We discuss the physics of the Efimov effect from a renormalization group viewpoint using the concept of limit cycles. Furthermore, we discuss recent experiments providing evidence for the Efimov effect in ultracold gases and its relevance for nuclear systems.

Keywords: universality; few-body physics; limit cycle; discrete scale invariance

1. Introduction

The Efimov effect was discovered by Vitaly Efimov 40 years ago [1] but its significance for low-energy universality was only realized in the last decade. It describes the appearance of a discrete scaling symmetry in the non-relativistic three-body system with a large two-body scattering length $a$. Efimov showed that there are infinitely many three-body bound states with an accumulation point at zero energy in the unitary limit of infinite scattering length ($a \to \pm \infty$). The ratio of the binding energies of two successive bound states approaches a universal number $\approx 515$ as one approaches the threshold.

The Efimov effect also has important consequences for finite scattering length. In particular, it provides the starting point for a systematic effective field theory (EFT) description of few-body observables when the two-body scattering length $a$ is much larger than the range $\ell$ of the underlying interaction. The EFT approach allows then to include the corrections owing to the finite range of the interaction perturbatively. A more detailed discussion of these aspects can be found in the reviews by Braaten & Hammer [2] and Platter [3].

The Efimov effect has recently received a significant amount of attention as modern experimental techniques have allowed the demonstration of its existence in systems that have a large scattering length. In particular, the ability to tune the scattering length in experiments with ultracold atoms has allowed the demonstration of the discrete scaling symmetry by measuring atom loss rates...
in ultracold gases [4]. The Efimov effect plays also an important role in nuclear physics [5]. The nuclei $^3$H and $^3$He can be considered ground states of an Efimov spectrum with only one state. Some halo nuclei, which consist of a tightly bound core and a halo of weakly bound nucleons, are also candidates for Efimov states.

The renormalization group (RG) provides a powerful method to understand the physics associated with the Efimov effect. This tool has improved our understanding of critical phenomena in condensed matter physics and of the asymptotic behaviour of field theories in particle physics [6]. In this review, we discuss recent progress in the theoretical description and experimental observation of the Efimov effect. We emphasize the importance of RG concepts as a tool for understanding Efimov physics. In §2, we briefly introduce the concept of the RG. We then discuss the connection between RG fixed points and limit cycles to continuous and discrete scaling symmetries. These ideas are illustrated using the inverse square potential as an example. In §3, we demonstrate that the Efimov effect is a result of an infrared limit cycle and discuss some of the observable features associated with the term Efimov physics. The consequences of Efimov physics in the four-body system are discussed in §4. In §5, we give a broad overview of Efimov physics in systems from atomic to nuclear physics. We end with conclusions and an outlook.

2. The renormalization group

The RG is one of the most important developments in modern physics [6]. The RG has been particularly important for the understanding of critical phenomena. This term is used to describe universal behaviour in second-order phase transitions. Near a critical point certain thermodynamic observables show power-law behaviour. This power-law behaviour at the critical point implies scale invariance of the system. For example, the correlation length $\xi$ diverges as $T \to T_c$:

$$\xi(T) \to \text{const. } |T - T_c|^{-\nu},$$

where $T_c$ is the critical temperature and $\nu$ is a critical exponent. As a consequence, the system looks the same at all distance scales. The critical exponents are the same for systems in the same universality class. An example of two such systems in the same universality class is liquid–gas systems and a ferromagnet with one easy axis of magnetization. Even though their dynamics at short distances is very different, they share the same critical exponents. From the perspective of the RG, this scaling behaviour is characterized by a fixed point. Such fixed points also play an important role in quantum field theories in particle physics. Asymptotically free theories like quantum chromodynamics (QCD), for example, have an ultraviolet fixed point that corresponds to zero coupling [7,8].

In order to understand the concept of RG fixed points, it is convenient to expand the Hamiltonian $\mathcal{H}$ governing the system in a basis of operators $\mathcal{O}_n$:

$$\mathcal{H} = \sum_n g_n \mathcal{O}_n.$$  

We refer to the basis coefficients $g_n$ as the coupling constants of the Hamiltonian. A particular Hamiltonian is represented by a point $\mathbf{g} = (g_1, g_2, \ldots)$ in the infinite dimensional vector space. If we now introduce an ultraviolet cutoff $\Lambda$, an RG
transformation is defined as a transformation that eliminates short-distance
degrees of freedom (lowers the cutoff \( \Lambda \)) while the long-distance observables
remain unchanged. We treat the cutoff \( \Lambda \) as a continuous variable and the RG
transformation defines thus a flow in the space of coupling constants. This flow
can be expressed through a differential equation for the coupling \( g \)
\[
\beta(g) = \Lambda \frac{d}{d\Lambda} g,
\]
(2.3)
where \( \beta(g) \) is the beta function. This equation can be used to define the concept
of a fixed point. A fixed point \( g_* \) in the space of coupling constants fulfils
\[
\beta(g_*) = 0.
\]
(2.4)
Wilson discovered that scale-invariant behaviour at long distances is a result of
a fixed point in the infrared limit \( \Lambda \to 0 \) (infrared fixed point). The Hamiltonian
\( \mathcal{H}_* \) then remains invariant under a change of the ultraviolet cutoff \( \Lambda \). Since \( \Lambda \)
defines the length scale of the system, the system is, therefore, scale invariant at
the fixed point.

Equation (2.4) can be linearized in the vicinity of a fixed point
\[
\Lambda \frac{d}{d\Lambda} g \approx B(g - g_*),
\]
(2.5)
where \( B \) is a linear operator. The eigenvalues of the operator \( B \) are the critical
exponents. Operators with positive, zero and negative critical exponents are called
relevant, marginal and irrelevant operators, respectively. Relevant operators
become more important for large \( \Lambda \), while irrelevant operators become less
important. Marginal operators are a special case and higher order corrections
are important to decide their fate. The critical exponent \( \nu \) in equation (2.1) is
related to the critical exponent of an appropriate operator in the Hamiltonian.

Another solution to equation (2.3) corresponds to a limit cycle. In this case,
the RG trajectory flows around a closed loop in the space of coupling constants.
A limit cycle can be understood as a family of Hamiltonians \( \mathcal{H}_*(\theta) \) that is closed
under the RG flow and can be parametrized by an angle \( \theta \) that runs from 0 to
2\( \pi \). All coupling constants in the Hamiltonian \( \mathcal{H} \) will return to their initial value
once the cutoff is changed by a factor \( S_0 \) (the preferred scaling factor):
\[
g(\Lambda) = g_* \left( \theta_0 + 2\pi \frac{\ln \left( \frac{\Lambda}{\Lambda_0} \right)}{\ln(S_0)} \right).
\]
(2.6)
A necessary condition for a limit cycle is invariance under discrete scale
transformations: \( x \to (S_0)^n x \), where \( n \) is an integer. This discrete scaling
symmetry is reflected in log-periodic behaviour of physical observables. With
a single coupling constant, a limit cycle can only occur if the coupling has
discontinuities. If this is the case, the RG equation has two complex conjugate
fixed point solutions, e.g. [9]. The preferred scaling factor \( S_0 \) is then determined
by the imaginary part of the fixed point coupling.

The possibility of RG limit cycles was first discussed by Wilson in a work
that applied the concepts of the RG to the strong interactions of elementary
particle physics [10]. The Efimov effect can be understood in terms of an RG limit

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cycle with discrete scaling factor \( S_0 \simeq 22.7 \) [11]. In the limit \( a \to \pm \infty \), there is an accumulation of three-body bound states near threshold with binding energies differing by multiplicative factors of \( (S_0)^2 \simeq 515.03 \) [1]. In the EFT formulation of Bedaque et al. [12], the limit cycle becomes explicit in the running of the leading-order three-body contact interaction required for renormalization. An RG analysis of the three-body problem was carried out by Barford & Birse [13].

We will use the quantum mechanical \( 1/r^2 \) potential as an example that exhibits both fixed point and limit cycle solutions. Our discussion follows [14,15]. Consider the three-dimensional Schrödinger equation with the potential \( V(r) \)

\[
V(r) = \frac{\alpha}{r^2},
\]

where we have set \( \hbar = m = 1 \) for convenience, such that \( \alpha \) is dimensionless. This potential is scale invariant at the classical level. Rescaling all distances by a factor \( l \) and rescaling time by a factor of \( l^2 \) leaves the Hamiltonian invariant up to a constant prefactor \( 1/l^2 \).

It is straightforward to show that the S-wave solution to the Schrödinger equation for \( \alpha_* < \alpha < \alpha_* + 1 \) and zero energy is given by

\[
\psi(r) = c_- r^{n-} + c_+ r^{n+} \quad \text{and} \quad n_{\pm} = -\frac{1}{2} \pm \sqrt{\alpha - \alpha_*},
\]

where

\[
\alpha_* \equiv -\frac{1}{4}.
\]

If either \( c_- = 0 \) or \( c_+ = 0 \), this solution is scale invariant. We can identify the solution \( c_- = 0 \) with an infrared fixed point and the solution \( c_+ = 0 \) with an ultraviolet fixed point. Formally, one can pick the desired solution by adding a counterterm potential with coupling constant \( g \) to the Hamiltonian. This is equivalent to putting a boundary condition on the wave function at short distances. The coupling constant \( g \) satisfies an RG equation with a \( \beta \)-function that takes the form [14]

\[
\beta(g; \alpha) = (\alpha - \alpha_*) - (g - g_*)^2.
\]

For \( \alpha > \alpha_* \), the fixed points of \( g \) are

\[
g_{\pm} = g_* \pm \sqrt{\alpha - \alpha_*},
\]

where \( g_- \) and \( g_+ \) correspond to the infrared and ultraviolet fixed points for the solutions with \( c_- = 0 \) and \( c_+ = 0 \), respectively.

As \( \alpha \) is decreased and approaches \( \alpha_* \) from above, the two fixed points approach each other and merge at \( g_{\pm} = g_* \). For \( \alpha < \alpha_* \), the fixed point equation \( \beta(g; \alpha) = 0 \) has two complex conjugate solutions

\[
g_{\pm} = g_* \pm i\sqrt{\alpha_* - \alpha},
\]

which correspond to a limit cycle with preferred scaling factor

\[
S_0 = \exp\left(\frac{\pi}{\sqrt{\alpha_* - \alpha}}\right).
\]

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The continuous scaling symmetry of the system is now broken to a discrete scaling symmetry by quantum fluctuations at small distances. This symmetry breaking can be interpreted as a quantum mechanical anomaly [16]. The discrete scaling symmetry is manifest in observables such as in the geometric bound-state spectrum
\[ B(n) = B_0(S_0)^{-2n}, \]  
(2.14)
as well as the log-periodic dependence of the S-wave scattering phase shift
\[ \delta(k) = \delta_0 - \frac{\pi \ln k}{\ln S_0}, \]  
(2.15)
and the corresponding cross section on the momentum \( k \).

3. Efimov effect

In 1970, Efimov analysed the three-nucleon system interacting through zero-range interactions. This system had been considered before, but he was the first to realize that one should not focus on the pathologies at large energies but on the universal physics at low energies, \( E \ll \hbar^2/m\ell^2 \). In this limit, where zero-range forces are adequate, he found some surprising results [1]. He pointed out that when \( |a| \) is sufficiently large compared with the range \( \ell \) of the potential, there is a sequence of three-body bound states whose binding energies are spaced roughly geometrically in the interval between \( \hbar^2/m\ell^2 \) and \( \hbar^2/ma^2 \). As \( |a| \) is increased, new bound states appear in the spectrum at critical values of \( a \) that differ by multiplicative factors of \( e^{\pi/s_0} \), where \( s_0 \) depends on the statistics and the mass ratios of the particles. In the case of spin-doublet neutron–deuteron scattering and for three identical bosons, \( s_0 \) is the solution to the transcendental equation
\[ s_0 \cosh \frac{\pi s_0}{2} = \frac{8}{\sqrt{3}} \sinh \frac{\pi s_0}{6}. \]  
(3.1)
Its numerical value is \( s_0 \approx 1.00624 \) and the preferred scaling factor is
\[ S_0 = e^{\pi/s_0} \approx 22.7. \]  
(3.2)
As \( |a|/\ell \to \infty \), the asymptotic number of three-body bound states is
\[ N \to \frac{1}{\ln S_0} \ln \frac{|a|}{\ell}. \]  
(3.3)
Equation (3.3) expresses the fact that in any system with a finite range, the number of Efimov states is bounded from above owing to the finite range of the interaction. Since Efimov’s approach requires the scattering length \( a \) to be significantly larger than the range of the interaction \( \ell \), Efimov states must have binding energies smaller than \( \hbar/\ell m^2 \). The three-nucleon systems \(^3\)H or \(^3\)He contain, therefore, only one Efimov state, which is also the ground state. In the limit \( a \to \pm \infty \), there are infinitely many three-body bound states.
Figure 1. The Efimov plot for the three-body problem. We show $K \equiv \text{sgn}(E)(m|E|)^{1/2}/\hbar$ versus the inverse scattering length $1/a$. The solid lines denote some of the infinitely many branches of Efimov states while the cross-hatching indicates the threshold for scattering states. Any one of the quantities $a_\star$, $a_\star'$ and $\kappa_\star$ can be used to specify the branch labelled with $n = n_\star$. All other states follow from the discrete scaling symmetry. (Online version in colour.)

with an accumulation point at the three-body scattering threshold with a geometric spectrum:

$$B_t^{(n)} = S_0^{-2(n-n_\star)} \frac{\hbar^2 \kappa_\star^2}{m},$$

where $m$ is the mass of the particles and $\kappa_\star$ is the binding wavenumber of the branch of Efimov states labelled by $n_\star$ (figure 1). The geometric spectrum in equation (3.4) is the signature of a discrete scaling symmetry with scaling factor $S_0 \approx 22.7$. It is independent of the mass or the structure of the identical particles and independent of the form of their short-range interactions. The Efimov effect can also occur in other three-body systems if at least two of the three pairs have a large S-wave scattering length but the numerical value of the asymptotic ratio may differ from the value $22.7^2 \approx 515$.

A formal proof of the Efimov effect was subsequently given by Amado & Noble [17,18]. The Thomas and Efimov effects are closely related. The deepest three-body bound states found by Thomas’ variational calculation can be identified with the deepest Efimov states [19].

The universal properties in the three-body system with large scattering length are not restricted to the Efimov effect. The dependence of three-body observables on the scattering length is characterized by scaling behaviour modulo coefficients that are log-periodic functions of $a$ [20,21]. This behaviour is characteristic of a system with a discrete scaling symmetry. We will refer to universal aspects associated with a discrete scaling symmetry as Efimov physics.

In 1981, Efimov proposed a new approach to the low-energy few-nucleon problem in nuclear physics that, in modern language, was based on perturbation theory around the unitary limit [22]. Remarkably, this programme works reasonably well in the three-nucleon system at momenta small compared with $M_\pi$. It has been used as the basis of an EFT for nuclear physics at very low energies [23,24].

Efimov formulated the three-body problem using hyperradial coordinates that are particularly well-suited for the analysis of few-body problems in coordinate space. In this approach, the six independent coordinates of the
problem are given by the hyper-radius and five angles. The hyper-radius \( R \) is defined as
\[
R^2 = \frac{1}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2), \tag{3.5}
\]
where \( r_{ij} = r_i - r_j \) is the separation between atoms \( i \) and \( j \). It is only small if all three particles are close together. Efimov showed that in these coordinates the three-body problem with a large scattering length reduces to a simple Schrödinger-like equation with a \( 1/R^2 \) potential. At short distances, the problem simplifies and the radial equation takes the form
\[
\frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial R^2} - \frac{s_0^2 + \frac{1}{4}}{R^2} \right) f_0(R) = E f_0(R), \quad R \ll a. \tag{3.6}
\]
As discussed in the previous section, the \( 1/R^2 \) potential requires a boundary condition on the wave function at short distances if the potential is sufficiently attractive. This boundary condition, which alternatively can be expressed through a short-range three-body force [12], can be fixed from any three-body observable.

In practical applications to the three-nucleon system, one uses either the spin-doublet neutron–deuteron scattering length or the triton binding energy as three-body input. If the deuteron binding energy and the spin-singlet scattering length are used as two-body input and if the boundary condition is fixed using the spin-doublet neutron–deuteron scattering length, the triton binding energy is predicted with an accuracy of 6 per cent [25]. The accuracy of the predictions can be further improved by taking into account the effective range as a first-order perturbation [26]. Thus, the triton can be identified as an Efimov state associated with the deuteron being a \( pn \) bound state with large scattering length [22].

The effective coupling of the potential in equation (3.6), \( \alpha = -(s_0^2 + 1/4) \), is smaller than \( \alpha_\ast = -1/4 \). This implies that the RG flow of the counterterm introduced for regularization of the problem will display limit cycle behaviour. This limit cycle has important consequences for observables. For example in the unitary limit, the binding energies scale as given in equation (3.4). For finite scattering length, observables such as binding energy and cross sections still scale with integer powers of \( S_0 = \exp(\pi/s_0) \) under this symmetry. For example, the binding energy of an Efimov trimer which is a function of \( a \) and \( k_\ast \) scales
\[
B_3^{(n)}(S_0^m a, \kappa_\ast) = S_0^{-2m} B_3^{(n-m)}(a, \kappa_\ast). \tag{3.7}
\]
This implies for positive scattering length
\[
B_3^{(n)}(a, \kappa_\ast) = F_n(2s_0 \ln(a\kappa_\ast)) \frac{\hbar^2 \kappa_\ast^2}{m}. \tag{3.8}
\]
The function \( F_n \) parametrizes the scattering length dependence of all Efimov trimers exactly in the limit of vanishing range. The function \( F_n \) satisfies
\[
F_n(x + 2m\pi) = S_0^{-2m} F_{n-m}(x). \tag{3.9}
\]
The scattering length dependence of the bound-state spectrum is shown in figure 1. We plot the quantity \( K \equiv \text{sgn}(E)(m|E|)^{1/2}/\hbar \) against the inverse scattering length. For bound states \( K \) corresponds to the binding momentum. The
solid lines denote the Efimov trimers while the scattering threshold is indicated by the hatched area. For $a < 0$, the relevant threshold is the three-particle threshold. For $a > 0$, the atom–dimer threshold is lower in energy. Only a few of the infinitely many Efimov branches are shown. A given physical system has a fixed scattering length value and the corresponding states lie on a vertical line. Changing the parameter $\kappa_*$ by a factor $S_0$ corresponds to multiplying each branch of trimers with this factor without changing their shapes. One important result is that three-bound states exist for positive and negative scattering length. This is remarkable for the latter case as the two-body subsystem is unbound for $a < 0$. At a negative scattering length denoted by $a'_*$, a bound state with a given $\kappa_*$ has zero binding energy. As the inverse scattering length is increased, the trimer binding energy gets larger until it crosses the atom–dimer threshold at the positive scattering length $a_*$. The quantities $a_*$, $a'_*$ and $\kappa_*$ are related to each other and can be used to quantify a branch of Efimov states. The other branches then follow from the discrete scaling symmetry.

4. The four-body legacy and beyond

We discussed in the previous section that one three-body parameter is required for a consistent description of the three-body system with zero-range interactions. It is, therefore, natural to ask how many parameters are needed for calculations in the $N$-body system. A first step towards answering this question was performed in the work by Platter et al. [27]. The authors of that work showed that the two-body scattering length and one three-body parameter are sufficient to make predictions for four-body observables. Results of a more detailed analysis [28] also lead to the conclusion that every trimer state is tied to two universal tetramer states with binding energies related to the binding energy of the next shallower trimer. In the unitary limit $1/a = 0$, the relation between the tetramer and trimer binding energies was found as

$$B_4^{(0)} \approx 5B_3 \quad \text{and} \quad B_4^{(1)} \approx 1.01B_3,$$

(4.1)

where $B_4^{(0)}$ denotes the binding energy of the deeper of the two tetramer states and $B_4^{(1)}$ the shallower of the two.

A recent calculation by von Stecher et al. [29] supports these findings and extends them to higher numerical accuracy. For the relation between universal three- and four-body bound states in the unitary limit, they found

$$B_4^{(0)} \approx 4.57B_3 \quad \text{and} \quad B_4^{(1)} \approx 1.01B_3,$$

(4.2)

which is consistent with the results given in equation (4.1) within the numerical accuracy. The universality of the deeper tetramer was also confirmed in a functional RG analysis by Schmidt & Moroz [30], but the shallower tetramer could not be resolved.

The results obtained by Hammer & Platter [28] were furthermore presented in the form of an extended Efimov plot, shown in figure 2. Four-body states have to have a binding energy larger than that of the deepest trimer state. The corresponding threshold is denoted by the dashed line in figure 2. At positive
scattering length, there are also scattering thresholds for scattering of two dimers and scattering of a dimer and two particles indicated by the dash-dotted and dash-dash-dotted lines, respectively.

An extended version of this four-body Efimov plot was presented by von Stecher et al. [29]. They calculated more states with higher numerical accuracy and extended the calculation of the four-body states to the thresholds where they become unstable. From these results they extracted the negative values of the scattering lengths at which the binding energies of the tetramer states become zero and found

$$a_{4,0}^* \approx 0.43 a_s' \quad \text{and} \quad a_{4,1}^* \approx 0.92 a_s'.$$

These numbers uniquely specify the relative position of three- and four-body recombination resonances. This was the key information for the subsequent observation of these states in ultracold atoms by Ferlaino et al. [31].

Recently, Deltuva [32] calculated four-body scattering observables. He solved the atom–trimer scattering problem in momentum space and calculated the real and imaginary parts of the atom–trimer scattering length and effective range:

$$a_T = (18.5 + i 0.890) \kappa_s^{-1} \quad \text{and} \quad r_T = (2.63 + i 0.014) \kappa_s^{-1}.$$  

He also obtained more precise values for the universal relations between an Efimov trimer and the associated tetramers:

$$B_4^{(0)} \approx 4.6108 B_3 \quad \text{and} \quad B_4^{(1)} \approx 1.00228 B_3.$$  

Interestingly, the widths of the excited tetramer states are also universal. They are 0.3 per cent of the binding energy for the deeper state and 0.02 per cent of the binding energy for the shallower state [32].
The bound-state properties of larger systems of bosons interacting through short-range interactions were considered by Hanna & Blume [33]. Using Monte Carlo methods they showed that universal correlations between binding energies can also be obtained for those properties.

Calculations for larger number of particles using a model that incorporates the universal behaviour of the three-body system were carried out by von Stecher [34]. His findings indicate that there is at least one $N$-body state tied to each Efimov trimer, and numerical evidence was also found for a second excited five-body state.

5. Physical systems

Efimov’s work was originally intended as a description of the three-nucleon system. However, experiments with ultracold atoms have proved to be an indispensable testing ground for universal predictions. Using Feshbach resonances, the scattering length of the atoms can be tuned experimentally by varying an external magnetic field (see [35] for more details). As a consequence, the discrete scaling symmetry can be tested by measuring the scattering length dependence of observables.

In the following two subsections, we first summarize recent experimental efforts that have provided evidence for the Efimov effect in ultracold atoms and then discuss nuclear systems in which Efimov physics is expected to play an important role.

(a) Atomic physics

The first experimental evidence for Efimov physics in ultracold atoms was presented by Kraemer et al. [36]. This group used $^{133}$Cs atoms in the lowest hyperfine spin state. They observed a resonant enhancement of the loss of atoms from three-body recombination that can be attributed to an Efimov trimer crossing the three-atom threshold. Such crossings occur when the scattering length is equal to $(S_0)^n a_\ast$ with $n$ an integer (cf. figure 1). The occurrence of recombination resonances at negative scattering length was predicted by Esry et al. [37]. The universal line shape for the resonance as a function of the scattering length was first derived by Braaten & Hammer [38]. Kraemer et al. also observed a minimum in the three-body recombination rate that can be interpreted as an interference effect associated with Efimov physics. One of the most exciting developments in the field of Efimov physics involves universal tetramer states. Ferlaino et al. recently observed two tetramers in an ultracold gas of $^{133}$Cs atoms [31] and confirmed the prediction of such states by Platter et al. [27], Hammer & Platter [28] and von Stecher et al. [29].

Recent experiments with other bosonic atoms have provided even stronger evidence of Efimov physics in the three- and four-body sectors. Zaccanti et al. [39] measured the three-body recombination rate and the atom–dimer loss rate in an ultracold gas of $^{39}$K atoms. They observed two atom–dimer loss resonances and two minima in the three-body recombination rate at large positive values of the scattering length. The positions of the loss features are consistent with the
universal predictions with discrete scaling factor 22.7. They also observed loss features at large negative scattering lengths. Barontini et al. [40] obtained the first evidence of the Efimov effect in a heteronuclear mixture of $^{31}$K and $^{87}$Rb atoms. They observed three-atom loss resonances at large negative scattering lengths in both the K–Rb–Rb and K–K–Rb channels, for which the discrete scaling factors are 131 and $3.51 \times 10^5$, respectively. A theoretical analysis of the heteronuclear case was presented in Helfrich et al. [41].

Gross et al. [42] measured the three-body recombination rate in an ultracold system of $^7$Li atoms. They observed a three-atom loss resonance at a large negative scattering length and a three-body recombination minimum at a large positive scattering length. The positions of the loss features, which are in the same universal region on different sides of a Feshbach resonance, are consistent with the universal predictions with discrete scaling factor 22.7. Pollack et al. [43] at Rice University measured the three-body recombination in a system of $^7$Li atoms in a different hyperfine state. They observed a total of 11 three- and four-body loss features. The features obey the universal relations on each side of the Feshbach resonance separately; however, a systematic error of approximately 50 per cent is found when features on different sides of the Feshbach resonance are compared. Recently, the Rice measurement was repeated by Gross et al. [44]. They found that recombination features were related across the Feshbach resonance as predicted by the universal relations. Gross et al. also remeasured the magnetic field strength at which the resonance occurs and found their result to be different from Pollack et al. [43]. They claim that this discrepancy might explain the deviations from universality in the Rice experiment.

Efimov physics has also been observed in the fermionic three-component system of $^6$Li atoms. For the three lowest hyperfine states of $^6$Li atoms, the three pair scattering lengths approach a common large negative value at large magnetic fields and all three have nearby Feshbach resonances at lower fields that can be used to vary the scattering lengths [45]. The first experimental studies of many-body systems of $^6$Li atoms in the three lowest hyperfine states were carried out by Ottenstein et al. [46] and by Huckans et al. [47]. Their measurements of the three-body recombination rate revealed a narrow loss feature and a broad loss feature in a region of a low magnetic field. Theoretical calculations of the three-body recombination rate supported the interpretation that the narrow loss feature arises from an Efimov trimer crossing the three-atom threshold [48–50]. Very recently, another narrow loss feature was discovered in a much higher region of the magnetic field by Williams et al. [51] and others. Williams et al. used measurements of the three-body recombination rate in this region to determine the complex three-body parameter that governs Efimov physics in this system. This parameter, together with the three scattering lengths as functions of the magnetic field, determines the universal predictions for $^6$Li atoms in this region of the magnetic field.

Another process that provides signatures for the Efimov effect is atom–dimer relaxation. This process can occur in a mixture of atoms and shallow dimers. There are two types of dimers in ultracold gases. First, there are the shallow dimers with a binding energy given by the universal formula

$$B_2 \approx \frac{\hbar^2}{ma^2},$$

(5.1)
Second, there are (many) deeply bound dimers with a binding energy of order $\frac{\hbar^2}{m\ell^2}$. These dimers are not universal and their properties depend on the details of physics at short distances. The atoms and shallow dimers can undergo inelastic collisions into atoms and deeply bound dimers. The difference in the binding energies of the shallow dimer and deep dimer is released as kinetic energy and the atom and deep dimer in the final state recoil from each other. The total relaxation rate into deep dimers displays log-periodic scaling and is enhanced whenever the binding energy of the shallow dimer equals the binding energy of an Efimov trimer. This happens when the scattering length is equal to $(S_0)^n a_*$ with $n$ an integer (cf. figure 1). In an experiment with a mixture of bosonic $^{133}$Cs atoms and dimers, Knoop et al. observed this resonant enhancement in the loss of atoms and dimers [52]. This loss feature could be explained by an Efimov trimer crossing the atom–dimer threshold [53].

The three-component $^6$Li system provides a new aspect to atom–dimer relaxation. If at least two of the three atom–atom scattering lengths in the three-component system are positive, atom–dimer relaxation can also proceed from the shallow dimer with the smaller binding energy to the one with the larger binding energy. In $^6$Li, there are, in principle, three different relaxation processes possible since there are three types of shallow dimers: 12, 23 and 13. Additionally, each of these relaxation rates can be decomposed into the partial relaxation rates into shallow and deep dimers, respectively.

Measurements of the atom–dimer relaxation rate were reported by Lompe et al. [54] and Nakajima et al. [55]. A theoretical analysis for the relaxation rate of the particular channel relevant to their experimental set-up was performed by Nakajima et al. [55]. A complete analysis for all possible partial relaxation rates was carried out in Hammer et al. [56]. In figure 3, we show their results.
for the relaxation rate constant $\beta_1$ for the 23-dimer and atom 1. We compare with the measurements of Lompe et al. [54] and Nakajima et al. [55]. The figure shows the full relaxation rate as well the individual contributions from shallow and deep dimers in the magnetic field region from 590 to 730 G. While a qualitative agreement can be obtained some questions concerning the quantitative description of the data remain [56]. Including non-universal corrections in the two- and three-body sector can partially resolve these questions [57].

All experiments discussed so far observed Efimov states indirectly through their signature in atom loss rates. Lompe et al. [58] recently reported on the association and direct observation of a trimer state consisting of three spin states of $^6\text{Li}$ atoms using radio-frequency spectroscopy. Their binding energy measurements are consistent with theoretical predictions, which include non-universal corrections.

(b) Nuclear physics

The properties of hadrons and nuclei are determined by QCD, a non-abelian gauge theory formulated in terms of quark and gluon degrees of freedom. At low energies, however, the appropriate degrees of freedom are the hadrons. Efimov physics and the unitary limit can serve as a starting point for EFTs describing hadrons and nuclei at very low energies. For convenience, we will now work in natural units where $\hbar = c = 1$.

In nuclear physics, there are a number of EFTs that are useful for a certain range of systems [23,24]. At very low energies, where Efimov physics plays a role, all interactions can be considered short range and even the pions can be integrated out. This so-called pionless EFT is formulated in an expansion of the low-momentum scale $M_{\text{low}}$ over the high-momentum scale $M_{\text{high}}$. It can be understood as an expansion around the limit of infinite scattering length or equivalently around threshold bound states. Its breakdown scale is set by one-pion exchange, $M_{\text{high}} \sim M_\pi$, while $M_{\text{low}} \sim 1/a \sim k$. For momenta $k$ of the order of the pion mass $M_\pi$, pion exchange becomes a long-range interaction and has to be treated explicitly. This leads to the chiral EFT whose breakdown scale $M_{\text{high}}$ is set by the chiral symmetry breaking scale $\Lambda_\chi$. The pionless theory relies only on the large scattering length and is independent of the short-distance mechanism generating it. At leading order, it is equivalent to the coordinate space formulation of the three-body problem with zero-range interactions. However, the ability of the EFT approach to account systematically for finite range effects makes it particularly interesting to the three-nucleon problem where the ratio of $\ell/|a| \approx 1/3$. This theory is, therefore, ideally suited to unravel universal phenomena driven by the large scattering length such as limit cycle physics [59,60] and the Efimov effect [1] but also to describe very low-energy few-nucleon processes to high accuracy.

In this section, we will focus on the aspects of nuclear EFTs related to Efimov physics. In the two-nucleon system, the pionless theory reproduces the well-known effective range expansion in the large scattering length limit. The renormalized S-wave scattering amplitude to next-to-leading order in a given channel takes the form [61,62]

$$ T(k) = \frac{4\pi}{m} \frac{1}{-1/a - ik} \left[ 1 - \frac{r_0 k^2 / 2}{-1/a - ik} + \ldots \right], \quad (5.2) $$
where $k$ is the relative momentum of the nucleons, $r_0$ denotes the effective range, and the dots indicate corrections of order $(M_{\text{low}}/M_{\text{high}})^2$ for typical momenta $k \sim M_{\text{low}}$. In the language of the RG, this corresponds to an expansion around the non-trivial fixed point for $1/a = 0$ [61,63]. The pionless EFT becomes very useful in the two-nucleon sector when external currents are considered and has been applied to a variety of electroweak processes. These calculations are reviewed in detail in Bedaque & van Kolck [23].

Since the nucleon–nucleon scattering lengths cannot be changed as in the case of atoms, the consequences of the limit cycle in the three-nucleon system cannot be measured directly. The scattering lengths could be manipulated if the quark masses that are parameters of the underlying theory of strong interactions (QCD) could be varied. This is in fact done in numerical simulations of QCD (lattice QCD) that aim at a calculation of the $NN$ scattering length from first principles. Simulations with heavier quark masses are computationally cheaper but their results can be extrapolated to the physical values using chiral EFT. It turns out that chiral EFT indicates the possibility that the $NN$ scattering lengths diverge simultaneously at a quark mass values slightly larger than the physical ones. Braaten & Hammer [59] conjectured, therefore, that a lattice QCD simulation of the three-nucleon system with quark masses close to this critical value would therefore lead to several bound states related to each other by Efimov’s scaling factor of 515. The consequences of this infrared limit cycle were studied in the works by Epelbaum et al. [64] and Hammer et al. [65].

Three-body calculations with external currents are still in their infancy. However, a few exploratory calculations have been carried out. Universal properties of the triton charge form factor were investigated by Platter & Hammer [66] and neutron–deuteron radiative capture was calculated by Sadeghi & Bayegan [67], Sadeghi et al. [68] and Sadeghi [69]. Electromagnetic properties of the triton were recently investigated by Sadeghi & Bayegan [70]. This work opens the possibility to carry out accurate calculations of electroweak reactions at very low energies for astrophysical processes.

The pionless approach has also been extended to the four-nucleon sector [71]. In order to be able to apply the Yakubovsky equations, an equivalent effective quantum mechanics formulation was used. The study of the cutoff dependence of the four-body binding energies revealed that no four-body parameter is required for renormalization at leading order. As a consequence, there are universal correlations in the four-body sector, which are also driven by the large scattering length. The best known example is the Tjon line: a correlation between the triton and alpha particle binding energies, $B_t$ and $B_{\alpha}$, respectively. Of course, higher order corrections break the exact correlation and generate a band.

Another interesting development is the application of the resonating group model to solve the pionless EFT for three- and four-nucleon systems [72]. This method allows for a straightforward inclusion of Coulomb effects. Kirscher et al. extended previous calculations in the four-nucleon system to next-to-leading order and showed that the Tjon line correlation persists. Moreover, they calculated the correlation between the singlet S-wave $^3\text{He}$–neutron scattering length and the triton binding energy. Preliminary results for the halo nucleus $^6\text{He}$ have also been reported by Kirscher et al. [73].
The pionless theory has also been applied within the no-core shell model approach. Here, the expansion in a truncated harmonic oscillator basis is used as the ultraviolet regulator of the EFT. The effective interaction is determined directly in the model space, where an exact diagonalization in a complete many-body basis is performed. In Stetcu et al. [74,75], the $0^+$ excited state of $^4\text{He}$ and the $^6\text{Li}$ ground state were calculated using the deuteron, triton, alpha particle ground states as input. The first $(0^+; 0)$ excited state in $^4\text{He}$ is calculated within 10 per cent of the experimental value, while the $^6\text{Li}$ ground state comes out at about 70 per cent of the experimental value in agreement with the 30 per cent error expected for the leading order approximation. These results are promising and should be improved if range corrections are included. Finally, the spectrum of trapped three- and four-body systems was calculated using the same method [74–78]. In this case, the harmonic potential is physical and not simply used as an ultraviolet regulator.

A special class of nuclear systems exhibiting universal behaviour are halo nuclei [79,80]. Halo nuclei consist of a tightly bound core surrounded by one or more loosely bound valence nucleons. The valence nucleons are characterized by a very low separation energy compared with those in the core. As a consequence, the radius of the halo nucleus is large compared with the radius of the core. A trivial example is the deuteron, which can be considered a two-body halo nucleus. The root mean square radius of the deuteron is about three times larger than the size of the constituent nucleons. Halo nuclei with two valence nucleons are particularly interesting examples of three-body systems. If none of the two-body subsystems are bound, they are called Borromean halo nuclei. This name is derived from the heraldic symbol of the Borromeo family of Italy, which consists of three rings interlocked in such a way that if any one of the rings is removed the other two separate. The most carefully studied Borromean halo nuclei are $^6\text{He}$ and $^{11}\text{Li}$, which have two weakly bound valence neutrons [79]. In the case of $^6\text{He}$, the core is a $^4\text{He}$ nucleus, which is also known as the alpha particle. The two-neutron separation energy for $^6\text{He}$ is about 1 MeV, small compared with the binding energy of the alpha particle that is about 28 MeV. The neutron–alpha particle ($n\alpha$) system has no bound states and the $^6\text{He}$ nucleus is, therefore, Borromean. There is, however, a strong P-wave resonance in the $J = 3/2$ channel of $n\alpha$ scattering that is sometimes referred to as $^5\text{He}$. This resonance is responsible for the binding of $^6\text{He}$. Thus, $^6\text{He}$ can be interpreted as a bound state of an alpha particle and two neutrons, both of which are in $P_{3/2}$ configurations.

Because of the separation of scales in halo nuclei, they can be described by extensions of the pionless EFT. One can assume the core to be structureless and treat the nucleus as a few-body system of the core and the valence nucleons. Corrections from the structure of the core appear in higher orders and can be included in perturbation theory. Cluster models of halo nuclei then appear as leading order approximations in this ‘halo EFT’. A new facet is the appearance of resonances as in the $n\alpha$ system that leads to a more complicated singularity structure and renormalization compared with the few-nucleon system discussed above [81].

The first application of EFT methods to halo nuclei was carried out by Bertulani et al. [81] and Bedaque et al. [82], where the $n\alpha$ system ($^5\text{He}$') was considered. They found that for resonant P-wave interactions both the scattering length and effective range have to be resummed at leading order. In this case, two
coupling constants in the effective Lagrangian are fine tuned to unnatural values. At threshold, however, only one fine tuning is required and the EFT becomes perturbative. Because the $n\pi$ interaction is resonant in the P-wave and not in the S-wave, the binding mechanism of $^6$He is not the Efimov effect. However, this nucleus can serve as a laboratory for studying the interplay of resonance structures in higher partial waves.

Three-body halo nuclei composed of a core and two valence neutrons are of particular interest owing to the possibility of these systems to display the Efimov effect [1]. Since the scattering length cannot be varied in halo nuclei, one looks for Efimov scaling between different states of the same nucleus. Such analyses assume that the halo ground state is an Efimov state. They have previously been carried out in cluster models and the renormalized zero-range model (e.g. [83–85]). A comprehensive study of S-wave halo nuclei in halo EFT was carried out by Canham & Hammer [86]. This work provided binding energy and structure calculations for various halo nuclei including error estimates. Confirming earlier results by Fedorov et al. [83] and Amorim et al. [84], $^{20}$C was found to be the only candidate nucleus for an excited Efimov state assuming the ground state is also an Efimov state. This nucleus consists of a $^{18}$C core with spin and parity quantum numbers $J^P = 0^+$ and two valence neutrons. The nucleus $^{19}$C is expected to have a $\frac{1}{2}^+$ state near threshold, implying a shallow neutron–core bound state and therefore a large neutron–core scattering length. The value of the $^{19}$C energy, however, is not known well enough to make a definite statement about the appearance of an excited state in $^{20}$C. An excited state with a binding energy of about 65 keV is marginally consistent with the current experimental information.

The matter form factors and radii of halo nuclei can also be calculated in the halo EFT [86,87]. As an example, we show the various one- and two-body matter density form factors $F_c$, $F_n$, $F_{nc}$, and $F_{nn}$ with leading-order error bands for the ground state of $^{20}$C as a function of the momentum transfer $k^2$ from Canham & Hammer [86] in figure 4. The effective theory description breaks down for momentum transfers of the order of the pion-mass squared ($k^2 \approx 0.5 \text{ fm}^{-2}$). The range corrections to these results were estimated in Canham & Hammer [88] and found to be generally small.

Scattering observables offer a complementary window on Efimov physics in halo nuclei and some recent model studies have focused on this issue. In the works by Mazumdar et al. [89] and Yamashita et al. [90] the trajectory of the possible $^{20}$C excited state was extended into the scattering region in order to find a resonance in $n$–$^{19}$C scattering. A detailed study of $n$–$^{19}$C scattering near an Efimov state was carried out by Yamashita et al. [91].

6. Outlook and conclusion

In this work, we have discussed the Efimov effect in the context of the RG. The discrete scaling invariance particular to the Efimov effect can be understood as the consequence of an RG limit cycle. The log-periodic scattering length dependence of observables is, therefore, a prominent signature of the limit cycle in

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1We note that it is also possible that only the excited state is an Efimov state while the ground state is more compact. This scenario cannot be ruled out but is also less predictive.
experimental observables. We have also reviewed the steadily growing number of observations of Efimov physics in ultracold atomic gases. In particular, the ability to precisely tune the scattering length using Feshbach resonances has made it possible to observe the log-periodic scaling in three-body loss processes.

The situation in nuclear physics is very different since the value of the scattering length is fixed. Only a consistent description of different three-body observables within the same universal EFT approach can, therefore, be regarded as evidence for Efimov physics. A number of such calculations has been performed. In particular, the properties of the triton have been thoroughly analysed in the pionless EFT. These calculations include various electromagnetic, scattering and bound-state properties. It is furthermore important to note that few-nucleon physics is not only interesting because of the universal aspects but also has important practical applications. Low-energy nuclear reactions are hard to measure but knowledge of them is important for the description of astrophysical processes such as big bang nucleosynthesis. This aspect requires a consistent inclusion of higher order corrections, e.g. from finite-range effects, and external currents.

Future lattice QCD simulations of few-nucleon systems might be able to provide more evidence for the possible existence of an infrared limit cycle in QCD [92]. In this approach, the observables are extracted from correlation functions calculated by evaluating the QCD path integral on a Euclidean space–time lattice using Monte Carlo methods. With high-statistics lattice QCD simulations of three-baryon systems within reach [93], the calculation of the structure and reactions of light nuclei directly from QCD appears feasible in the intermediate future.

Halo nuclei are an additional class of systems that might provide examples of Efimov states. They consist of a tightly bound core surrounded by one or more weakly bound valence nucleons and can be considered effective few-body
systems. The properties of a number of three-body halo nuclei have been analysed. In particular, bound-state properties such as matter radii and form factors have been calculated. In the two-body sector, various scattering and breakup processes have been considered. In the future, the EFT approach might also be helpful in analysing reactions involving three-body halo nuclei since it reduces the number of degrees of freedom included in a controlled manner and higher order corrections can be included systematically.

Since the discovery of the Efimov effect approximately 40 years ago, it has fascinated an ever growing number of physicists. It is not only a remarkable example of discrete scale invariance on the quantum level but it also relates physical systems across different branches of physics including atomic, nuclear and particle physics. It serves as paradigm for a universal binding mechanism in weakly bound few-body systems with binding energies from nanoelectron volts to megaelectron volts. Furthermore, it provides a starting point for an EFT that can facilitate the calculation of low-energy observables to high accuracy.

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