Gravitation governs the expansion and fate of the universe, and the growth of large-scale structure within it, but has not been tested in detail on these cosmic scales. The observed acceleration of the expansion may provide signs of gravitational laws beyond general relativity (GR). Since the form of any such extension is not clear, from either theory or data, we adopt a model-independent approach to parametrizing deviations to the Einstein framework. We explore the phase space dynamics of two key post-GR functions and derive a classification scheme, and an absolute criterion on accuracy necessary for distinguishing classes of gravity models. Future surveys will be able to constrain the post-GR functions' amplitudes and forms to the required precision, and hence reveal new aspects of gravitation.
within this Meeting Issue for some possible guiding principles). Instead, we take
a phenomenological, or (ideally) model-independent, approach in analogy to the
manner in which compact source gravity uses the parametrized post-Newtonian
formalism. Such model independence may put us in good stead to catch signs
revealing the underlying nature of surprising observations.

Building on a robust framework of the conservation and continuity equations,
we parametrize functions from the equations of motion most closely tied to photon
and matter density perturbation observables. This is equivalent to starting from
the metric potentials themselves, although not directly from an action. In §2,
we describe the parametrization, and consistency structure of the system of
equations, in more detail. Discussing the types and reach of data enabled by
future surveys such as BigBOSS, Euclid and WFIRST, in §3 we explore the
degree of constraints that may be placed on these extended gravity, or post-
GR, quantities in comparison with our knowledge today. We draw connections
between this model-independent approach and representatives of strong coupling
and dimensional reduction classes of gravity (\( f(R) \) and DGP, respectively) in §4,
as well as exploring a different tack to distinguishing between gravity models
through a phase space analysis similar to that used for dark energy.

2. Framing gravity

In the equations of motion for cosmological perturbations in a homogeneous,
isotropic background, four quantities enter: the time–time and space–space metric
potentials (equal to each other within GR), the mass density perturbation field
and the velocity perturbation field (taking a perfect fluid, e.g. ignoring pressure
and anisotropic stress). Conservation of stress-energy gives the continuity
equation relating the density and velocity fields, and the Euler equation
relating the velocity and time–time metric potential; so we are left with
two free connecting equations. These can be chosen, for example, to be the
gravitational slip between the two metric potentials and a modified Poisson
equation relating the space–space potential to the density field, or they could
be two Poisson-like equations relating one potential to the density field and
the sum of the potentials to the density. The latter choice turns out to
give greater complementarity between the parametrized functions, with one
closely tied to the growth of matter structures and the other closely related to
photon perturbations such as gravitational lensing deflection and the integrated
Sachs–Wolfe (ISW) effect.

Table 1 shows several different forms of model-independent parametrization,
and how they translate one into the other, adapted and extended from
Daniel et al. [2].

Recently, several groups [1,2,7,9] have advocated the ‘light/growth’ forms we
will use here, owing to their close relations with observables and their near
orthogonality. The defining equations are
\[
-k^2(\phi + \psi) = 8\pi G_N a^2 \bar{\rho}_m \Delta_m \times \mathcal{G} \tag{2.1}
\]
and
\[
-k^2 \psi = 4\pi G_N a^2 \bar{\rho}_m \Delta_m \times \mathcal{V}, \tag{2.2}
\]

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Table 1. Translation between several different parametrizations of extended gravity and the light/growth functions $G$ and $V$.

<table>
<thead>
<tr>
<th>functions $\mathcal{G}$, $\mathcal{V}$</th>
<th>parametrization</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 2\mathcal{G} - \mathcal{V}$, $\sigma = \frac{2\mathcal{V} - 2\mathcal{G}}{2\mathcal{G} - \mathcal{V}}$</td>
<td>bins in $k$, $z$</td>
<td>[1]</td>
</tr>
<tr>
<td>$\mu, \sigma \sim a^s$ or bins in $z$</td>
<td>see translation table in Daniel et al. [2]</td>
<td></td>
</tr>
<tr>
<td>PPF: $f_G = \mathcal{G}^{-1} - 1$, $g = \frac{2\mathcal{G}}{2\mathcal{G} - \mathcal{V}}$</td>
<td>model-dependent</td>
<td>[3,4]</td>
</tr>
<tr>
<td>MGCAMB: $\gamma = \frac{2\mathcal{G}}{\mathcal{V}} - 1$, $\mu = \mathcal{V}$</td>
<td>$\gamma, \mu = \frac{1 + \beta_{\gamma,\mu} \kappa_{\gamma,\mu} k^2 a^s}{1 + \lambda_{\gamma,\mu} k^2 a^s}$</td>
<td>[5,6]</td>
</tr>
<tr>
<td>$\Sigma = \mathcal{G}$, $\mu = \mathcal{V}$</td>
<td>$\Sigma = 1 + \Sigma_s a^s, \mu = 1 + \mu_s a^s$</td>
<td>[7,8]</td>
</tr>
<tr>
<td>PCA</td>
<td></td>
<td>[9]</td>
</tr>
</tbody>
</table>

where $\psi$ is the time–time metric potential, $\phi$ is the space–space metric potential (in conformal Newtonian gauge), $G_N$ is Newton’s constant, $\bar{\rho}_m$ is the homogeneous matter density, $\Delta_m$ the perturbed matter density (gauge invariant), $k$ is the wavenumber and $a$ is the scale factor. The functions $G(k, a)$ and $V(k, a)$ generically are length-scale-dependent and time-dependent. Arising from the sum of potentials, $G$ (meant to evoke an effective Newton’s constant) predominantly governs photon perturbations through light deflection and the ISW effect. Coming from the velocity equation, $V$ predominantly governs growth and motion of structure. Thus, they probe reasonably distinct areas of extension to standard gravity.

These functions also map onto cosmological observations in somewhat orthogonal ways. Cosmic microwave background (CMB) data are mostly sensitive to the ISW effect and hence $G$ (as we want the gravitational modifications to be responsible for current acceleration, we assume at high redshift, e.g. at CMB last scattering, the theory acts like GR without modifications). Weak gravitational lensing involves $G$ through the light deflection law, and $V$ to some extent through the growth of structure. Galaxy distributions and motions are most sensitive to $V$. Note that the phenomenological gravitational growth index parameter $\gamma$ of Linder [10] as well as Linder & Cahn [11] is directly related to $V$ [2].

Cosmological data, now or in the near future, will not have sufficient leverage to reconstruct the general functions $G(k, a)$ and $V(k, a)$. Just as with the dark energy equation of state $w(a)$ describing the cosmic expansion, one can only constrain a very limited number of parameters describing the functions. Some assume a particular time dependence, and possibly neglect scale dependence [8]. More generally, one can use principal component analysis to determine the best constrained eigenmodes of the functions [12]. Here, we will use a similar but simpler approach of dividing the functions into bins of redshift and wavenumber, as this provides a more direct physical interpretation of the results: gravity is modified in a certain way at low/high redshift and smaller/larger scales. We find that two bins in redshift and two in wavenumber, for each of the two post-GR functions (we call this the $2 \times 2 \times 2$ gravity model), are the extent of the leverage that next-generation surveys will provide; so more complex parametrizations are not useful.
3. Constraining gravity

To effectively constrain the post-GR parameters of $G(k_i, z_i)$ and $V(k_i, z_i)$, where $i = 1, 2$ represent the two bins, we need observational data that are sensitive to both the effects on the photons (for $G$) and the matter growth (for $V$). Distance measures such as type Ia supernova distances or baryon acoustic oscillation scales are useful for determining background quantities such as the matter density $\Omega_m$ that might have covariance with the post-GR parameters.

For the current state-of-the-art data, we can consider CMB photon perturbation spectra (WMAP [13]), supernova distances [14], galaxy clustering [15], weak gravitational lensing (CFHTLS [16], COSMOS [17]) and CMB temperature–galaxy cross correlation [18,19]. The results, discussed in detail in [1,2] (also see [20], and [21,22] for DGP constraints, Lombriser et al. [23] for $f(R)$ constraints), show that while $G$ is currently bounded to lie within 10–20% of the GR value for each of the four combinations of low/high wavenumber and low/high redshift, $V$ is only weakly limited to within ±1 of the GR value. This indicates that current growth probes do not have sufficient leverage to look for deviations from GR on cosmic scales. Moreover, the two weak lensing datasets do not agree with each other, with COSMOS showing consistency with GR while CFHTLS gives up to 99% CL deviations at high wavenumbers and low redshift. This may be owing to difficulties in extracting accurate weak lensing shear measurements on small angular scales where the density field is more nonlinear.

To place significant limits on $V$ and growth, future datasets including galaxy clustering and weak lensing surveys covering much more sky area, accurately, and ideally to greater depth are required. (Another, nearer term probe will be measurement of the CMB lensing deflection field.) Surveys such as BigBOSS [24,25], Euclid [26,27], KDUST [28], LSST [29] and WFIRST [30] offer great potential gains. Clear understanding of redshift space distortions would enable probes of the matter velocity field in addition to the density field and serve as a method to measure the growth rate (see below), giving further windows on extensions to gravitation theory.

Figure 1 illustrates one example of future leverage possible on the $2 \times 2 \times 2$ post-GR model-independent parametrization testing gravitation on cosmic scales. The constraints on the growth and $V$ tighten to the few–10% precision level, giving tests of eight different post-GR variables (and all their cross correlations) to better than 10 per cent. That could deliver strong guidance on the nature of cosmic gravity.

To compactify all the information on testing gravity into a single variable (which is not always desirable), we can also examine the gravitational growth index parametrization approach and the constraints that future data will be able to place on $\gamma$. Remember that this is only a partial characterization of extensions to gravity, but can serve as an alert to deviations from GR (or to matter coupling) if the derived value of $\gamma$ shows time or scale dependence or is inconsistent with $\gamma_{GR} \approx 0.55$.

One promising method to measure $\gamma$ is redshift space distortions in the galaxy power spectrum [31,32]. This depends on the growth rate $f \equiv d \ln \Delta_m/d \ln a \approx \Omega_m(a)^\gamma$, as well as the growth itself $\Delta_m(a)$ and the galaxy bias $b(a)$. One needs strong knowledge of the bias and modelling of the redshift distortion form (beyond linear theory), or excellent data (clear angular dependence maps or higher order
correlations) to separate out $\gamma$ without assuming a form for the bias. If the shape of the bias is fixed, keeping the amplitude as a fit parameter, then next-generation galaxy surveys such as BigBOSS, Euclid or WFIRST can measure $\gamma$ to approximately 7 per cent, simultaneously with fitting the expansion history and neutrino mass effects on growth [25].

Another probe is weak gravitational lensing, either by itself (in which case the mass is measured and galaxy bias does not enter) or in cross correlation with the galaxy density field (where one can form ratios of observables to separate out the galaxy bias). Recall that extensions to gravity act on weak lensing in multiple ways: the growth of the matter power spectrum alters (which can be phrased in terms of $\gamma$), but also the light deflection law changes, involving the post-GR parameter $\mathcal{G}$ separate from $\gamma$. This must also be included in the fit, except that many classes of extended gravity (such as DGP and $f(R)$ gravity) actually have $\mathcal{G} = 1$ on cosmological scales. One other subtlety is that the relation of the matter power spectrum to the photon temperature power spectrum changes with the altered growth, modifying the mapping between the primordial photon
Figure 2. Weak gravitational lensing probes both the growth history and expansion history of the universe, so failure to account for possible extended gravity effects on the growth overestimates the tightness of constraints on the expansion history parameters $w_0$ and $w_a$. Determination of $w_0$ and $w_a$ simultaneously with fitting for the growth index $\gamma$ is nearly independent of the value of $\gamma$, here shown with typical fiducial values for GR (0.55, black line and dotted line), DGP gravity (0.68, dashed line) and $f(R)$ gravity (0.42, dashed-dotted line). The 68% CL constraints on $\gamma$ are about 0.11 for the case shown of space-based weak gravitational lensing alone over $4000\,\text{deg}^2$. For the impact of other growth effects (e.g. neutrino mass, spatial curvature) on weak lensing, see Das et al. [33]. (Online version in colour.)

perturbation amplitude $A_s$ and the present mass amplitude $\sigma_8$. When both weak lensing and cosmic microwave background data are included in the constraints, the treatment of $A_s$ and $\sigma_8$ must be made consistent.

Figure 2 illustrates the effects of fitting the gravitational growth index $\gamma$ simultaneously with determining the effective dark energy equation of state $w(a) = w_0 + w_a(1 - a)$, i.e. the expansion history. The area figure of merit in the $w_0$–$w_a$ plane decreases by 45 per cent, but as Huterer & Linder [34] pointed out the consequences of neglecting to fit for the growth index are worse. For an assumption mistaken by $\Delta \gamma$ and weak lensing alone as a probe the derived value of $w_a$ would be biased as $\Delta w_a \approx 8\Delta \gamma$. When fitting for $\gamma$ simultaneously, however, the estimations of $w_0$ and $w_a$ do not depend strongly on the fiducial $\gamma$. This is a reflection of $\gamma$ being defined specifically to separate the expansion history influence on growth from any ‘beyond general relativity’ effects [10], making it a key method to test gravity.

4. Paths of gravity

While the model-independent approach allows exploration without assuming a particular theory, it is of interest as well to consider some specific models and their mapping into the post-GR parameters we have described. One can talk about
three broad classes of extensions to gravity in terms of the physics restoring them to GR in solar system conditions, as observations require ([35], also see [36–38]): dimensional reduction where below a Vainshtein radius the theory acts like GR (e.g. DGP or cascading gravity), strong coupling where extra degrees of freedom freeze out through gaining a large mass in a chameleon mechanism (e.g. \( f(R) \) or scalar–tensor gravity), or screening where the extra degrees of freedom decouple and vanish through symmetry restoration (symmetron gravity). On cosmic scales, the first and third classes behave similarly; so we examine two representative cases: DGP gravity and \( f(R) \) gravity.

In both of these cases, the light variable \( G \) is simply equal to unity, the GR value. However, the growth variable \( V \) is affected. The expressions become

\[
V_{\text{DGP}} = \frac{2 + 4\Omega_m^2(a)}{3 + 3\Omega_m^2(a)} \tag{4.1}
\]

and

\[
V_{f(R)} = \frac{3 + 4\kappa^2(k, a)}{3 + 3\kappa^2(k, a)}, \tag{4.2}
\]

where \( \Omega_m(a) = 8\pi G_N\bar{\rho}_m(a)/[3H^2(a)] \) is the dimensionless matter density, \( H = \dot{a}/a \) is the Hubble expansion rate and \( \kappa = k/[aM(a)] \), where \( M(a) \approx (3d^2f/dR^2)^{-1/2} \) is the effective scalar field mass. Note that DGP gravity does not have scale dependence on cosmic scales above the Vainshtein scale; gravity is scale-free (e.g. the force is a power law with distance) on both the large scale (five-dimensional gravity) and small-scale (four-dimensional gravity) limits and only the Vainshtein scale defined from the five-dimensional to four-dimensional crossover breaks this. On the other hand, \( f(R) \) gravity has both scale and time dependence, though tied together in a specific manner.

Recall that the gravitational growth index \( \gamma \) is related to \( V \). For DGP gravity, \( \gamma = 0.68 \) \([10,11,39]\) is an excellent approximation to use for calculating the matter density linear growth factor as a function of redshift, good to 0.2 per cent. A mild time dependence can be incorporated into \( \gamma \) (though this is not necessary) through eqn (27) of Linder & Cahn \([11]\). For \( f(R) \) gravity, \( \gamma \) is not generally as well approximated by a constant in time, and has non-negligible scale dependence at redshifts \( z \approx 1–3 \) \([40–42]\) although the details depend on the specific \( f(R) \) model and parameters.

From equations (4.1) and (4.2), we can create phase plane diagrams of the evolution of the post-GR function \( V \). This is analogous to the dark energy phase plane \( w–w' \) for the dark energy equation of state and its time variation, where prime denotes \( d/d\ln a \). For dark energy, such diagrams led to clear distinction of certain physical classes \([43]\) as well as calibration of the compact and accurate parametrization \( w(a) = w_0 + w_a(1 - a) \) \([44]\). (Also see Song \textit{et al.} \([7]\) for comparison of \( G \) and \( V \) at a fixed time in extended gravity versus interacting dark energy.)

\textbf{Figure 3} shows the results in the \( V–V' \) plane for DGP and \( f(R) \) gravity. For DGP gravity, the equation for the phase space trajectory is

\[
V' = \frac{-4\Omega_m^2(1 - \Omega_m)}{(1 + \Omega_m)(1 + \Omega_m^2)^2} \tag{4.3}
\]
Figure 3. The phase space trajectories of DGP gravity and $f(R)$ gravity resemble thawing dark energy, except here the theories move away from GR in the post-GR growth parameter plane, instead of thawing from a cosmological constant. Unlike canonical dark energy, the theories thaw in opposite directions: DGP moves to weaker gravity than GR while $f(R)$ moves to stronger gravity. The phase space location today for each of the theories is shown by the squares (for $\Omega_m = 0.27$).

(Online version in colour.)

\[
V' = -18(1 - V) \left( V - \frac{2}{3} \right) \left[ 1 + \sqrt{\frac{3V - 2}{4 - 3V}} \right]^2, \tag{4.4}
\]

where we used the relations

\[
\Omega'_m = 3w\Omega_m(1 - \Omega_m) = -3\Omega_m \frac{1 - \Omega_m}{1 + \Omega_m}, \tag{4.5}
\]

and $\Omega_m$ is the time-dependent matter density.

For $f(R)$ gravity, the phase space trajectory is

\[
V' = \frac{2\kappa k'}{3(1 + \kappa^2)^2} \rightarrow \frac{2(s - 1)\kappa^2}{3(1 + \kappa^2)^2} = -6(s - 1)(V - 1) \left( V - \frac{4}{3} \right), \tag{4.7}
\]

where in the second line we parametrize $M(a) = M_0 a^{-s}$. If we wanted to look at the phase space evolution with respect to inverse length $k$ rather than time $a$, then the equation still holds, with $s = 2$.

Both classes of gravity theory act as thawing cases in the nomenclature of dark energy phase space: they are frozen in the GR state $(V, V') = (1, 0)$ in the past, then as the Hubble parameter or Ricci scalar curvature drop from the cosmic expansion the theory thaws and moves away from GR. The theories eventually
freeze to an asymptotic attractor with $\mathcal{V}' = 0$ in the future, with $\mathcal{V} = 2/3$ in the case of DGP gravity (weaker gravity) and $\mathcal{V} = 4/3$ in the case of $f(R)$ gravity (stronger gravity). Note that interpreting $f(R)$ as a scalar–tensor theory agrees with the expectation that gravity should strengthen, as forces carried by a scalar field are attractive. The clear separation of phase space territory for the two classes, as for thawing and freezing fields of dark energy, shows how observations could distinguish the nature of gravity. This also defines a science requirement that $\mathcal{V}$ should be measurable to an accuracy better than 0.1 for a $3\sigma$ distinction of gravity theories.

However, there is one further point regarding $f(R)$. While the form $M(a) \sim a^{-s}$ is a reasonable description for the past behaviour of the scalar field mass [5,6,42], in the future we expect $M$ to freeze to a constant (e.g. as $R$ itself does when the theory goes to the de Sitter attractor state; thanks to Stephen Appleby for pointing this out). If instead we parametrize the scalar field mass as $M(a) = M_1 a^{-s} + M_\ast$ (accurate to $\sim 1\%$ for at least some models), then the phase space trajectory does not asymptote to $(4/3, 0)$ but rather returns to the GR limit of $(1, 0)$ as $a \gg 1$ and $\kappa = k/(aM) \to 0$ in the future. Figure 4 illustrates this behaviour. As $M_\ast$ increases (dotted curve) or for wavenumbers near the mass scale $M$ (dotted-dashed curve), the cosmic version of the chameleon mechanism begins to operate and the trajectory heads back towards GR. Also note that because $M$ is no longer a power law in $a$, different wavenumbers $k$ do not simply correspond to a rescaling of $a$ and so the single $f(R)$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{In the $f(R)$ case with a de Sitter future the scalar field mass $M$ freezes at a non-zero constant, causing the theory to restore GR on cosmic scales just like with the chameleon mechanism on small scales. The heavy black curve recreates the $M_\ast = 0$, vanishing mass case of figure 3 that freezes at $\mathcal{V} = 4/3$; however the finite mass cases return to $\mathcal{V} = 1$. While the wavenumber dependence of the post-GR parameter at $z = 1$ (squares) is appreciable, by $z = 0$ the $k$-dependence is negligible (stars). Since the gravitational growth index $\gamma$ is directly related to $\mathcal{V}$, this explains the scale independence of $\gamma$ at the present in such $f(R)$ models. (Online version in colour.)}
\end{figure}
trajectory in figure 3 breaks up into varied paths for different $k$. The evolution along a path varies as well. This can be seen by the different values of $\mathcal{V}$ at $a = 0.5$ for $k/M_1 = 10$ (blue square near the peak of the figure) versus $k/M_1 = 100$ (red square near the right side). However, by the present (and for several e-folds to the future) this $k$ dependence has vanished, with $\mathcal{V}(a = 1)$ shown with stars for the two cases agreeing to within 0.5 per cent. This holds for all $k/M_1 \gg 1$ (note that the scalar mass $M_1$ is likely to be of the order of the Hubble constant for observationally allowed models). Only for $k/M_1 \approx 1$ does scale dependence today enter (and the trajectory as a whole deviate). As the gravitational growth index $\gamma$ is directly related to $\mathcal{V}$, this explains the scale independence of $\gamma$ in such $f(R)$ models at the present $[40,42]$.

5. Conclusions

Cosmic acceleration may be a sign that gravitation deviates from GR on large scales, pointing the way to a deeper theory of gravity and perhaps the nature of space–time and fundamental physics. Even apart from this, cosmological observations now have the capability to test gravity on scales barely probed, and we should certainly do so.

In the absence of a compelling specific model, and to remain receptive to surprises in how gravity behaves, a model-independent approach in terms of parametrizing the relations between the metric potentials and matter density and velocity observables has advantages. We have explored here the ‘$2 \times 2 \times 2$’ approach parametrizing light/growth functions $G$, $V$ motivated by observable effects on photon and matter perturbations. These are divided into bins sensitive to both scale and time dependence.

We find strong complementarity between $G$ and $V$, with each probing specific aspects of extensions to gravity, and both capable of being constrained by a variety of cosmological methods. Current surveys are making some inroads on determining $G$, but constraining $V$ requires future large surveys such as BigBOSS, Euclid or WFIRST. Cross correlations between different probes will be valuable as well, and geometric measures such as supernova and BAO distances will be essential to fit simultaneously the cosmic expansion history. Conversely, fitting extended gravity growth parameters such as $\gamma$ when using probes such as weak lensing to measure the equation of state is necessary to avoid bias.

Viewing the post-GR parameters in a dynamical phase plane yields insights similar to its use for dark energy. Models such as DGP and $f(R)$ gravity act as thawing fields, although evolving in opposite directions away from GR. Phase diagrams also can illustrate the scale dependence of $\gamma$ at various epochs, and most importantly deliver a science requirement on distinguishing classes of gravity: future surveys should aim to determine $V$ to better than 0.1. Planned next-generation experiments such as BigBOSS, Euclid and WFIRST can indeed potentially reach this level, with the caveat that expansion history should be tested as well, such as through supernova distances immune to gravitational modifications.

Gravity can and should be tested on all scales, on laboratory, solar system, compact object, galactic, cluster, cosmological and horizon scales. The approach discussed here is designed for model independence on cosmological scales,
smaller than the horizon. On horizon scales, the light/growth functions become more complicated or insufficient as other terms in the equations become important [3, 6, 45].

Another model-independent approach is to check consistency relations within general relativity [46–48]. These can provide an alert to deviations, and then one must adopt specific models or more detailed parametrizations such as discussed here to characterize the physics.

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