2, \theta_3, \theta_4). \]

Let us analyse this function. Momentum conservation should allow us to eliminate one angle. Actually it allows us more [12,15] because the momenta come from a very thin annulus near \( K_F \). Look at the left half of figure 2. Assuming that the cut-off has been reduced to the thickness of the circle in the figure, it is clear that if two points, 1 and 2, are chosen from it to represent the incoming lines in a four-point coupling, the outgoing ones are forced to be equal to them (not in their sum, but individually) up to a permutation, which is irrelevant for spinless fermions. Thus, we have in the end just one function of two angles, and by rotational invariance, their difference

\[ u(\theta_1, \theta_2, \theta_3, \theta_4) = F(\theta_1 - \theta_2) \equiv F(\theta). \quad (3.10) \]

About 40 years ago, Landau [17] came to the very same conclusion that a Fermi system at low energies would be described by one function defined on the Fermi surface. He did this without the benefit of the RG, and for that reason, some
of the leaps were hard to understand. Later, detailed diagrammatic calculations justified this picture [18]. The RG provides yet another way to understand it. It also tells us other things, as we will now see.

A significant exception to the preceding kinematical argument arises when the incoming and outgoing momenta are equal and opposite. Now the total momentum $P$ vanishes. The direction of the final pair is no longer slaved to that of the incoming pair. As is clear from the right half of figure 2, the final pair can point in any direction as long they are also equal and opposite. This leads to one more allowed marginal coupling besides $F$, this time in the Bardeen Cooper Schrieffer (BCS) channel,

$$u(\theta_1, -\theta_1, \theta_3, -\theta_3) = V(\theta_3 - \theta_1) \equiv V(\theta).$$

### 4. Renormalization group at one loop

Since $F$ and $V$ are marginal at tree level, we must go to one loop to determine their ultimate fate. So we draw the usual diagrams shown in figure 3 to eliminate an infinitesimal momentum slice of thickness $dA$ at $k = \pm A$. While $k$’s of the propagators are restricted as above, the angles go all around the shell and frequencies are integrated between $\pm \infty$. In other words, we are reducing the single-particle Hilbert space, but keeping the theory local in time, i.e. extracting an effective Hamiltonian and not just action [12,15].

These diagrams have the topology familiar to other quartic field theories, but behave differently.

Consider the renormalization of $F$ due to the first one (called zero sound (ZS)). The external momenta have zero frequencies and lie on the Fermi surface since $\omega$ and $k$ are irrelevant. The momentum and frequency transfer at either vertex is exactly zero. So the integrand has the following schematic form:

$$\delta F \simeq \int d\theta \int dk \int d\omega \left( \frac{1}{(i\omega - \epsilon(K))} \frac{1}{(i\omega - \epsilon(K))} \right).$$

The loop momentum $K$ lies in one of the two shells being eliminated. Since there is no energy difference between the two propagators, the poles in $\omega$ lie in the same half-plane and we get zero upon closing the contour in the other half-plane.
Figure 3. One-loop diagrams for the flow of $F$ and $V$. The last at the bottom shows that a large momentum $Q$ can be absorbed only at two particular initial angles (only one of which is shown) if the final state is to lie in the shell being eliminated.

In other words, this diagram can only contribute if it is a particle–hole diagram, but given zero momentum transfer, we cannot convert a hole at $-A$ to a particle at $+A$.

In the ZS diagram, we have a large momentum transfer, called $Q$ in the inset at the bottom of figure 3. This is of order $K_F$ and much bigger than the radial cut-off, a phenomenon unheard of in say $\phi^4$ theory, where all momenta and transfers are bounded by $A$. This, in turn, means that the loop momentum is restricted not only in the magnitude to a sliver $dA$, but also in the angular direction also of width $dA$ in order to be able to absorb this huge momentum $Q$ and land up in the other shell being eliminated. So we have $\delta F \simeq dt^2$, where $dt = -dA/A$. The same goes for the BCS diagram. Thus, $F$ does not flow at one loop.

Let us now turn to the renormalization of $V$. The first two diagrams are useless for the same reasons as before, but the last one is special. Since the total incoming momentum is zero, the loop momenta are equal and opposite, and no matter what direction $K$ has, $-K$ is guaranteed to lie in the same shell being eliminated. However, the loop frequencies are now equal and opposite so that the poles in the two propagators now lie in opposite half-planes. We now get a flow, dropping constants $[12,14,15]$,

$$\frac{dV(\theta_1 - \theta_3)}{dt} = - \int d\theta \ V(\theta_1 - \theta) V(\theta - \theta_3).$$

(4.2)
Here is an example of a flow equation for a coupling function. However, by expanding in terms of angular momentum eigenfunctions, we get an infinite number of flow equations for the coefficients $V_m$,

$$\frac{dV_m}{dt} = -V_m^2,$$  \hspace{1cm} (4.3)

one for each coefficient. These equations tell us that if the potential in angular momentum channel $m$ is repulsive, it will get renormalized down to zero (a result derived many years ago by Morel & Anderson [19], while if it is attractive, it will run off, causing the BCS instability. This is the reason why the $V$’s are not a part of the Landau theory, which assumes we have no phase transitions. This is also a nice illustration of what was stated earlier: one could begin with a large positive coupling say $V_3$ and a tiny negative coupling $V_5$. After much renormalization, $V_3$ would shrink to a tiny value and $V_5$ would dominate.

5. Large-$N$ approach to Fermi liquids

Not only did Landau say we could describe Fermi liquids with an $F$ function, he also managed to compute the response functions at small $\omega$ and $q$ in terms of the $F$ function, even when it was large, say 10, in dimensionless units. Again the RG gives us one way to understand this. To see how, we need to recall the key ideas of ‘large-$N$’ theories.

These theories involve interactions between $N$ species of objects. The largeness of $N$ renders fluctuations (thermal or quantum) small and enables one to make approximations that are not perturbative in the coupling constant, but are controlled by the additional small parameter $1/N$.

As a specific example, let us consider the Gross–Neveu model [20], which is one of the simplest fermionic large-$N$ theories. This theory has $N$ identical massless relativistic fermions interacting through a short-range interaction. The Lagrangian density is

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i \slashed{D} \psi_i + \frac{\lambda}{N} \left( \sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^2.$$  \hspace{1cm} (5.1)

Note that the kinetic term conserves the internal index, as does the interaction term: any index that goes in comes out. You do not have to know much about the Gross–Neveu model to follow this discussion, which is all about the internal indices.

Figure 4 shows the first few diagrams in the expression for the scattering amplitude of particle of isospin indices $i$ and $j$ in the Gross–Neveu theory. The bare vertex comes with a factor $\lambda/N$. The one-loop diagrams all share a factor $\lambda^2/N^2$ from the two vertices. The first one-loop diagram has a free internal summation over the index $k$ that runs over $N$ values, with the contribution being identical for each value of $k$. Thus, this one-loop diagram acquires a compensating factor of $N$ that makes its contribution of order $\lambda^2/N$, the same order in $1/N$ as the bare vertex. However, the other one-loop diagrams have no such free internal summation and their contribution is indeed of order $1/N^2$. Therefore, to leading order in $1/N$, one should keep only diagrams that have a free internal summation.
Figure 4. Some diagrams from a large-$N$ theory.

for every vertex, that is, iterates of the leading one-loop diagram, which are called bubble graphs. For later use, remember that in the diagrams that survive (do not survive), the indices $i$ and $j$ of the incoming particles do not (do) enter the loops. Let us assume that the momentum integral up to the cut-off $\Lambda$ for one bubble gives a factor $-\Pi(A, q_{\text{ext}})$, where $q_{\text{ext}}$ is the external momentum or frequency transfer at which the scattering amplitude is evaluated. To leading order in large $N$, the full expression for the scattering amplitude is

$$G(q_{\text{ext}}) = \frac{1}{N} + \frac{\lambda}{1 + \lambda \Pi(A, q_{\text{ext}})}.$$  \hfill (5.2)

Once one has the full expression for the scattering amplitude (to leading order in $1/N$), one can ask for the RG flow of the ‘bare’ vertex as the cut-off is reduced by demanding that the physical scattering amplitude $\Gamma$ remains insensitive to the cut-off. One then finds (with $t = \ln(A_0/\Lambda)$)

$$\frac{d\Gamma(q_{\text{ext}})}{dt} = 0 \Rightarrow \frac{d\lambda}{dt} = -\lambda^2 \frac{d\Pi(A, q_{\text{ext}})}{dt} \simeq -\lambda^2,$$  \hfill (5.3)

which is exactly the flow one would extract at one loop. Thus, the one-loop RG flow is the exact answer to leading order in a large-$N$ theory. All higher order corrections must therefore be subleading in $1/N$.

(a) Large $N$ applied to Fermi liquids

Consider now the $\bar{\psi}\psi - \bar{\psi}\psi$ correlation function (with vanishing values of external frequency and momentum transfer). Landau showed that it takes the form

$$\chi = \frac{\chi_0}{1 + F_0},$$  \hfill (5.4)

where $F_0$ is the angular average of $F(\theta)$ and $\chi_0$ is the answer when $F = 0$. Note that the answer is not perturbative in $F$.

Landau got this result by working with the ground-state energy as a functional of Fermi-surface deformations. The RG provides us with not just the ground-state energy, but an effective Hamiltonian (operator) for all of low-energy physics. This operator problem can be solved using large-$N$ techniques.

The value of $N$ here is of order $K_F/\Lambda$, and here is how it enters the formalism [12]. (For an alternate route, see [21].) Imagine dividing the annulus in figure 1 into $N$ patches of size ($\Lambda$) in the radial and angular directions. The momentum
of each fermion $K_i$ is a sum of a 'large' part ($O(K_F)$) centred on a patch labelled by a patch index $i = 1, \ldots, N$ and a 'small' momentum ($O(A)$) within the patch.

Consider a four-fermion Green’s function, as in figure 4. The incoming momenta are labelled by the patch index (such as $i$), and the small momentum is not shown but implicit. We have seen that as $A \to 0$, the two outgoing momenta are equal to the two incoming momenta up to a permutation. At small but finite $A$, this means the patch labels are the same before and after. Thus, the patch index plays the role of a conserved isospin index as in the Gross–Neveu model.

The electron–electron interaction terms, written in this notation (with $k$ integrals replaced by a sum over patch index and integration over small momenta over patches of size $A \times A$) also come with a pre-factor of $1/N (\approx L/K_F)$.

It can then be verified that in all Feynman diagrams of this cut-off theory, the patch index plays the role of the conserved isospin index exactly as in a theory with $N$ fermionic species. For example, in figure 4, in the first diagram, the external indices $i$ and $j$ do not enter the diagram (small momentum transfer only) and so the loop momentum is nearly the same in both lines and integrated fully over the annulus, i.e. the patch index $k$ runs over all $N$ values. In the second diagram, the external label $i$ enters the loop, and there is a large momentum transfer ($O(K_F)$). In order for both momenta in the loop to be within the annulus, and to differ by this large $Q$, the angle of the loop momentum is limited to a range $O(L/K_F)$. (This just means that if one momentum is from patch $i$, the other has to be from patch $j$.) Similarly, in the last loop diagram, the angle of the loop momenta is restricted to one patch. In other words, the requirement that all loop momenta in this cut-off theory lie in the annulus singles out only the bubble diagrams that survive in the large-$N$ limit.

The sum of bubble diagrams, singled out by the usual large-$N$ considerations, reproduces Landau’s Fermi liquid theory [22,23]. For example, in the case of the compressibility $\chi$, one obtains a geometric series that sums to give $\chi = \chi_0/(1 + F_0)$.

Since in the large-$N$ limit, the one-loop $\beta$-function for the fermion–fermion coupling is exact, it follows that the marginal nature of the Landau parameters $F$ and the flow of $V$ in equation (4.3) are both exact as $A \to 0$.

6. Extensions and application in related fields

The theory is readily extended to non-circular Fermi surfaces in $d = 2$. While the final momenta will still equal the initial ones, the fixed-point couplings will be a function of the two incoming angles and not just their difference. If the Fermi surface is nested at half-filling, i.e. every point on the Fermi surface maps to another point on it upon adding a fixed vector, say $(Q = \pi, \pi)$, the theory predicts a charge density wave instability. What happens is that besides $F$ and $V$, a third marginal coupling $W$ arises in which the two final momenta differ from the initial ones by $Q = \pi, \pi$ (allowed since $2Q$ is a reciprocal lattice vector), and this $W$ flows to strong coupling for repulsive interactions.

There is a natural way to extend the theory to higher spatial dimensions and to include spin. In $d > 2$, the fixed point will allow non-forward scattering [12]. The Landau parameters that correspond to the forward scattering amplitude will emerge when we compute energies for deforming the Fermi surface. The flow in the BCS channel will still separate into angular momentum channels.

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In addition to giving us a way to understand Fermi liquid theory and its instabilities, the RG allows other applications. One can ask how a non-Fermi liquid of the type Anderson [24] envisaged may emerge by asking how the above derivation could fail. One way is at strong coupling where the perturbative RG does not converge, say owing to singular Fermi liquid parameters [25]. Another option is to consider fermions coupled to gauge fields either in strongly correlated electrons [26] or in the fractional quantum Hall problem at half-filling [27]. One can ask whether a repulsive interaction can spawn superconductivity. This idea has been implemented many times, generally in two stages: integrate high-energy modes perturbatively (to one loop say) to get an effective set of interactions and then run the flow as above starting with these as initial couplings [28–32]. One can also try to derive the ideas espoused here more rigorously [33], though as of now, no small parameters besides $A/K_F = 1/N$ or weak coupling has emerged to justify approximations that inevitably follow in any practical calculation.

The $1/N$ expansion may be used to study problems where the interaction has frequency dependence, as in the BCS problem with retardation [34].

It is possible to use the RG, even if bosons are simultaneously present, provided the dynamical exponent $z = 1$ [35].

The theory may be applied to nuclear matter or stars [36], as well as quarks at finite density [37,38].

One can also vary the power law of the force between fermions continuously to a controllable fixed point [39].

The theory may also be extended to quantum dots to address a problem with finite size, disorder and interactions [40].

If the present approach is applied to fermions in $d$ spatial dimensions, the scaling is done only for $\omega$ and $k = K - K_F$, since the other angles (like $\theta$) on the Fermi surface behave like internal or isospin degrees of freedom. Thus, the Fermi liquid behaves like a $(1+1)$-dimensional field theory for all $d \geq 2$. A new generalization has been proposed [41] to Fermi surfaces with more co-dimensions on which the energy depends linearly. For example, in three dimensions, a line Fermi surface has co-dimension 2 and behaves like a $(2+1)$-dimensional theory. It is also possible to vary these extra dimensions in an $\epsilon$ expansion and obtain new weak coupling fixed points.

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