Hydroelastic wave diffraction by a vertical cylinder

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A linear three-dimensional problem of hydroelastic wave diffraction by a bottom-mounted circular cylinder is analysed. The fluid is of finite depth and is covered by an ice sheet, which is clamped to the cylinder surface. The ice stretches from the cylinder to infinity in all lateral directions. The hydroelastic behaviour of the ice sheet is described by linear elastic plate theory, and the fluid flow by a potential flow model. The two-dimensional incident wave is regular and has small amplitude. An analytical solution of the coupled problem of hydroelasticity is found by using a Weber transform. We determine the ice deflection and the vertical and horizontal forces acting on the cylinder and analyse the strain in the ice sheet caused by the incident wave.

Keywords: hydroelastic waves; wave–structure interaction; elastic plate

1. Introduction

This paper is concerned with the interaction between incident hydroelastic waves in an ice sheet and structures frozen in the ice. The study is motivated by the expected need for structures, such as drilling rigs, oil–gas production platforms and offshore wind farms, to be built far north in ice-covered waters, where new oil–gas fields were discovered recently. This problem was intensively studied for hydroelastic waves without such large structures as rigs, and for water waves without an ice cover but in the presence of massive floating or bottom-mounted structures. The aim of this paper is to combine the theory of hydroelastic waves in an ice cover with the theory of water-wave interaction with offshore structures, to evaluate the wave loading on a structure frozen in the ice, and to investigate the dependence of the ice loads on various parameters of the ice cover and characteristics of the incident wave.

The hydroelastic behaviour of ice sheets is well studied. The first foray into this diverse and fascinating field is largely accredited to Greenhill [1] who considered an elastic beam model to describe the behaviour of the ice sheet. The idea was extended by Ewing & Crary [2], beginning a series of papers that experimentally studied flexural waves in large ice sheets. Evans & Davies [3] used the Wiener–Hopf technique to solve a linear problem of water-wave interaction with a semi-infinite ice sheet. Recently, their work has been revisited by a number of

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Authors, including Tkacheva [4]. Sea ice in the marginal ice zone near New Zealand has motivated much study in the field; the interaction of ocean waves and sea ice determines the formation of ice floes [5]. The response of ice floes to ocean waves was also studied by Fox & Squire [6], among others. Recently, this subject has experienced a resurgence in interest. The discovery that very large floating platforms (such as floating airports) can be modelled using similar methods led to further interest in the field; Andrianov & Hermans [7] and Tkacheva [4] are excellent examples.

Nowadays, more complex models of ice response are available: hydroelastic waves in the presence of cracks in ice sheets have been studied by Evans & Porter [8]; waves in an ice cover of variable thickness were studied by Bennets et al. [9]; and hydroelastic waves propagating over variable bottom topography by Porter & Porter [10]. For a more complete bibliography of historical or current advances in this field, consult the excellent reviews by Squire et al. [11] and Squire [12]. However, not many studies have been conducted on the interaction of hydroelastic waves with large structures such as offshore platforms. The present three-dimensional problem of a vertical cylinder mounted to the sea bottom is an example of modelling such a structure.

The problem of free-surface water waves interacting with a vertical cylinder is well studied. The cylinder is intended to represent the supporting foundations of an offshore platform, such as an offshore wind farm [13] or oil rig. Water-wave scattering by a vertical cylinder was first examined by Omer & Hall [14], and later MacCamy & Fuchs [15]. Mei [16] obtained a solution by decomposing the potentials of the incident and reflected waves into Fourier series with respect to the azimuthal coordinate. Consideration of arrays of vertical cylinders is now commonplace, pioneered by such authors as Spring & Monkmeyer [17] and Linton & Evans [18]. Methods of solutions such as eigenfunction expansions and multi-pole expansions for infinite arrays are predominantly used, as described in the excellent book by Linton & McIver [19], accompanied by many bibliographical notes. The inclusion of an ice cover to diffraction problems involving a vertical cylinder has been studied considerably less. Malenica & Korobkin [20] considered the problem of water-wave interaction with a vertical cylinder frozen into a circular finite ice floe. The efficient technique of eigenfunction expansions in the region covered by the ice flow and in the open-water region was used. The velocity potentials were obtained in these two regions separately and then matched over the circular boundary.

In the present paper, the ice is of infinite extent, and the method of integral transforms is used instead of the method of eigenfunction expansions used by Malenica & Korobkin [20]. The integral transform used in the present paper is a modified Weber transform, which itself is an extension of the Hankel transform with a more general kernel. The Hankel transform is particularly useful for problems with circular symmetry. It can be derived by introducing polar coordinates in a Fourier transform. Examples of its applications include optics, acoustics and geophysics. The Weber transform is useful for solving Laplace’s equation in cylindrical coordinates with boundary conditions imposed on cylindrical surfaces. The particular form of the Weber kernel outlined in §3 has been used by Emmerhoff & Sclavounos [21] in the context of arrays of vertical cylinders, as well as by Ghosh [22] in an axisymmetric wavemaker problem. As we will show, the transform simplifies the fourth-order elastic
plate equation into an algebraic equation, which can be readily solved. The present technique offers several advantages to the eigenfunction solution of Malenica & Korobkin [20]. Firstly, in Malenica & Korobkin [20], the system of algebraic equations for the unknown coefficients resulting from the eigenfunction expansion was truncated. The method in the present paper provides an exact solution to the original linear problem. Secondly, the Weber transform method leads to a concise expression for the horizontal force in terms of integral quadrature, and a simple algebraic expression for the vertical shear force on the structure caused by the incident waves. These simple expressions allow the behaviour of the forces under variation of relevant physical parameters to be investigated easily.

In the present study, the ice sheet is modelled as a thin elastic plate, the deflection of which is governed by the Bernoulli–Euler thin-plate equation (see [11]) involving fourth-order derivatives of the plate deflection in space (see §2). This model was justified by Squire et al. [23], who showed that it performs well compared with the results of field measurements. The fluid is assumed incompressible and inviscid. The problem of incident hydroelastic wave interaction with a vertical cylinder is formulated within the linear theory of hydroelasticity with the ice sheet being clamped to the surface of the frozen cylinder. The boundary conditions are linearized and imposed on the equilibrium positions of the liquid boundary. The dynamic boundary condition comes from the thin-plate equation, the high order of which is largely responsible for the difficulties researchers encounter in the problems of hydroelasticity.

The two-dimensional version of the problem considered here was studied by Brocklehurst et al. [24], who investigated hydroelastic wave reflection by a vertical wall to which the ice sheet was frozen. The authors were interested in whether the ice would maintain the frozen connection or whether the strain in the ice sheet would become too high for this to be possible. It was concluded that the strains at the wall could break the ice-sheet connection at that point unless the incident wave amplitude is sufficiently small. Both the horizontal and vertical forces on the wall owing to the flexure of the ice were also studied. The solution was found by using a Fourier cosine transform. In the present paper, similar techniques and formulations are used. Some of the functions that appeared in the solution of Brocklehurst et al. [24] appear in the analysis of the present three-dimensional problem.

The structure of the paper is as follows. Section 2 describes the mathematical formulation of the problem, and relevant physical parameters are introduced. Progress is made by decomposing the solution into a Fourier series with respect to the azimuthal coordinate with the coefficients to be determined. The boundary-value problems for these coefficients are shown at the end of this section. Section 3 introduces the Weber transform. The solutions for the ice deflection and velocity potential of the flow are derived. Section 4 presents numerical results, both for the ice deflection and velocity potential. The strain in the ice sheet is analysed to determine if the ice clamping boundary condition will be maintained. Important physical forces acting on the cylinder are calculated, which are of concern for the design of structures to be built in ice-covered water. These include the horizontal force acting on the cylinder and the vertical shear force caused by the ice clamping condition. Conclusions are drawn in §5.

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2. Mathematical formulation

The diffraction of two-dimensional hydroelastic waves by a circular cylinder frozen in an ice sheet is studied within the linear theory of hydroelasticity (see [25]). The cylinder is vertical and bottom mounted. In this section, the physical parameters of the problem are introduced and the governing equations of hydroelasticity are outlined. The solution of the formulated problem is decomposed with respect to time and angular coordinate, and into three components corresponding to incident wave, far-field diffracted wave and a near-field component. The evanescent near-field component describes the flow and ice-sheet response close to the cylinder. With the help of this three-step decomposition, the original problem is split into a series of boundary-value problems of lower dimension, which are solved analytically in §3.

The geometry of the problem is shown in figure 1. The problem is formulated in cylindrical polar coordinates \((r, \theta, z)\), with the origin at the centre of the circular cylinder. The vertical \(z\)-axis is directed upwards. The plane \(z = 0\) corresponds to the lower surface of the ice sheet at rest, and the plane \(z = -H\) corresponds to the bottom of the liquid. The surface of the bottom-mounted cylinder is at \(r = b\), where \(b\) is the radius of the cylinder. The relationship between cylindrical coordinates and cartesian \((x, y)\) coordinates is given by the equations \(x = r \cos(\theta)\) and \(y = r \sin(\theta)\). An incident hydroelastic wave approaches the cylinder along the \(x\)-axis from \(x = -\infty\). The incident wave parameters are: \(a\), wave amplitude; \(\omega\), wave frequency; and \(k\), wavenumber. The liquid is assumed inviscid and incompressible, with irrotational flow. The liquid flow is described by the velocity potential \(\phi(r, \theta, z, t)\), where \(t\) is time, and the ice deflection by the equation \(z = w(r, \theta, t)\), where \(r > b\) and \(0 \leq \theta \leq 2\pi\). The ice has mass per unit area \(M = \rho_i h\), where \(\rho_i\) is the ice density and \(h\) the ice thickness. The flexural rigidity of the

Figure 1. Schematic of the physical set-up for the problem and coordinate system.
Table 1. Physical parameters. Values of typical parameters taken from measurements at McMurdo Sound, Antarctica [26]. Section 2 defines each parameter.

<table>
<thead>
<tr>
<th>parameter</th>
<th>typical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>9.8 m s$^{-2}$</td>
</tr>
<tr>
<td>$H$</td>
<td>350 m</td>
</tr>
<tr>
<td>$h$</td>
<td>1.6 m</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>$E$</td>
<td>$4.2 \times 10^9$ N m$^{-2}$</td>
</tr>
<tr>
<td>$J$</td>
<td>0.375 m$^3$</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>917 kg m$^{-3}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1026 kg m$^{-3}$</td>
</tr>
<tr>
<td>$M$</td>
<td>1467.2 kg m$^{-2}$</td>
</tr>
</tbody>
</table>

Ice sheet is denoted $EJ$, where $E$ is Young’s modulus, $J = h^3/[12(1 - \nu^2)]$ and $\nu$ is Poisson’s ratio. Typical values of each parameter can be found in table 1, taken from measurements in McMurdo Sound in Antarctica [26] and used here as the reference dataset. The velocity potential satisfies Laplace’s equation in the flow region,

$$\nabla^2 \phi = 0 \quad (-H < z < 0, \ r > b). \quad (2.1)$$

The linearized Bernoulli equation gives the pressure in the fluid as

$$p(r, \theta, z, t) = -\rho \phi_t - \rho gz, \quad (2.2)$$

where $\rho$ is the liquid density and $g$ is gravitational acceleration. Subscripts indicate partial derivatives. Boundary conditions for Laplace’s equation (2.1) are due to the rigid vertical wall of the cylinder, the fluid bed and the linearized kinematic condition on the ice cover, respectively,

$$\phi_r = 0 \quad (-H < z < 0, \ r = b), \quad (2.3)$$
$$\phi_z = 0 \quad (z = -H, \ r > b) \quad (2.4)$$

and

$$\phi_z = w_t \quad (z = 0, \ r > b). \quad (2.5)$$

The deflection of the ice cover is governed by the Bernoulli–Euler thin elastic plate equation

$$EJ \nabla^4 w + Mw_{tt} = p \quad (r > b), \quad (2.6)$$

with the boundary conditions

$$w = 0 \quad (r = b) \quad (2.7)$$

and

$$w_r = 0 \quad (r = b). \quad (2.8)$$

These boundary conditions imply that the ice sheet is clamped to the vertical cylinder. The ice deflection is due to incident hydroelastic waves approaching the cylinder from $x = -\infty$.
Both the ice deflection and the generated flow are assumed periodic in time with the frequency $\omega$ of the incident wave. The solution of the coupled problem (2.1)–(2.8) is sought in the form (see [11])

$$\Phi(r, \theta, z, t) = \text{Re}(\Phi(r, \theta, z)e^{-i\omega t})$$

and

$$w(r, \theta, t) = \text{Re}\left(\frac{i}{\omega} W(r, \theta)e^{-i\omega t}\right).$$

The velocity potential $\Phi_{\text{inc}}(r, \theta, z)$ and the deflection $W_{\text{inc}}(r, \theta)$ of the incident waves have the form

$$\Phi_{\text{inc}}(r, \theta, z) = \frac{a\omega}{k} \frac{\cosh(k(z + H))}{\sinh(kH)} e^{ikr \cos(\theta)}$$

and

$$W_{\text{inc}}(r, \theta) = a\omega e^{ikr \cos(\theta)},$$

where the wave frequency $\omega$ and wavenumber $k$ are related by the dispersion relation for hydroelastic waves (see [6])

$$\omega^2 \left( M + \frac{\rho}{k \tanh(kH)} \right) = \rho g + EJk^4.$$ 

In the far field as $r \to \infty$, the velocity potential $\Phi(r, \theta, z)$ in equation (2.9) and the deflection $W(r, \theta)$ in equation (2.10) consist of two components, one representing the incident wave and another representing the hydroelastic waves diffracted by the cylinder. The second diffracted components of the velocity potential and ice deflection in the far field are unknown and must be calculated as part of the solution. It is convenient to define these components in dimensionless form.

The following non-dimensional variables and parameters are introduced, denoted by an asterisk,

$$r^* = \frac{r}{H}, \quad z^* = \frac{z}{H}, \quad b^* = \frac{b}{H}, \quad t^* = t\omega, \quad k^* = kH,$$

$$W^*(r^*, \theta, t^*) = \frac{W(r, \theta)}{a\omega} \quad \text{and} \quad \Phi^*(r^*, \theta, z^*) = \frac{\Phi(r, \theta, z)}{Ha\omega},$$

where $k^*$ is the dimensionless incident wavenumber and $b^*$ is the dimensionless radius of the vertical cylinder. Note that the liquid depth $H$ is taken as the length scale, $1/\omega$ as the time scale and $a\omega$ as the scale of the liquid velocity. In the following, the asterisks, are dropped and only non-dimensional quantities are used, if not stated otherwise.

In the non-dimensional variables, the velocity potential of the incident wave (2.11) can be rewritten as [16]

$$\Phi_{\text{inc}} = f_0(z) \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \cos(n\theta),$$

where $J_n(kr)$ is the Bessel function of the first kind, $\epsilon_0 = 1, \epsilon_n = 2$ for $n \geq 1$ and

$$f_0(z) = \frac{\cosh(k(z + 1))}{k \sinh(k)}.$$
It is well known (see [16]) that cylindrical outgoing waves are represented by the Hankel function of the first kind, defined by

$$H_n^{(1)}(kr) = J_n(kr) + i Y_n(kr),$$

where $Y_n(kr)$ is the Bessel function of the second kind. Only Hankel functions of the first kind appear in this problem and we drop the superscript in equation (2.15). Therefore, the velocity potential and ice deflection can be decomposed as

$$\Phi(r, \theta, z) = \sum_{n=0}^{\infty} \epsilon^n i^n \cos(n\theta) (f_0(z) J_n(kr) + a^D_n f_0(z) H_n(kr) + \varphi_n(r, z))$$

and

$$W(r, \theta) = \sum_{n=0}^{\infty} \epsilon^n i^n \cos(n\theta) (J_n(kr) + a^D_n H_n(kr) + w_n(r)),$$

where the complex coefficients $a^D_n$ and the new unknown functions $\varphi_n(r, z)$ and $w_n(r)$ are to be determined. The terms including $a^D_n$ in equation (2.16) represent the outgoing waves diffracted by the cylinder. The terms with $J_n(kr)$ and $H_n(kr)$ in equation (2.16) satisfy equations (2.1)–(2.6) and correctly describe the far-field behaviour of the hydroelastic waves. However, they do not satisfy the boundary conditions (2.7)–(2.8). The functions $\varphi_n(r, z)$ and $w_n(r)$ are introduced to describe both the flow and ice deflection in the vicinity of the cylinder owing to the particular conditions (2.7)–(2.8) on the contact line between the ice sheet and the cylinder surface. The unknown functions $\varphi_n(r, z)$ and $w_n(r)$ are such that they do not give contributions to the wave pattern in the far field,

$$\sqrt{r} \varphi_n(r, z) \to 0, \quad \sqrt{r} w_n(r) \to 0 \quad \text{and} \quad r \to \infty.$$ 

In order to derive the boundary-value problems for the unknown functions $\varphi_n(r, z)$ and $w_n(r)$, where $n \geq 0$, we substitute equations (2.9), (2.10) and (2.16) into equations (2.1)–(2.8) written in dimensionless variables and make use of the properties of the Fourier series in $\theta$ and the Bessel functions $J_n(kr)$ and $H_n(kr)$. The resulting boundary-value problems are formulated as

$$S_n(\varphi_n) + \varphi_{zzz} = 0 \quad (-1 < z < 0, r > b),$$

$$\gamma S_n^2(w_n) + \left( \frac{1}{k \tanh(k)} - \gamma k^4 \right) w_n = \varphi_n \quad (z = 0, r > b),$$

$$\varphi_{nz} = 0 \quad (z = -1, r > b),$$

$$\varphi_{nr} = -f_0(z) k (J_n'(kb) + a^D_n H_n'(kb)) \quad (-1 < z < 0, r = b),$$

$$\varphi_{nz} = w_n \quad (z = 0, r > b),$$

$$w_n = -J_n(kb) - a^D_n H_n(kb) \quad (r = b)$$

and

$$w_{nr} = -k J_n'(kb) - a^D_n k H_n'(kb) \quad (r = b).$$
Here, primes indicate differentiation with respect to argument \( \gamma = EJ/(\rho \omega^2 H^5) \), and \( S_n \) is the differential operator defined as

\[
S_n(\varphi_n(r, z)) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) \varphi_n(r, z).
\]  

(2.25)

The boundary-value problems (2.18)–(2.24) include the complex coefficients \( a_n^D \), which should be determined as part of the solution.

### 3. Solution by Weber transform

In order to handle the fourth-order derivative in equation (2.19), we employ a modified Weber transform [21]. We may then use the helpful properties of this transform to greatly simplify the differential equations in equations (2.18)–(2.24) and facilitate a solution. The Weber transform is defined as

\[
F(s) = \int_b^\infty r f(r) Z_n(r, s) \, dr,
\]

(3.1)

where a function \( f(r) \) is defined in \( r > b \) and such that \( \sqrt{r} f(r) \to 0 \) as \( r \to \infty \). Here, \( Z_n \) is given by

\[
Z_n(r, s) = J_n(sr) Y'_n(sb) - J'_n(sb) Y_n(sr), \quad Z_n(b, s) = \frac{2}{\pi sb}
\]

and

\[
\frac{\partial Z_n}{\partial r}(b, s) = 0, \quad Z_n \to 0 \quad \text{as} \quad r \to \infty.
\]

The corresponding inverse transform reads

\[
f(r) = \int_0^\infty \frac{F(s) s Z_n(r, s) \, ds}{(J'_n(sb))^2 + (Y'_n(sb))^2}.
\]

(3.2)

Defining the integral operator in equation (3.1) as \( F(s) = \text{Web}[f(r)] \), it is possible to prove the following formulae:

\[
\text{Web}[S_n(f(r))] = -s^2 \text{Web}[f(r)] - \frac{2}{\pi s} f'(b)
\]

(3.3)

and

\[
\text{Web}[S_n^2(f(r))] = s^4 \text{Web}[f(r)] + \frac{2s}{\pi} f'(b) - \frac{2}{\pi s} \frac{\partial}{\partial r} (S_n(f(r)))_{r=b}.
\]

(3.4)

Equation (3.3) (see [21]) can be obtained by applying the Weber transform (3.1) to the operator \( S_n \) and repetitively applying integration by parts. Equation (3.4) is a double application of the Weber transform that can be derived by using equation (3.3) to calculate \( S_n(g(r)) \), where \( g(r) = S_n(f(r)) \). Applying the Weber transform to equation (2.18), where \( \text{Web}[\varphi_n(r, z)] = \psi_n(s, z) \), and using equations...
(3.3) and (2.21), we arrive at the ordinary differential equation

$$\psi_{nzz} - s^2 \psi_n = \frac{2}{\pi s} \varphi_{nr}(b, z) = \frac{P_n}{s} \cosh(k(z + 1)), \quad (3.5)$$

where the coefficients $P_n$ are defined as

$$P_n = \frac{-2}{\pi \sinh(k)} (J'_n(kb) + a_n^D H'_n(kb)). \quad (3.6)$$

Applying the Weber transform to equations (2.20) and (2.22), we obtain

$$\psi_{nz} = 0 \quad (z = -1) \quad (3.7)$$

and

$$\psi_{nz} = W_n \quad (z = 0), \quad (3.8)$$

where $W_n(s) = \text{Web}[w_n(r)]$. The solution of the boundary-value problem (3.5)–(3.8) is given by

$$\psi_n(s, z) = \cosh(s(z + 1)) \left( W_n - \frac{P_n k \sinh(kH)}{s(k^2 - s^2)} \right) + \frac{P_n \cosh(k(z + 1))}{s(k^2 - s^2)}, \quad (3.9)$$

where $W_n$ and the coefficients $a_n^D$ are still to be determined.

We now apply the Weber transform to the plate equation (2.19) and make use of equation (3.4) to yield

$$\gamma \left( s^4 W_n + \frac{2s}{\pi} w_{nr}(b) - \frac{2}{\pi s} \bar{V}_n \right) + \left( \frac{1}{k \tanh(k)} - \gamma k^4 \right) W_n = \psi_n(s, 0). \quad (3.10)$$

The function

$$\bar{V}_n(k, b) = \frac{\partial}{\partial r} (S_n \langle w_n(r) \rangle)_{r=b} \quad (3.11)$$

will be used below in calculations of the shear force acting on the cylinder owing to the clamped condition at $r = b$. Equation (3.10) can be simplified by noting that

$$w_{nr}(b) = -k J'_n(kb) - k a_n^D H'_n(kb) = \frac{1}{2} \pi k \sinh(k) P_n, \quad (3.12)$$

which follows from equations (2.24) and (3.6). Inserting equations (3.12) and (3.9) into the transformed plate equation (3.10) and rearranging the result to factorize $W_n(s)$ on the left-hand side, we find

$$W_n \left( \gamma(s^4 - k^4) + \frac{1}{k \tanh(k)} - \frac{1}{s \tanh(s)} \right) = \frac{2\gamma}{\pi s} \bar{V}_n - \gamma sk P_n \sinh(k) + P_n \sinh(k) \left( \frac{1}{s(k^2 - s^2)} \left( \frac{1}{\tanh(k)} - \frac{k}{s \tanh(s)} \right) \right). \quad (3.13)$$

Clearly, the left-hand side of equation (3.13) is zero at $s = k$. Hence, to avoid a singularity of the function $W_n(s)$ at this point, we require that the limit of the right-hand side is also zero as $s \to k$. This limit can be calculated by applying
Figure 2. The function $Q(s)$ plotted against $s$ for: solid line, $k = 5$; dashed line, $k = 10$; dotted line, $k = 15$. Data in table 1 were used.

l’Hôpital’s rule to the right-hand side of equation (3.13). Equating the right-hand side of equation (3.13) to zero at $s = k$, we obtain the formula for $\bar{V}_n(k, b)$,

$$\bar{V}_n = \frac{\pi}{2} P_n \sinh(k) \left( k^3 + \frac{V(k)}{\gamma} \right), \quad (3.14)$$

where the function

$$V(k) = \frac{1}{2k^2 \tanh(k)} + \frac{1}{2k \sinh^2(k)}$$

was introduced by Brocklehurst et al. [24].

Inserting equation (3.14) into equation (3.13) and noting that the right-hand side is now also equal to zero at $s = k$, we calculate

$$W_n(s) = \frac{P_n \sinh(k)}{s} Q(s). \quad (3.15)$$

Here, $Q(s)$ is the same function that appeared in the solution of the equivalent two-dimensional problem [24],

$$Q(s) = \frac{s \tanh(s)(V(k) + \gamma k(k^2 - s^2)) + \frac{1}{k^2 - s^2} \left( \frac{s \tanh(s)}{\tanh(k)} - k \right)}{s \tanh(s) \left( \gamma(s^4 - k^4) + \frac{1}{k \tanh(k)} \right) - 1}. \quad (3.16)$$

The function is smooth and its properties are well known from the analysis of the problem of hydroelastic wave diffraction by a vertical wall. The function $Q(s)$ is plotted in figure 2. The inverse transform (3.2) must be performed to obtain $w_n(r)$, which may then be substituted into equation (2.16) to give the final solution for the total ice deflection. The inverse transform exists owing to the rapid decay of the function $Q(s)$, which is of the order $O(s^{-2})$ as $s \to \infty$. 

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Second derivatives of the deflection \( w(r, \theta, t) \) with respect to \( r \) and \( \theta \) are required to calculate the strain distribution in the ice cover. In order to evaluate the derivatives, we need to improve the convergence of the integral solution for \( w_n(r) \). This has been done using the technique outlined in Brocklehurst et al. [24].

The solution (3.15) still contains the undetermined coefficients \( a_n^D \). These coefficients are calculated by using the condition (2.23), where \( w_n(b) \) is given as

\[
    w_n(b) = \int_0^\infty \frac{s W_n(s) Z_n(b, s) \, ds}{(J'_n(sb))^2 + (Y'_n(sb))^2}.
\]

(3.17)

Substituting equations (3.17), (3.15) and (3.6) into equation (2.23), we have

\[
    -J_n(kb) - a_n^D H_n(kb) = P_n \sinh(k) \frac{2}{\pi b} \int_0^\infty \frac{Q(s)}{s((J'_n(sb))^2 + (Y'_n(sb))^2)} \, ds
\]

\[
    = -\frac{2}{\pi} (J'_n(kb) + a_n^D H_n(kb)) \frac{2}{\pi b} \tau_n,
\]

(3.18)

where

\[
    \tau_n = \int_0^\infty \frac{Q(s)}{s((J'_n(sb))^2 + (Y'_n(sb))^2)} \, ds.
\]

By resolving equation (3.18) into real and imaginary parts and defining

\[
    \chi_n = \frac{4\tau_n}{\pi^2 b} J'_n(kb) - J_n(kb)
\]

\[
    \eta_n = \frac{4\tau_n}{\pi^2 b} Y'_n(kb) - Y_n(kb),
\]

(3.19)

we can write

\[
    a_n^D = \frac{i\chi_n}{\eta_n - i\chi_n}.
\]

(3.20)

The solution of the original boundary-value problem (2.1)–(2.8) is now derived in terms of integral quadratures. The ice deflection, bending stresses in the ice sheet and the hydrodynamic and ice loads acting on the cylinder can now be readily evaluated with a prescribed accuracy.

### 4. Numerical results

There are many physical parameters in the current problem; when presenting results, the default dataset in table 1 is used unless stated otherwise. In this section, non-dimensional parameters such as \( k^* = kH \) and \( b^* = b/H \), and non-dimensional variables such as \( r^* = r/H \) and \( w^* = w/a \) are once again denoted by an asterisk for clarity. We are interested in varying the dimensionless incident wavenumber \( k^* \) to investigate how waves of different lengths interact with the vertical cylinder. When dealing with waves in ice sheets, we are mostly interested in reasonably long waves in accordance with the linear theory. Hence, an appropriate range of \( k^* \) to consider is from \( k^* = 2 \) (corresponding to a wave period of 27.1s) to \( k^* = 10 \) (corresponding to a wave period of 11.5s). Since many of the physical properties of the ice sheet and the fluid are contained
within the dimensionless parameter $\gamma = EJ/(\rho \omega^2 H^5)$, we are also interested in varying $\gamma$. Though $\gamma$ is typically small ($\gamma \approx 10^{-6}$ for the default dataset with $k^* = 5$), it has a large impact on the behaviour of the ice sheet and the forces on the cylinder.

The only numerical procedures required are the calculation of integral quadratures arising from the inverse Weber transform (3.2). These quadratures have an oscillating integrand owing to the presence of Bessel functions. Using asymptotic expressions, the oscillating part of the integrand can be extracted from the integrand and dealt with separately, improving the convergence of the integrals. The remaining integrals are calculated numerically using standard techniques. In the Fourier series for the ice deflection (2.16), between 10 and 20 terms were required for convergence of the series.

(a) Deflection of the ice sheet

The non-dimensional ice deflection $w^*(r^*, \theta, t^*)$ is calculated by the formulae (2.10) and (2.16), where the coefficients $a_n^D$ are given by equation (3.20) and $w_n^*(r^*)$ are numerically computed by the inverse Weber transform (3.2) applied to $W_n^*(s^*)$ from equation (3.15). The distribution of the ice deflection around the cylinder is shown in figure 3 at $t^* = 3$ for wavenumber of the incident wave $k^* = 5$ and cylinder radius $b^* = 0.01$. For a water depth $H = 350$ m from table 1, these dimensionless parameters correspond to a cylinder radius of $b = 3.5$ m and length of incident wave $\lambda = 440$ m. The calculations were performed for $\gamma = 2.1 \times 10^{-6}$, which corresponds to an ice thickness $h = 1.6$ m and the incident hydroelastic wave frequency $\omega = 0.37$ s$^{-1}$.

The deflection and slope of the deflection are both zero at $r = b$, as defined by the ice clamping condition. Recall that the incident wave approaches from $r = \infty$, $\theta = \pi$. It is seen that there are perturbations in the wave peaks owing to the diffraction of waves by the cylinder. The same ice sheet is shown from a different angle in figure 4 to demonstrate this effect more clearly. The largest

Figure 3. The deflection of the ice sheet $w^*(r^*, \theta, t^*)$ is plotted for $b^* = 0.01$ and $t^* = 3$. The incident hydroelastic wave approaches from $\theta = \pi$ and is diffracted by the vertical cylinder.
wave peaks occur directly downstream of the vertical cylinder. Figure 5 shows the same ice sheet closer to the cylinder to outline the behaviour of the deflection in the vicinity of the vertical structure.

(b) Strain in the ice sheet

The propagation of waves in the ice sheet causes elastic strain in the ice. Many factors influence the strain distribution and its magnitude, such as the amplitude and wavenumber of the incident waves and the properties of the ice itself. If the strain becomes too high, the ice could fracture and cause cracks to propagate [11]. The strain is of particular interest in the present problem as we are interested in whether the clamped ice condition at $r = b$ can be maintained. In this study, we are interested in the radial strain component $\varepsilon_r$, defined as

$$\varepsilon_r = \frac{h}{2} \frac{\partial^2 w}{\partial r^2}. \quad (4.1)$$
The calculation of the strain is not straightforward. The inverse Weber transform for the deflection of the ice sheet involves an integrand that behaves as $s^{-2}$ when $s \rightarrow \infty$. Direct calculation of the second derivative with respect to $r$ required for the strain in equation (4.1) would lead to this integrand being multiplied by $s^2$, leaving the integral undefined. However, it is possible to find asymptotic expressions for the functions in the integrand as $s \rightarrow \infty$ and integrate the resulting asymptotics analytically, improving the convergence of the integral so that its second derivative with respect to $r$ can now be numerically evaluated. This procedure is outlined in Brocklehurst et al. [24] and can be applied to the present integrals by using well-known asymptotic expressions for Bessel functions.

The highest strain is expected to occur at $r = b$, where the ice is frozen to the cylinder. Figure 6a shows the polar distribution of $\varepsilon_r$ around the cylinder. Note that we present the maximum of $\varepsilon_r$ with respect to the time periodicity of the problem; $\varepsilon_r$ at any time is subject to a phase shift that may be easily calculated. We see that the highest values of $\varepsilon_r$ occur at $\theta = \pi$, where the incident wave impacts on the cylinder, and $\varepsilon_r$ decreases as we move towards the rear of the cylinder. In figure 6a, the solid line corresponds to an ice thickness of $h = 1.6$ m and the dashed line to a thickness of $h = 0.5$ m. We see that thin ice causes smaller values of $\varepsilon_r$ as well as less polar variation. Figure 6b shows $\varepsilon_r$ plotted against the non-dimensional incident wavenumber $k^*$ at $\theta = \pi$. We find that $\varepsilon_r$ increases with $k^*$ owing to the higher curvature caused by shorter waves. Two values of the cylinder radius $b$ are included to demonstrate that a larger cylinder causes a decrease in the maximum value of $\varepsilon_r$.
The yield strain of the ice is defined to be the level of strain beyond which the ice begins to deform plastically. Ice that is subject to strain beyond its yield value is more likely to fracture. Brocklehurst et al. [24] take an estimate of the yield strain as $8 \times 10^{-5}$. Upon investigation of figure 6, it seems likely that the connection at $r = b$ will not be maintained unless physical parameters take certain values. For example, the strain amplitude is proportional to the amplitude of the incident wave within the linear theory of hydroelasticity, and hence if the wave amplitude is small enough, the strain may stay below yield. The strain in the ice sheet also has a sensitive dependence to the parameters $k^*$ and $b^*$. Hence, if the incident waves are long enough, or the cylinder large enough, the strain may stay below yield level and the ice sheet will remain frozen to the cylinder.

(c) Vertical shear force

We now turn our attention to the vertical shear force, the upwards lifting force caused by the ice clamping condition. The flexure of the ice sheet caused by the incident hydroelastic wave drives the cylinder vertically, threatening to break the connection between the cylinder and the sea bed if the connection is not strong enough. Here, we assess the magnitude of this lifting force and factors influencing it. The vertical shearing force $Q_r$ acting on the cylinder is defined in non-dimensional variables as (see [27])

$$Q_r^* = -\frac{\partial}{\partial r^*} \left( \nabla^2 w^* \right)_{r^* = b^*}, \tag{4.2}$$

where the scale of the shear force is $EJa/H^{3}$. Equations (2.10) and (2.16) at $r^* = b^*$ provide

$$Q_r^*(\theta, t^*) = \text{Re} \left( -ie^{-i\theta} \sum_{n=0}^{\infty} \epsilon^n i^n \cos(n\theta) \left( \frac{\partial}{\partial r^*} S_n(J_n(k^* r^*))_{r^* = b^*} + a^n \frac{\partial}{\partial r^*} S_n(H_n(w^* r^*))_{r^* = b^*} \right) \right). \tag{4.3}$$

The derivatives on the right-hand side may be simplified by noting

$$S_n(J_n(k^* r^*)) = -k^* J_n(k^* r^*) \quad \text{and} \quad S_n(H_n(k^* r^*)) = -k^* H_n(k^* r^*),$$

and we may substitute

$$\frac{\partial}{\partial r^*} S_n(w^* r^*)_{r^* = b^*} = \bar{V}_n, \tag{4.4}$$

where $\bar{V}_n$ is given by equation (3.14). It may be shown that the largest shear force occurs at $\theta = \pi$, where the incident wave impacts upon the cylinder.

The total shear force in non-dimensional variables is obtained by integrating $Q_r^*(\theta, t^*)$ with respect to the angular coordinate $\theta$, with the result

$$b^* \int_0^{2\pi} Q_r^*(\theta, t^*) \, d\theta = Q_{tot}^* \cos(t^* + \delta), \tag{4.5}$$

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Figure 7. The amplitude of the total dimensional shear force for two ice thicknesses, where the ice thickness is given by: solid line, $h = 1.6$ m; dashed line, $h = 0.5$ m. The incident wave amplitude is fixed at $a = 2$ cm and all other parameters are fixed at their default value. (a) The total shear force is plotted against $k^*$ for $b = 3.5$ m. (b) The total shear force is plotted against the cylinder radius for $k^* = 5$.

where

$$Q_{\text{tot}}^* = 2\pi b^*(u^2 + v^2)^{1/2},$$

$$u = J_0'(k^* b^*) \left( 2k^{*3}(\beta_0 - 1) + \frac{V^*(k^*)\beta_0}{\gamma} \right) + Y_0'(k^* b^*) \left( 2k^{*3}\alpha_0 + \frac{V^*(k^*)\alpha_0}{\gamma} \right),$$

$$v = J_0'(k^* b^*) \left( -2k^{*3}\alpha_0 - \frac{V^*(k^*)\alpha_0}{\gamma} \right) + Y_0'(k^* b^*) \left( 2k^{*3}\beta_0 + \frac{V^*\beta_0}{\gamma} \right),$$

and $a_0^D = \alpha_0 + i\beta_0$.  

Note that $Q_{\text{tot}}^*$ is the maximum total shear force with respect to time, and the phase shift $\delta$ is not studied here. Also note that in the integration of $Q_{\text{tot}}^*(\theta, t^*)$ in equation (4.5), the only contribution to the result comes from the terms with $n = 0$. The scale of $Q_{\text{tot}}^*$ is equal to $EJa/H^2$.

Figure 7a shows the behaviour of the total dimensional shear force as a function of the non-dimensional wavenumber $k^*$. The solid line corresponds to an ice thickness of $h = 1.6$ m and the dashed line to an ice thickness of $h = 0.5$ m. The total shear force is highest at $k^* = 2$ for both values of ice thickness considered, implying long waves induce more shear force than short waves. For an ice thickness of $h = 1.6$ m, a minimum occurs close to $k^* = 7$ before the shear force begins rising slightly. It is clear that thin ice causes less shear force than thick ice. Figure 7b shows the total shear force plotted against the radius of the cylinder. An increase in the radius of the cylinder causes a monotonic increase in the total shear force, as expected.
Horizontal force

We now investigate another important force acting on the cylinder, the horizontal force caused by the incoming incident waves. The horizontal force is obtained by integrating the hydrodynamic pressure over the surface of the cylinder. The horizontal force component in the \(y\)-direction is zero owing to the symmetry of the problem. The horizontal force component in the \(x\)-direction, \(F_x\), is given in dimensional variables as

\[
F_x(t) = -b \int_0^{2\pi} p_D(b, \theta, z, t) \cos(\theta) d\theta dz,
\]

where \(p_D\) is the hydrodynamic pressure, which can be calculated from equations (2.2), (2.9) and (2.16) as

\[
p_D^*(b^*, \theta, z^*, t^*) = \text{Re} \left( -ie^{-it^*} \sum_{n=0}^{\infty} e^n i^n \cos(n\theta) (f_0^*(z^*) J_n(k^*b^*) + a_n D^n f_0^*(z^*) H_n(k^*b^*))) \right). \tag{4.8}
\]

When integrating in equation (4.7) with respect to \(\theta\), we note that only the terms in equation (4.8) with \(n = 1\) contribute to the result. The contribution from \(\phi_1^*(b^*, z^*)\) in equation (4.8) is calculated by using equation (3.9) and an inverse Weber transform. After manipulations, we find the non-dimensional horizontal force as

\[
F_x^*(t^*) = \bar{F} \cos(t^* + \delta_f),
\]

where

\[
\bar{F} = \frac{16}{k^*3b^* \pi^2(\chi_1^2 + \eta_1^2)^{1/2}} \int_0^{\infty} \frac{(s^2 - k^*2) Q^*(s^*) + k^*}{s^3((J_1'(s^*b^*))^2 + (Y_1'(s^*b^*))^2)} ds^* \tag{4.9}
\]

is the maximum horizontal force with respect to the time periodicity of the problem. The scale of the force \(F_x\) is \(a\rho H^2\omega^2\). The quantities \(\chi_1\) and \(\eta_1\) are given by equation (3.19).

Figure 8a shows the dimensional horizontal force plotted against the non-dimensional wavenumber \(k^*\) for two values of the cylinder radius \(b\). There exists a maximum horizontal force for both values of the cylinder radius, and it is clear that a larger radius causes greater force on the cylinder and the dependence is sensitive.

We wish to compare the horizontal force in the presence of an ice sheet with the hydrodynamic force acting on a vertical cylinder with no ice sheet present. From Mei [16], the horizontal force on a vertical cylinder owing to incident free surface waves is given by (in the notation of the present paper and in dimensionless variables)

\[
F_x^{\text{fs}} = \frac{4 \tan(h^*)}{k^*2((J_1'(k^*b^*))^2 + (Y_1'(k^*b^*))^2)}, \tag{4.10}
\]

and the dimensional force is obtained by multiplying \(F_x^{\text{fs}}\) by \(a\rho H^2\). Figure 8b plots the horizontal force for various ice thicknesses along with the free-surface
Figure 8. The incident wave amplitude is fixed at $a = 2$ cm and all other parameters are fixed at their default value unless stated. (a) The maximum total horizontal force is plotted against the incident wavenumber $k^*$. Here, the cylinder radius is given by: solid line, $b = 2$ m; dashed line, $b = 3$ m. (b) The maximum total horizontal force is plotted against the wave frequency $\omega$. Here, $b = 3.5$ m and the ice thickness is: solid line, $h = 1.6$ m; dashed line, $h = 0.75$ m; dotted line, $h = 0.1$ m; the dashed-dotted line represents the free-surface case.

5. Conclusion

The linear diffraction of hydroelastic waves by a vertical cylinder has been studied. The ice is frozen to the vertical cylinder. The problem was solved by Weber transform. This method leads to the solution being explicitly written in terms of integral quadratures. The decomposition of the azimuthal coordinate combined with the modified Weber transform allows for a relatively simple solution. Also calculated are expressions for the vertical shear and horizontal force components. It is concluded that these forces can reach large magnitudes, even for small-amplitude long waves. The forces must, therefore, be considered when designing ocean structures in the presence of an ice cover. The behaviour of these forces under variation of parameters of the problem was investigated. The strain distribution in the ice sheet was also investigated. It was shown that the strain is highest at the surface of the vertical cylinder where the ice is clamped. These strains are generally high, but under certain conditions on the wave amplitude and wavelength, the connection may be maintained. It is unlikely that the ice will
remain frozen to the cylinder under the constant swell of large-amplitude ocean waves, but in applications involving frozen lakes where one expects smaller wave amplitude, the clamped boundary condition proves realistic. In the event of the ice breaking off from the surface of the cylinder, the problem can be reformulated and solved using the same method, with free-edge conditions being considered instead of the fixed-edge conditions of the present problem.

An alternative method of solution that could have been used to tackle the present problem is an eigenfunction expansion used by Malenica & Korobkin [20]. The velocity potential in both the free-surface and ice-covered regions are expressed in terms of eigenfunction expansions, the eigenvalues of which are the solutions of the corresponding dispersion relation. Because the dispersion relations consist of an infinite number of imaginary roots, the resulting series must be truncated, leading to a linear system of equations for unknown coefficients. The Weber transform and its inherent propensity for symmetry leads to several advantages for solving the present problem. Firstly, the important function $Q(s)$ also appeared in the equivalent two-dimensional problem considered by Brocklehurst et al. [24], and its properties and behaviour are, therefore, well known. $Q(s)$ and its behaviour govern every aspect of the problem, hence familiarity is useful. Also, the Weber transform solution expresses the unknown coefficients $a_n^D$ in terms of exact integral quadrature. The velocity potential and ice deflection are expressed in terms of an infinite sum, but the solution converges extremely quickly, meaning few Fourier modes are required. The present method also leads to very concise expressions for the forces on the wall. The total horizontal force given by equation (4.9) is expressed in terms of integral quadrature, the integrand of which decays rapidly. The total vertical shear force is given by equation (4.6), and owing to auspicious cancellation during the Weber transform, we are not required to calculate the third derivative, which potentially could have led to convergence issues. The force is instead expressed algebraically once the coefficient $a_1^D$ is obtained.

Intended future work includes the problem of cylinder arrays in the presence of an ice sheet. Also, a mixed boundary-value problem would be a valuable consideration, whereby the ice is frozen to only part of the cylinder and is free to move on other parts. The ice model should be improved by moving away from the linear, idealized thin elastic plate model by including the heterogeneity of the ice and other important effects that occur naturally in ice sheets.

References

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