The role of the international prototype of the kilogram after redefinition of the International System of Units

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Since 1889, the international prototype of the kilogram has served to define the unit of mass in what is now known as the International System of Units (SI). This definition, which continues to serve mass metrology well, is an anachronism for twenty-first century physics. Indeed, the kilogram will no doubt be redefined in terms of a physical constant, such as the Planck constant. As a practical matter, linking the quantum world to the macroscopic world of mass metrology has, and remains, challenging although great progress has been made. The international prototype or, more likely, a modern ensemble of reference standards, may yet have a role to play for some time after redefinition, as described in this paper.

Keywords: International System of Units; kilogram; redefinition; international prototype; quantity calculus; mass metrology

1. Introduction

In the present International System of Units (SI) [1], the kilogram is defined as the mass of a particular object, known as the international prototype of the kilogram (IPK), which was put into service in 1889 and which has been used sparingly since [2,3]. It seems inevitable that, owing to advances in science and technology, this definition will soon be superseded by one based on physical constants. The Planck constant \( h \) combined with the two constants that already define the size of the SI second and metre have been suggested [4].

Ultimately, the IPK will cease to retain any metrological importance. This would obviously occur if, for example, realizations of the new definition reach an accuracy of 1 part in \( 10^9 \). However, we will examine below the possibility that, before meeting its ultimate fate, the IPK may have a continuing role to play for a limited time after the redefinition of the kilogram.

The following will make clear that the IPK already plays multiple roles, one of which is to define the SI kilogram, another is to realize the kilogram unit with the aid of a commercially available mass comparator and a third is to minimize uncertainties in mass metrology by traceability of the mass unit to a common source. The following will discuss to what extent these functions can, or should,
be separated. The discussion will necessarily examine the uncertainty to which the redefined kilogram can be realized in practice and the prospects for realization of the kilogram after its redefinition. We will note that the IPK, in its role as a useful object for mass metrology, could be replaced by an even more useful ensemble of objects.

We also re-examine the interesting question of how well \( h \) should be known prior to redefinition of the kilogram and the consequences to physics and metrology if the uncertainties of the present measurements of \( h \) have been significantly underestimated.

We will use quantity calculus as a tool to discuss issues related to the change in units. This helps elucidate which SI units are in play without the need to go into great technical detail regarding various complex experiments. The latter are dealt with by other papers given at this meeting.

2. Quantity calculus and the present definition of the kilogram

The present definition of the kilogram (kg) is simply stated: ‘The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram’ [1].

Note that the first part of the definition identifies the base unit ‘kilogram’ with the corresponding base quantity, ‘mass’. The second part of the definition refers to a specific object that we will refer to as the IPK in this paper, and specifies that the numerical value of the mass of this object is exactly 1 when expressed in kilograms.

We now introduce a useful notation from quantity calculus [5] that clarifies distinctions between a physical quantity and the value of the quantity in terms of SI coherent units. Suppose \( Q \) represents a physical quantity, then \([Q]\) is the SI coherent unit of \( Q \) and \( \{ Q \} \) is the numerical value of \( Q \) when expressed in units of \([Q]\). The SI value of \( Q \) in coherent units is therefore given by

\[
Q = \{Q\}[Q].
\] (2.1)

(In the case that \( Q \) is dimensionless, \([Q]\) = 1 implicitly, as explained in detail in [1].) The first part of the kilogram definition states that for any entity \( X \) to which a mass value can be assigned,

\[
[m(X)] = \text{kg}.
\] (2.2)

The second part of the definition specifies that for a particular \( X \), which we are calling the IPK in this discussion,

\[
\{m(\text{IPK})\} = 1 \text{ (exactly)}.\] (2.3)

It follows that the relation

\[
m(X) = \{m(X)\}[m(X)] = \frac{m(X)}{m(\text{IPK})}\text{kg}
\] (2.4)

governs the dissemination of the kilogram unit to any \( m(X) \). Most of the challenges of a practical realization of the definition arise from implementation of equation (2.4) [3].

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3. Some features of the present definition

We now examine implications of equation (2.4).

— The ratio represents a measurement with an experimental uncertainty.
— The experimental uncertainty of the ratio, being identical to the
  experimental uncertainty of \( \{m(X)\} \), equals the uncertainty of the mass
  of \( X \) in units of kilogram.
— The IPK is particularly convenient for mass metrology because its size
  allows dissemination to other 1 kg weight pieces by means of commercially
  available 1 kg mass comparators having standard deviations of the order
  of 1 \( \mu \)g or below. As a consequence, a dissemination chain has been
  established such that stainless steel weight pieces of the highest class
  specified by the International Organization of Legal Metrology (OIML) [6]
  can be purchased as stock items. The OIML recommends that the relative
  maximum permissible error (MPE) upon verification be 5 parts in \( 10^7 \) for
  nominal masses ranging from 0.1 to 50 kg.
— The mass of a weight piece is adjusted during manufacture to approximate
  its nominal mass. Beyond this, weight pieces can then be calibrated
  through a traceability chain represented by equation (2.4) in order
  to have a more exact measure than the MPE of their mass. The
  smallest relative standard uncertainties routinely available from national
  laboratories are several parts in \( 10^8 \) for the calibration of weight pieces
  having nominal masses in the range 0.1–50 kg (for example, Calibration
  and Measurement Capabilities: mass and related quantities, Germany,
  Physikalisch-Technishe Bundesanstalt (PTB) at http://kcdb.bipm.org/
  AppendixC/country_list_search.asp?CountSelected=DE&iservice=M/
  Mass.1.1.1). Relative uncertainties increase outside this range.

That the IPK is conserved at the Bureau International des Poids et Mesures
(BIPM) and is only made available for use during rare calibration campaigns
[2] presents a more serious problem. In general, \( m(X) \) must be known through
a traceability chain that begins with the IPK, ends with \( X \) and may include
numerous intermediate steps, as shown in great detail by Borys and co-workers
of the PTB [7]. An extensive system of mass metrology is in place to ensure that
end users have the required traceability to the IPK. But this is not the most
serious problem.

The most problematic feature of equation (2.4) is the convention that
\( \{m(IPK)\} = 1 \) at any given time, once any surface contamination has been
removed from the IPK by cleaning (from appendix 2 of BIPM [1], Practical
realization of the definition of the kilogram; available uniquely at http://
comparison is possible between \( m(IPK) \) as it exists at a particular time \( t' \), let us
call this quantity \( m(IPK)_{t'} \), with its initial value in 1889, \( m(IPK)_0 \) or indeed with
its value at any other time \( t \) in between. Some values of the function \( F(t, t') \), where

\[
F(t, t') = \frac{m(IPK)_{t'}}{m(IPK)_t}
\] (3.1)
can, however, be inferred. For instance, the same physical constants (the
maximum density of distilled water at atmospheric pressure, the density of a
particular sample of mercury at standard conditions, etc.) have been measured
at different time epochs. To date, the inferences that can be drawn have, until
relatively recently, had large uncertainties [3,8].

Instability of the mass of standard mass pieces is a general problem
in metrology. It is worth quoting how this is dealt with in a key OIML
recommendation [6], where the local standard of mass is referred to as the
reference weight:

The uncertainty due to instability of the reference weight...can be estimated from observed
mass changes after the reference weight has been calibrated several times. If previous
calibration values are not available, the estimation of uncertainty has to be based on
experience. [6, p. 67]

This is good advice, which can be followed all the way through the dissemination
chain until the IPK is reached. But \(m(\text{IPK})=1\) by definition, a value that
has no uncertainty even though \(m(\text{IPK})\) can be unstable with respect to a
fundamentally constant mass, such as the rest mass of the electron.

4. Generalization of equation (2.4) to measure fundamental constants

Equation (2.4) can easily be generalized to cases where the unknown is not the
mass of some entity \(X\) but rather any physical quantity whose units contain the
kilogram. Obviously, one important example is the measurement of the Planck
constant, \(h\), whose SI derived unit, J s, is a special name for the combination
of base units: kg m\(^2\) s\(^{-1}\). (In terms of quantity calculus, \([h]=\text{kg m}^2\text{s}^{-1}\).) Let us
represent an experiment to determine the SI value of \(h\) as

\[
h = m(\text{IPK}) \cdot Q_h. \tag{4.1a}
\]

The analogues of equations (2.2) and (2.4) may be written as

\[
\left\{ \frac{h}{m(\text{IPK})} \right\} = \{Q_h\} \tag{4.1b}
\]

and

\[
\left[ \frac{h}{m(\text{IPK})} \right] = [Q_h] = \text{m}^2\text{s}^{-1}. \tag{4.1c}
\]

The dimensions of \(Q_h\) imply that the experiment represented by equation (4.1a)
can be parametrized as \(Q_h = \left(\frac{c^2}{f}\right) R_h\), where \(c\) is the speed of light in vacuum,
which has a fixed numerical value in the SI, and \(f\) is an experimental frequency,
which is traceable to the definition of the SI second. In addition to these
dimensioned quantities, we must include a dimensionless quantity \(R_h\), which may
be the product of ratios of like quantities (e.g. ratios of frequencies), phase shifts,
dimensionless constants, etc.

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The role of the international prototype

The relative uncertainty in the experimental determination of \( h \) from equation (4.1a) is evidently

\[ u_r(h) = u_r(Q_h) = u_r \left( \frac{R_h}{f} \right). \]

We can carry out a similar programme for the Avogadro constant, \( N_A \), which has dimensions of \( \text{mol}^{-1} \),

\[ N_A = \frac{Q_N}{m_{\text{IPK}}}, \]  

(4.2a)

\[ \{N_A m_{\text{IPK}}\} = \{Q_N\} \]  

(4.2b)

and

\[ [N_A m_{\text{IPK}}] = [Q_N] = \text{kg mol}^{-1}. \]  

(4.2c)

The appearance of the kilogram on the right-hand side of equation (4.2c) is not problematic because in the present SI, the molar mass of carbon-12, \( M^{(12\text{C})} \), has an exactly defined numerical value of 0.012, with units of \( \text{kg mol}^{-1} \) [9]. (The molar mass constant, \( M_u \), is simply \( M^{(12\text{C})}/12 \).) Therefore, the dimensions of \( Q_N \) imply that the experiment represented by equation (4.2a) can be parametrized as \( Q_N = M^{(12\text{C})}R_N \), where we must include a dimensionless quantity, \( R_N \), which may be composed of the product of ratios of like quantities, dimensionless constants, etc.

The relative uncertainty in the experimental determination of \( N_A \) from equation (4.2a) is evidently

\[ u_r(N_A) = u_r(Q_N) = u_r(R_N). \]

An essential relation follows from the definition of the mole [1]:

\[ N_A m^{(12\text{C})} = M^{(12\text{C})}, \]  

(4.3a)

where \( m^{(12\text{C})} \) is the mass of a carbon-12 atom (\([m^{(12\text{C})}] = \text{kg}\)). Therefore, equation (4.2a) may also be written, perhaps, more aptly for the present discussion, as

\[ m^{(12\text{C})} = m_{\text{IPK}} R_N^{-1}. \]  

(4.3b)

The dimensional analysis given above hints at what might be possible experimentally. Additional information on the structure of \( R_h \) and \( R_N \) is given in appendix A.

It is up to the skill and ingenuity of scientists to devise appropriate experiments to yield a highly accurate value for \( R_h/f \) in the case of the Planck constant (see Stock [10] for an example of the watt balance) or \( R_N \) in the case of the Avogadro constant (see Becker & Bettin [11] for an example of the silicon-crystal method). Nor can dimensional analysis give us any insight into the importance of these constants to physics. For that, the reader is directed to Bordé [12]. We can, however, infer from dimensional analysis that the product \( N_A h \), known as the molar Planck constant, has dimensions

\[ [N_A h] = (m)(\text{m s}^{-1})(\text{kg mol}^{-1}). \]  

(4.4)

It is easily shown that equation (4.3a) can be combined with the Rydberg relation [9],

\[ h c R_\infty = \frac{1}{2} m_e (\alpha c)^2, \]  

(4.5)
where \( R_\infty \) is the Rydberg constant (\([R_\infty] = m^{-1}\)), \( m_e \) is the mass of the electron (\([m_e] = \text{kg}\)) and \( \alpha \) is the fine structure constant, which is dimensionless (\([\alpha] = 1\)), to yield

\[
N_A h = \left(\frac{1}{R_\infty}\right) (c)(M(^{12}\text{C})) \left(\frac{m_e}{m(^{12}\text{C})}\right) \left(\frac{\alpha^2}{2}\right). \tag{4.6}
\]

The right-hand side of equation (4.6) has been written as a product of five terms. The first three terms have SI dimensions in parallel to equation (4.4), and the remaining two terms are dimensionless. The speed of light and the molar mass of carbon-12 have exact numerical values in the present SI. The relative uncertainty of the product of the remaining three terms is dominated by \( u_r(\alpha^2) = 2u_r(\alpha) = 1.4 \times 10^{-9} \) [9,13].

Owing to the small uncertainty of \( N_A h \) relative to either \( N_A \) or \( h \) taken individually, any experiment described by equation (4.1a) as a determination of \( h \) can today be considered to be an experiment described by equation (4.2a) as a determination of \( N_A \),

\[
N_A = \frac{M(^{12}\text{C})}{m(\text{IPK})} \left(\frac{fR_h^{-1}}{cR_\infty}\right) \left(\frac{m_e}{m(^{12}\text{C})}\right) \left(\frac{\alpha^2}{2}\right) \tag{4.7}
\]

and vice versa

\[
h = m(\text{IPK}) \left(\frac{c^2}{cR_\infty}\right) R_N^{-1} \left(\frac{m_e}{m(^{12}\text{C})}\right) \left(\frac{\alpha^2}{2}\right). \tag{4.8}
\]

At present, uncertainties of the right-hand sides of equations (4.1a), (4.2a), (4.7) and (4.8) are dominated by the smallest relative uncertainties of \( R_h/f \) or \( R_N \) obtained from experiments. For this reason, one often sees results of determinations of \( N_A \) via the silicon-crystal method plotted as experimental values of \( h \) (fig. 6 of Stock [10]) and vice versa (fig. 5 of Andreas et al. [14]). Such graphs illustrate whether results from independent experiments and/or methods are consistent to within their stated uncertainties. That is, whether \( R_h/f \) values obtained from various watt balance experiments are consistent among themselves and are consistent with \( R_N \) values obtained by the International Avogadro Coordination project.

5. The new definition of the kilogram

It seems that the problematic features of the present definition of the kilogram would be eliminated by simply redefining the SI unit of mass in terms of an appropriate fundamental constant [15]. If \{\( h \)\} was defined to have a fixed value, then \{\( m(\text{IPK}) \)\} could be derived via equations (4.1a), (4.1b) or (4.8). Or the numerical value \{\( N_A \)\} could be defined and \{\( m(\text{IPK}) \)\} could be derived via equations (4.2a), (4.2b) or (4.7). The second solution would essentially define the unit of mass in terms of \( m(^{12}\text{C}) \). After much discussion, the Consultative Committee for Units and the International Committee for Weights and Measures (CIPM) have endorsed the proposal that the next evolution of the SI should fix the value of both \{\( h \)\} and \{\( N_A \)\}, as was proposed in Mills et al. [16].
Table 1. The effect of the redefinition of the kilogram on the mass of the international prototype, $m$(IPK), the Planck constant, $h$, and the experimental value, $Q$.

<table>
<thead>
<tr>
<th></th>
<th>immediately before redefinition</th>
<th>immediately after redefinition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h/m$(IPK) = $Q$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>relative uncertainty of $Q = u_r(Q)$</td>
<td></td>
<td>relative uncertainty of $Q = u_r(Q)$</td>
</tr>
<tr>
<td>value of $m$(IPK) = 1 kg</td>
<td></td>
<td>value of $m$(IPK) = 1 kg</td>
</tr>
<tr>
<td>relative uncertainty of $m$(IPK) = 0</td>
<td></td>
<td>relative uncertainty of $m$(IPK) = $u_r(Q)$</td>
</tr>
<tr>
<td>relative uncertainty of $h = u_r(Q)$</td>
<td></td>
<td>relative uncertainty of $h = 0$</td>
</tr>
</tbody>
</table>

(In order not to over-constrain the revised SI, $M^{(12)C}$ would then have a measured value determined, for example, from equation (4.6) with $u_r(M^{(12)C}) = 2u(\alpha)$ [17].)

As part of a major updating of the SI, a draft text for the ‘possible’ future redefinition of the kilogram has been submitted by the CIPM for consideration at the next of the quadrennial meetings of the General Conference on Weights and Measures (CGPM), in October 2011. This submission [18], which should be compared with the present definition, contains the following relevant text, which we will refer to as part A and part B:

> The kilogram will continue to be the unit of mass, but its magnitude will be set by fixing the numerical value of the Planck constant to be equal to exactly $6.62606\times10^{-34}$ when it is expressed in the SI unit $m^2kg^{-1}s^{-1}$, which is equal to J s. (Part A) [18, p. 2]

Here, X represents the final digits that have yet to be determined. It is further noted in the draft document that, on the day the redefinition takes effect,

> The mass of the international prototype of the kilogram...will be exactly 1 kg but with a relative uncertainty equal to that of the recommended value of $h$ just before redefinition and that subsequently its value will be determined experimentally. (Part B) [18, p. 4]

Note that this ‘possible’ redefinition could change before the CGPM gives its final approval to redefinitions at a future General Conference.

It is important to understand in detail the consequences of parts A and B. The key features are summarized in table 1. In this case, $\{Q\}$ represents the value of $\{h\}$ recommended by the Committee on Data for Science and Technology (CODATA) Task Group on Fundamental Constants (TGFC) just prior to the redefinition of the kilogram (when $\{m$(IPK)$\}$ is still identically equal to 1), and $u_r(Q)$ is the relative uncertainty of $Q$ as recommended by the TGFC. Presumably, the TGFC will follow its usual practice of carrying out a critical evaluation of all input data available at the time of its well-publicized cut-off date.

Let us briefly look at the content of table 1. That $u_r(Q)$ is unchanged by the new definitions proposed for the SI is easily explained: (i) the value of $Q$ derives from experimental results reported up to a fixed deadline prior to the new definitions and (ii) $[Q] = m^2s^{-1}$, and the metre and second are unaffected by the proposed changes to the SI. That $\{m$(IPK)$\} = 1$ (exactly) before the changes...
to the SI is essentially a restatement of the present definition of the kilogram. That \( \{m(\text{IPK})\} = 1 \) immediately after the redefinition is an obvious requirement to ensure that mass measurements traceable to the IPK before the changes to the SI will have the same mass values after the redefinition. Part B refers to the \( \{m(\text{IPK})\} \) just after the redefinition takes effect as being ‘exactly’ 1, as before (the continuity condition). Nevertheless, in the same sentence, the text of part B refers to a finite ‘relative uncertainty’ of \( m(\text{IPK}) \) after the new definition takes effect. Since \( Q \) is derived from experiments, its value must have a finite relative uncertainty, \( u_r(Q) \). In the final TGFC recommendations prior to the new definitions, the uncertainty \( u_r(Q) \) will continue to attach to the value of \( h \). Once \( \{h\} \) has been given an exact value (part A of the recommendation cited above), this relative uncertainty is transferred to \( m(\text{IPK}) \). (The possibility that \( u_r(Q) \) might be expanded immediately following the redefinition is addressed in §9a.) It is expected that the IPK will be used during the period preceding the redefinition in order to ensure that laboratories contributing to the final measured value of \( h \) will be as closely tied to \( m(\text{IPK}) \) as possible.

Part B adds that subsequent to the new definition, \( m(\text{IPK}) \) must be determined experimentally.

6. Consequences for mass metrology of a kilogram defined by physical constants

Clearly, the proposed definition of the kilogram has the potential to improve mass metrology at the macroscopic level. Dissemination of the mass unit to an arbitrary, macroscopic entity \( X \) will be traceable to a physical constant. There is no longer any special status for a particular artefact (the IPK). Instructions for the practical realization at the highest metrological level of the kilogram unit, known in metrology as the mise en pratique for the definition of the kilogram are available to all laboratories. Carrying out this programme to the benefit of mass metrology essentially depends on how the new uncertainty component, \( u_r(Q) \), is dealt with by mass metrology and whether laboratories that claim to realize the proposed definition will be sufficient in number and will be able to demonstrate that they obtain equivalent results.

The size of \( u_r(Q) \) and its initial calculation are pertinent to this discussion. Although we cannot know the details of future analyses, we can at least describe how the TGFC arrived at \( u_r(Q) \) in their 2002 and 2006 analyses and assume that similar principles will be followed in the future. As a first step, a weighted mean based on the a priori uncertainties of all relevant input data is calculated (taking into account correlations among the data). Often the TGFC will observe that the data under consideration are not self-consistent, as evidenced by the resulting Birge ratio [19]. When this happened for the Planck constant [9,20], the a priori uncertainties of the critical dataset were increased by the same multiplicative factor, which resulted in a posteriori uncertainties that give a satisfactory Birge ratio. By applying this procedure, the statistical weights assigned to the critical input values do not change; the value of the weighted mean of these values does not change appreciably, but the estimated standard uncertainty of the weighted mean is expanded by approximately the a posteriori factor. In the study of Mohr & Taylor [20], the uncertainties of the critical input data that contributed...
to a value of $h$ were increased by a factor of 2.325; for the 2006 dataset, a factor of 1.50 was deemed to be justified [9]. The 2010 adjustment is expected to be published in 2011 and will have to deal with the situation that has been shown graphically in fig. 6 of Stock [10]. Of course, the situation for later CODATA adjustments is unknowable at present. We do, however, have a clear statement of the basic philosophy of the TGFC, including a ‘downside’ which is pertinent to this discussion:

> to provide the user community with values of the constants having the smallest possible uncertainties consistent with the information available at the time. The motivation for this approach is that it gives the most critical users of the values of the constants the best possible tools for their work based on the current state of knowledge. The downside is that the information available may include an error or oversight. Nevertheless, the CODATA Task Group rejects the idea of making uncertainties sufficiently large that any future change in the recommended value of a constant will likely be less than its uncertainty—simply put, the Task Group shuns the use of a ‘safety factor’ as employed in some data compilations (emphasis added).

[19, p. 2112]

7. The needs of mass metrology

The present definition of the kilogram accommodated the needs of mass metrology in the late nineteenth century and it has continued to serve mass metrology reasonably well until now. However, trends that have been seen since the 1950s in mass comparisons between the IPK, its official copies and the national prototypes constitute proof that the system is not perfect [3]. Indeed, figure 1 has by now achieved almost iconic status. It shows that taking the mass of the IPK to be exactly 1kg (as required by the SI definition of the unit of mass), there is a general tendency for the mass of the oldest 16 national prototypes, as determined by equation (2.4), to increase by an amount of order 1 part in 10$^9$ per year. Unfortunately, this does not give us solid information about the quantity $F(t,t')$ as defined in equation (3.1). Indeed, not even the sign of $F(t,t')$ can be determined from figure 1 because the average mass of the whole set of artefacts, all made of the same alloy, might be drifting at an unknown rate with respect to constants of nature. However, it should be noted that remarkable precision with which 1kg artefact mass standards can be compared reveal the trends in figure 1. A similar trend is seen for the official copies of IPK, which are stored at the BIPM and with prototypes that have been manufactured more recently.

The Consultative Committee for Mass and Related Quantities (CCM), which advises the CIPM in matters related to the mass metrology, reacted to data such as those shown in figure 1 by producing a Recommendation that eventually resulted in Resolution 5 of the 20th CGPM in 1995. This resolution, noting the progress being made in developing methods for monitoring the stability of mass standards, ‘recommends that national laboratories pursue their work on these experiments, and develop new ones, with a view to monitor the stability of the IPK and in due course opening the way to a new definition of the unit of mass based upon fundamental or atomic constants’ [21]. The sense of urgency [15] to carry out this programme came later, led by concerns in other areas of science and metrology [10]. (To put figure 1 in perspective, fig. 6 of Stock [10] shows two results with very small uncertainty—from the National Institute of Standards and

Phil. Trans. R. Soc. A (2011)
Figure 1. Reproduction of fig. 3 from Davis [3] showing the changes in mass of the 16 oldest national prototypes of the kilogram with respect to the mass of the international prototype. There are only three data points for each line.

Technology (USA) watt balance and the International Avogadro Coordination project. These differ by 170 parts in $10^9$, an envelope that would contain the entire calibration history of the majority of national prototypes over a century of data.)

For its part, the OIML specifies that reference mass standards ‘have to be calibrated or adjusted, thus guaranteeing traceability to the international prototype of the kilogram…’ [22]. The issue of the instability of artefact masses, including $m$(IPK), is mitigated by the fact that $u_r(m$(IPK)) = 0. Mass metrology chooses to ignore the very weak limits that can, at present, be placed on $F(t,t')$ when $t'-t$ is of order 40 years, the time between the last two extensive comparisons of national prototypes of the kilogram with respect to the IPK. The ‘innocent until proven guilty’ approach has served mass metrology well but, as recognized earlier [21], a better solution must be sought.

After the redefinition, dissemination described by equation (2.4) of the kilogram unit will have an additional uncertainty component $u_r(Q)$. The nature of this uncertainty, and how mass metrology will deal with it, will now be discussed.

8. Consequences to physics and mass metrology owing to the redefinition of the kilogram

Ultimately, only those types of experiments shown in fig. 5 of Stock [10] contribute in any significant way to the present value of $h$. All these results can be interpreted as linking atomic or subatomic masses to the IPK. Consequently, these experiments represent the only areas of metrology and physics that could be significantly affected by choosing a fixed value for $h$.

Two questions are sometimes asked. What if we do not change the definition of the kilogram and, in succeeding years, it turns out that new and better experiments come online, making it evident that the values of $h$ and $u_r(h)$

Phil. Trans. R. Soc. A (2011)
The role of the international prototype

9. A way forward

If a number of apparatus exist, capable of realizing the new definition of the kilogram to an experimental uncertainty within 2 parts in 10^8, and if all of these results are consistent, then each laboratory with such an apparatus would be capable of disseminating the kilogram, and the world of metrology would be well satisfied [7]. However, if such conditions are not met, mass metrology could still be improved compared with its present state by the following strategy, which comprises three steps of increasing complexity.

(a) Continued use of the international prototype of the kilogram

Results of all experiments capable of realizing the proposed definition of the kilogram should be compared, and this should be linked as closely as possible to m(IPK). The uncertainty assigned to m(IPK) would, in all probability, be significantly smaller than the uncertainty of any individual realization. The kilogram could then be disseminated from the IPK as before, retaining most of
the present advantages of that method. The IPK, or more likely a pool of artefacts (see §9b), might thus be used following redefinition of the kilogram. According to CCM Recommendation G1 of 2010 [25], the uncertainty assigned to the IPK immediately following the redefinition might differ from the CODATA value for \( u_r(h) \) immediately before the redefinition, although the CODATA uncertainty would be given ‘suitable’ consideration when assigning the initial uncertainty to \( m(\text{IPK}) \). That is to say, the CCM might decide that a safety factor is necessary for metrology, even though the TGFC rejects such a factor for research into fundamental constants.

\((b)\) Pool of reference standards

The CCM Recommendation foresees a pool of reference standards established and maintained at the BIPM to facilitate the dissemination of the new definition of the kilogram. The goal here is to create a distributed artefact that is more stable and reliable than the IPK. The pool of reference standards would be comprised up to twelve 1 kg artefacts made of three different materials: the same platinum–iridium alloy used for the IPK, single crystals of silicon and a suitable alloy of non-magnetic stainless steel. The average mass of this group is more likely to be stable than the mass of a single component. In addition, the chance that mass values of reference standards made of several completely different materials would have a common instability should be much reduced. Finally, when not being used, the components of the pool would be stored either in vacuum or in a non-reactive gas, such as nitrogen or argon. The BIPM is proceeding to assemble such a pool [26].

\((c)\) Dealing with \( u_r(Q) \)

The CCM also recommends that ‘the BIPM and a sufficient number of National Metrology Institutes continue to develop, operate or improve facilities or experiments that allow the realization of the kilogram to be maintained with a relative standard uncertainty not larger than 2 parts in 10^8’ and that ‘the uncertainty component arising from the practical realization of the unit be suitably taken into account’. The last sentence indicates a reluctance to create a practical mass scale for mass metrology, in analogy to the practical scale that already exists in thermometry, and which will be unaffected by the planned redefinition of the kelvin [27] and the conventional electrical quantities presently in use [10]. Mills et al. [15] had already proposed such a scheme for mass metrology, where they suggest giving a ‘conventional’ value to the mass of the IPK and carrying on exactly as before the redefinition of the kilogram by omitting \( u_r(Q) \). In essence, the proposal is to create a ‘conventional’ quantity whose unit is the kilogram, which would be a close approximation to the new SI kilogram, but would be traceable to mass of the IPK. Unfortunately, the term ‘conventional mass’ is already widely used in legal metrology to define a much different quantity, also measured in units of kilogram [22], and thus the suggestion was to introduce a second such quantity. All this can be avoided if \( u_r(Q) \) remains sufficiently small [7], but this would be a daunting technological challenge that may never be met. At present, mass metrology benefits from a different convention, namely that \( u_r(m(\text{IPK})) = 0 \).
If

— all macroscopic mass standards were disseminated from a single source, either the IPK or the weighted average of a pool of reference standards,

and

— that source had an uncertainty $u_r(Q)$, based on a weighted average of the best possible links to all experiments capable of realizing the proposed definition of the kilogram,

then

— every mass piece in the dissemination chain would be subject to the identical uncertainty component, $u_r(Q)$.

These are well-known consequences of measurement results being correlated through calibration from a common source [28]. Extra bookkeeping regarding uncertainties is required and can lead to confusion and error if, taking our discussion as a specific example, $u_r(Q)$ is non-negligible.

The CCM and others have recommended that redefinition of the kilogram be delayed until $u_r(Q)$ is negligible for the needs of mass metrology [8,25]. But suppose that this goal remains elusive. At some point, one might be required to take a hard look at how $u_r(Q)$ could be best accommodated in a less than ideal system.

It would first be necessary to examine the uncertainty to which the mass values of reference standards: (i) must be traceable all the way to the SI definition of the unit of mass or (ii) need only be traceable to a common source, such as the IPK or a pool of reference standards. Category (i) applies to measurements of physical reference quantities, such as the densities of standard materials; to measurements of physical constants by other than quantum-based means; and to the measurement or generation of derived physical quantities related to mass, such as force and pressure. In all such cases, a limit of $u_r(Q) < 10^{-7}$ would seem to have a negligible effect on existing tables of reference values and claims of traceability to the SI.

Category (ii) applies to control of industrial processes or general maintenance of quality, where traceability to a fundamental constant of physics is not necessarily valuable. Owing to the correlation described above, the uncertainty of comparisons between secondary mass standards, each of which is traceable to the same source (for example, the average mass of the pool of reference standards to be maintained at the BIPM), is independent of $u_r(Q)$ regardless of its magnitude. It then seems possible to implement the proposal given in Mills et al. [15] by reporting uncertainties in mass metrology exclusive of the common $u_r(Q)$ component, adding a note that any user in category (i) would need to add an uncertainty component that, for instance, would be publicly available on a BIPM website.

This is a programme that the CCM was not prepared to accept at its 2010 meeting, preferring to encourage advances in experimental techniques in order to render $u_r(Q)$ sufficiently small.
10. Conclusion

We have shown that there may be a role for the IPK or, more likely, a pool of reference standards maintained at the BIPM after the redefinition of the kilogram. Details depend on the uncertainties that will be achieved in determinations of $h$ at different laboratories in the next few years, as well as whether these values of $h$ are consistent within their a priori uncertainties.

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Appendix A

(a) $R_h$ and $R_N$

We now show schematically the measurable quantities that comprise $R_h$ and $R_N$.

(b) Analysis of $R_h$

From equation (4.1a),

$$h = m(\text{IPK}) Q_h = m(\text{IPK}) \left( \frac{c^2}{f} \right) R_h. \quad \text{(A1)}$$

From eqn (6) of Stock [10], the watt balance equation can be written as

$$h = m \frac{g v}{f_1 f_2} C_{el}^{-1}, \quad \text{(A2)}$$

where $m$ is the mass (of order 0.1–1kg) of the mass piece placed on the watt balance, $g$ is the local acceleration of gravity, $v$ is a velocity, $f_1$ and $f_2$ are frequencies associated with two Josephson voltage measurements and $C_{el}$ is a dimensionless factor.

The mass, $m$, is traceable to the IPK, so that

$$a_m = \frac{m}{m(\text{IPK})}, \quad \text{(A3)}$$

where $a_m$ is a dimensionless ratio derived from a mass calibration traceable to the present definition of the kilogram.

The gravitational acceleration, $g$, is traceable to $c T^2 / f_g$ where $c$ is the speed of light, whose numerical value is fixed in the SI, $f_g$ is an experimental frequency and $T$ is a time delay. Ultimately,

$$g = a_g \frac{T^2}{f_g}, \quad \text{(A4)}$$

where $a_g$ is dimensionless [12]. The necessary observables are derived from a gravimeter measurement where one measures the acceleration of a body—a mirror or an ultra-cold atom—that is in free fall with respect to the laboratory frame of reference.
The role of the international prototype

The velocity, \( v \), is traceable to \( c \), the one velocity whose numerical value is defined in the SI,

\[
v = a_v c,
\]

where \( a_v \) is a dimensionless experimental result \([12]\).

Thus,

\[
h = m(\text{IPK}) \frac{c^2}{f_if_j T^2} (a_m a_g a_v C_{cl}^{-1}).
\]

One could, at least in principle, measure the combination \( f_if_j T^2 \) without recourse to the SI and this, multiplied by the last term in parentheses, would constitute the dimensionless quantity \( R_h \). In practice, it is more convenient for all frequency and time measurements to be traceable to the SI definition of the second.

\((c)\) Analysis of \( R_N \)

From equation (4.2a),

\[
N_A = \frac{Q_N}{m(\text{IPK})} = \frac{M(^{12}\text{C})}{m(\text{IPK})} R_N.
\]

From Becker & Bettin \([11]\),

\[
N_A = \frac{nM(^{28}\text{Si})}{\rho a^3},
\]

where \( n = 8 \), the number of atoms in a unit cell of crystalline silicon; \( M(^{28}\text{Si}) \) is the molar mass of silicon-28; \( \rho \) is the density of a perfect crystal of silicon-28; and \( a^3 \) is the volume of a unit cell of this crystal at the reference values of temperature and pressure for which \( \rho \) is known.

The density, \( \rho \), is determined from the ratio \( m_{\text{sph}}/V \), where \( m_{\text{sph}} \) is the mass of a particular crystal used in the determination (which is of order 1 kg) and \( V \) is the volume of the same crystal (measured at the same reference conditions as \( a^3 \)).

As in the case of the watt balance, we can write

\[
a_{\text{sph}} = \frac{m_{\text{sph}}}{m(\text{IPK})}.
\]

Additionally,

\[
a_{M\text{Si}} = \frac{M(^{28}\text{Si})}{M(^{12}\text{C})} = \frac{m(^{28}\text{Si})}{m(^{12}\text{C})}.
\]

The last ratio is identical to 1/12 the relative atomic mass of \(^{28}\text{Si} \), \( A_r(^{28}\text{Si})/12 \). This is known to high very accuracy from measurements with Penning traps \([9]\).

Thus,

\[
N_A = \frac{M(^{12}\text{C})}{m(\text{IPK})} \left( \frac{V}{a^3} \right) \left( \frac{na_{M\text{Si}}}{a_{\text{sph}}} \right).
\]

One could, in principle, determine the ratio of volumes independently of the SI. The last two terms of equation (A11) taken as a product constitute the dimensionless quantity \( R_N \).
Appendix B

(a) Consequences to tests of $E = mc^2$ of fixing the value of $\{h\}$

An experiment was reported in 2005 that has been interpreted either as the most accurate experimental test to date of the Einstein relation, $E = mc^2$ [29], or as a consistency check [30] of two different experimental routes to the same energy value. It is interesting to see that the proposed changes to the SI would have no effect on the interpretation of these results, even if they are significantly improved by future measurements.

The experiment compares a subatomic rest mass, $m(X)$, of an entity, $X$, with the electromagnetic frequency, $f$, which results from complete conversion of $m(X)$ to electromagnetic radiation (in general, $f$ represents a sum of frequencies and $X$ represents a linear combination of masses). Thus, the experiment is described schematically by

$$hf = m(X)c^2. \quad (B1)$$

Since the rest mass is determined in this experiment from precisely known relative atomic masses [9], it is easily shown that

$$f = \left(\frac{m(X)}{m(^{12}\text{C})}\right)c^2\left(\frac{M(^{12}\text{C})}{N_Nh}\right), \quad (B2)$$

where the first term on the right-hand side is 1/12 the relative atomic mass of $X$. In the notation of CODATA, the first term would be written as $A_r(X)/12$, where $A_r(X)$ is the relative atomic mass of $X$, known from measurements with Penning traps and $M(^{12}\text{C})$ would be replaced by $12M_n$. The observables are essentially $f$ and $A_r(X)$. Present uncertainties in this experiment are limited to parts in $10^7$ by the determination $f$ as derived from wavelength measurements [30].

The final term on the right-hand side is known to very high accuracy via equation (4.6), as discussed in §4. The uncertainty of this term will not be affected by the proposed evolution of the SI: at present $M(^{12}\text{C})$ has a fixed numerical value and $N_Nh$ has a small experimental uncertainty. In the proposed SI, the numerical value of $N_N$ will be fixed in order to redefine the mole. Therefore, the product $N_Nh$ will have a fixed numerical value and its previous, small uncertainty will transfer to $M(^{12}\text{C})$ (see equation (4.6)). Thus, the proposed changes to the SI are not an issue for this experiment or any other current experiment that links a frequency to an atomic or subatomic rest mass.

References


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The role of the international prototype


