

Far-field theory of wave power capture by oscillating systems

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A new derivation is given of an equation, relating the capture width of a wave power converter to the polar diagram of the waves generated by the device. The pattern of waves in the lee of the device is calculated in detail.

Keywords: wave power converter; capture width; wave generation; angular distribution; polar diagram

1. Introduction

This study concerns the capture of power from monochromatic plane waves in the sea by any system of bodies of finite extent, oscillating at the wave frequency, referred to as a wave energy converter (WEC). It is assumed that linear superposition applies and second-order effects can be neglected. The amount of power captured is conveniently expressed by the ‘capture width’ (CW), which is defined as the width of the incoming wavefront that has the same power as that being absorbed by the device. The CW can be larger than the size of the WEC. This can be achieved if the WEC generates waves which interfere destructively with the surrounding sea waves, so less power is left in the sea and the difference is transferred to the WEC. Here, we explore this interference mechanism in detail.

The CW of a WEC is often obtained from the forces acting upon it, the motions induced in the device and how power is transferred to some internal damping mechanism. Here, we use an alternative approach. The CW is specified by the energy balance in the sea far away from the device (the ‘far field’), which is entirely determined by the waves generated by the WEC. The WEC produces waves by its unmoving presence, its motions and through the forces it exerts on the sea, variously referred to as scattering, diffraction, radiation, etc. But for this study we need not distinguish between these processes, nor understand the mechanisms by which they are produced. The detailed motion of the WEC and the forces driving it do not enter into the calculation. At a distance, the waves emitted by the various parts of the WEC by its various processes all combine

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One contribution of 18 to a Theo Murphy Meeting Issue ‘The peaks and troughs of wave energy: the dreams and the reality’.

into a single wave, propagating away from the WEC, which we will call the 'generated wave'. The amplitude and phase of the generated wave may vary with direction. It will be shown that the CW is entirely specified by this total combined generated wave.

The generated wave can interfere coherently with the original sea wave, reducing its amplitude and energy. It can only do this over a useful area if the two waves have the same frequency and are travelling in the same direction. The waves emitted by the WEC inevitably propagate away from it. Therefore, they can only interfere usefully with some part of the original wave which is also propagating away from the WEC. Paradoxically, from this point of view, it seems that the WEC extracts power from sea waves which are travelling away from it! In reality, the incoming wave interacts locally with the WEC, losing energy as it passes and continues to the far field with reduced amplitude. Because of linear superposition, the ongoing amplitude can be obtained from the vector sum of the original wave and the total wave amplitude generated by the WEC.

2. Wave energy converters in the open sea

Let us first consider a WEC, comprising one or more small bodies, close together in the open sea. The far field is dominated by the incident wave that has travelled past the system with little change in amplitude. The WEC generates waves of the same frequency as the incident wave and they interfere with the original wave stream. Of crucial importance is the interference between the generated wave and the ongoing original wave, both in the forward direction (the direction the incident waves are travelling). As they are both propagating in the same direction, the phase relationship once established continues indefinitely, giving rise to cancellation and a continuing change in the average power level in the lee of the device. This turns out to be independent of the distance from the WEC.

Depending upon the phase of the generated wave, the resultant wave propagating forwards can have less power than the incident wave, the difference being captured by the WEC. The power captured is equal to the power subtracted from the incident waves by interference, minus the power spreading out in all directions. This leads to a relation between the CW and the angular distribution of the generated waves, which has been derived in various ways by Newman [1–3], Mei [3,4] and others [5–9]. But the wave amplitudes used in these formulae are not all the same and there has been disagreement as to whether it is the forward emission (in the direction the incident waves are travelling) or the backward emission (in the opposite direction) that determines the CW. Rainey [10,11 (eqn (2))] has emphasized that the forward emission is paramount in subtracting energy from the incident waves. The aim of this study is to clarify this issue.

In this study, I derive an alternative, and more general, variant of the Newman–Mei equation in a new, hopefully more transparent, way and show that for machines in the open sea it is the wave emitted forwards by the WEC that determines the amount of power captured. Waves emitted in the backward direction combine with the incoming wave to give standing waves; in some locations they reinforce, in others they cancel: the average power level in this region is unchanged, so the wave emitted backwards cannot influence the CW.

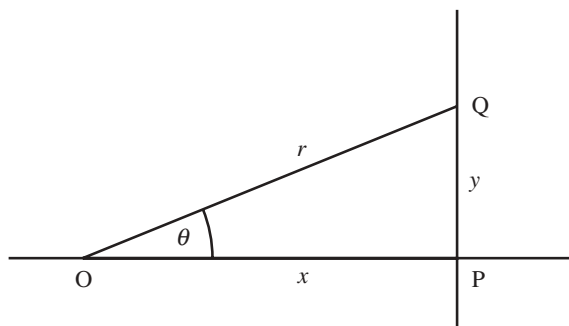


Figure 1. The wave comes from the left, the generator is at O, calculate the net wave at Q.

3. Forward interference pattern

Figure 1 shows the situation to be considered with the variables to be used. Assume that linear monochromatic waves travelling in the direction OP of wavelength λ are incident from the left on a wave generator at the same frequency, located at O. We calculate the wave power at any point Q, y being the distance PQ along a line transverse to the wave direction. r is the diagonal distance from the generator at O to the point Q. The amplitude of the original wave reaching the line PQ is taken as 1. For simplicity, the corresponding wave power per metre width of wavefront is taken as 1 power unit.

Assume that the generator at O generates a wave of amplitude A at distance r given by

$$A = -a\sqrt{\frac{\lambda}{r}} e^{ikr} e^{i\beta} f(\theta). \quad (3.1)$$

The angular function $f(\theta)$ gives the polar diagram of the generator relative to the propagation direction of the incident plane waves, taken as $\theta = 0$ (the forward direction). In general, this angular function $f(\theta)$ is a complex number specifying the polar diagram in amplitude and phase, but without loss of generality it is convenient initially to assume that $f(0)$ is real and add β to give the phase of the radiation at point P relative to the phase of the original wave. This allows us to adjust the phase to optimize the absorbed power. a is an arbitrary real positive numerical factor which will be used to adjust the amplitude of the generated wave. With the negative sign in (3.1), the generated wave will tend to cancel the original wave at P if $\beta = 0$. But we shall see below that this is not the optimum phase for overall power absorption. Essentially, this is because a circular wave spreading out from the WEC is interfering with the original linear wave; on average, the circular wave has to travel slightly further before it overlaps with the original wave.

To calculate the resultant wave at Q, we need the extra phase delay ϕ owing to the distance r being larger than the distance x . Using Pythagoras in the triangle OPQ

$$\phi = k(r - x) = \frac{ky^2}{2x}, \quad (3.2)$$

in which $k = \lambda/2\pi$ is the wavenumber. If the line PQ is far away, then only waves emitted at small angles θ will be relevant, so the angular function required is $f(0)$. The real part of the generated wave at Q is

$$-a\sqrt{\frac{\lambda}{r}} \cos(\phi + \beta)f(0). \quad (3.3)$$

Add this to the original wave of amplitude 1 and square it to get the net wave power per metre at Q, which is

$$J = 1 - 2af(0)\sqrt{\frac{\lambda}{x}} \cos(\phi + \beta), \quad (3.4)$$

in our power units. Here, we are approximating $r \approx x$ as far as the wave amplitude is concerned. We also neglect terms in λ/x which become negligible at large x when compared with the leading term in $\sqrt{\lambda/x}$. We are only interested in the ongoing wave power at very large x .

The parameter β , specifying the phase of the generator relative to the incident wave, is the same for all points Q, but ϕ changes with the distance $y = PQ$. Integrating over all y the total power subtracted from the original wave is

$$J_- = 2af(0)\sqrt{\frac{\lambda}{x}} \left[\cos(\beta) \int_{-\infty}^{\infty} \cos\left(\frac{ky^2}{2x}\right) dy - \sin(\beta) \int_{-\infty}^{\infty} \sin\left(\frac{ky^2}{2x}\right) dy \right]. \quad (3.5)$$

The integrals are Fresnel integrals of the form

$$\int_{-\infty}^{\infty} \cos\left(\frac{\pi v^2}{2}\right) dv = \int_{-\infty}^{\infty} \sin\left(\frac{\pi v^2}{2}\right) dv = 1. \quad (3.6)$$

Set $v = y\sqrt{k/\pi x}$ to find

$$\int_{-\infty}^{\infty} \cos\left(\frac{ky^2}{2x}\right) dy = \int_{-\infty}^{\infty} \sin\left(\frac{ky^2}{2x}\right) dy = \sqrt{\frac{\pi x}{k}}. \quad (3.7)$$

Combining (3.5) and (3.7) gives the total power subtracted from the wave

$$J_- = \{\cos(\beta) - \sin(\beta)\} \times \sqrt{2}\lambda f(0). \quad (3.8)$$

The distance x of the line PQ from the generator cancels out, showing that the ongoing power is independent of this distance. J_- is a maximum when $\beta = -\pi/4$ in which case the power subtracted from the wave is

$$J_- = 2a\lambda f(0). \quad (3.9)$$

The amount of power subtracted is independent of the distance x of the line PQ from the generator. As the wave continues on its way, the interference pattern changes in width and amplitude, but the integrated effect remains the same. Figure 2 shows the power in the wave, the integrand of (3.5), as a function of the transverse distance y with $\beta = -\pi/4$ for two values of x . The dashed line shows the original power level in each case. If x is doubled, then the pattern is stretched sideways by the factor $\sqrt{2}$ but its amplitude is reduced by the same factor, so that the area between the curve and the dashed line remains the same.

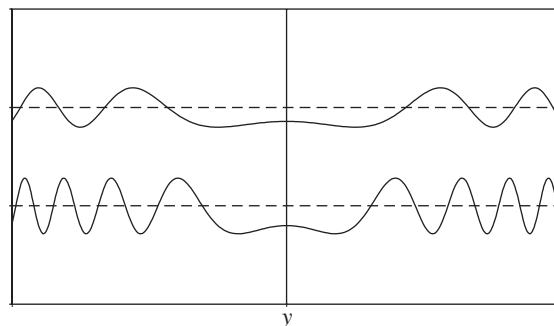


Figure 2. Interference pattern of wave energy versus y at two distances x from the generator with $\beta = -\pi/4$. The integrand in (3.5) gives the wave power behind the wave energy converter (WEC) as a function of the transverse distance y . The lower black curve gives the net power at some arbitrary distance x behind the WEC, whereas the lower dashed line shows the original wave power. There is a decrease in the centre and interference fringes on both sides. If the distance from the WEC is doubled, then we get the upper two lines. The corresponding original wave power (dashed line) is the same as before but both these lines have been displaced upwards for clarity. With x increased by the factor 2, the amplitude of the interference pattern is reduced by the factor $\sqrt{2}$, but it is stretched sideways by the same factor; the total energy subtracted from the wave is unchanged.

This means that with $\beta = -\pi/4$ and $f(0)$ real the generated wave, averaged over the whole ongoing wavefront, is exactly in anti-phase with the ongoing wave and reduces its amplitude. The phase of the generated wave is specified by (3.1) and could well be different. For example, if $\beta = +\pi/4$ (or equivalently if $\beta = -\pi/4$ but $f(0)$ is imaginary) equation (3.8) shows that no power is subtracted from the ongoing wave. In this case, the generated amplitude is, on average, orthogonal to the amplitude of the ongoing wave and alters its phase but does not reduce its amplitude. In the most general case, keeping $\beta = -\pi/4$, $f(0)$ can have real and imaginary components. The real part takes power from the ongoing wave, whereas the imaginary part changes its phase. (It will be realized that the choice of reference phase is entirely arbitrary; if we had chosen to work with $\beta = +\pi/4$ instead of $-\pi/4$, then the role of the real and imaginary components of $f(0)$ would be interchanged.)

Note that the circular wave spreading out from the WEC is interfering with the ongoing linear wave. It appears from the above that, on average, the generated wave has to travel an extra path length of $\lambda/8$, which corresponds to setting $\beta = -\pi/4$ in (3.8).

In figure 2, we see that the net power subtracted from the wave is dominated by the dip in the centre. Further out, the plus/minus wiggles largely cancel and contribute little to the power balance. Note that when the generated phase is optimum, the minimum in the ongoing power is not in the middle, but slightly off-centre. Also note that further off-centre, the ongoing power is actually increased! A small advantage may be gained by locating a second wave energy machine not centrally behind the first but slightly off axis, in a V-formation: reminiscent of the formation adopted by some birds in flight, perhaps for similar reasons.

As the power pattern is a function of $y^2/2x$, the wake field behind the device is parabolic, roughly plotted in figure 3. This agrees with the wake field illustration in Yemm *et al.* [12 (figs 4b and d)].

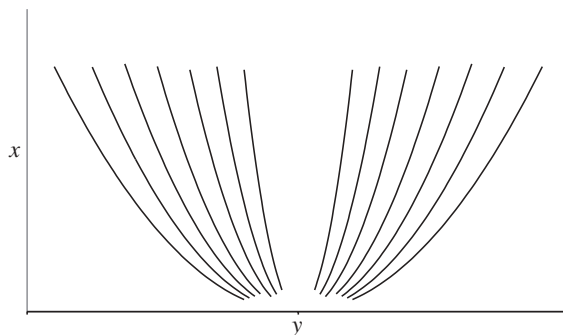


Figure 3. The parabolic wake field pattern behind any wave energy converter.

At large distances, any small possibly complex system of generators operating at the wave frequency must emit a combined wave with determined phase and angular distribution $f(\theta)$, which will have real and imaginary components. So the above and following calculations are valid for any complex system of generators, all oscillating at the wave frequency.

As the distance x from the source increases, the interference pattern expands laterally as \sqrt{x} , so it actually contracts in angle. This means that at any reasonable distance only the forward-generated amplitude $f(0)$ is relevant as assumed above. Also, once the distance from the generators is large compared with their size, the interference pattern, plotted in figures 2 and 3, will be the same for all wave energy devices, however complex, if they have been optimized in phase.

4. Optimum generator amplitude for maximum power capture

In an ideal system, the generator motions, with the ideal phase and amplitude, are excited by the incoming wave and the power subtracted from the incident wave as a result of interference is given by J_- in (3.9). Some of this energy is emitted in other directions and using (3.1) the total energy emitted by the generator is

$$J_{\text{out}} = a^2 \lambda I = a^2 \lambda \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta, \quad (4.1)$$

in which I is the integral defined by this equation. The remaining energy is absorbed by the WEC (presumed to have some internal damping mechanism); so combining (3.9) and (4.1), the energy captured by the WEC is

$$J_c = J_- - J_{\text{out}} = 2a\lambda f(0) - a^2 \lambda I. \quad (4.2)$$

The generated amplitude is controlled by the parameter a which we now vary to maximize the energy captured: this occurs when the first term is twice the second. At the optimum

$$a = \frac{f(0)}{\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta}. \quad (4.3)$$

Substituting in (4.2), the captured energy is

$$J_c = \lambda \frac{|f(0)|^2}{\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta}. \quad (4.4)$$

We have been assuming that a wave of unit amplitude gives unit energy in our units, so this implies that the optimum CW is given by

$$\text{CW} = \lambda \frac{|f(0)|^2}{\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta}. \quad (4.5)$$

This applies when the generated wave has the optimum amplitude and the optimum phase specified by $\beta = -\pi/4$.

We have seen above that the generated wave may also have another component with $f(0)$ imaginary. Equation (3.8) shows that this subtracts no power from the sea. In the most general case, keeping $\beta = -\pi/4$, $f(0)$ can have real and imaginary components. In this case, only the real component should be included in (4.2) which becomes

$$J_c = J_- - J_{\text{out}} = 2a\lambda\Re f(0) - a^2\lambda I. \quad (4.6)$$

On the other hand, both components of $f(0)$ contribute to (4.1) which remains valid. Optimizing as before, in the general case, the CW becomes

$$\text{CW} = \lambda \frac{|\Re f(0)|^2}{\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta}. \quad (4.7)$$

5. Discussion

Whether the real part of $f(0)$ should appear in the numerator of (4.7) or the imaginary part, or perhaps some combination of the two, clearly depends on the definition of $f(\theta)$ in (3.1) and the chosen value of β which is quite arbitrary. Allowing for this, (4.7) agrees exactly with the formulae for CW derived by Farley [7,8].

With our choice of β , only the real part of the total wave generated by the WEC appears in the numerator of (4.7). This is the component that, averaged over the whole wavefront, reduces the amplitude of the ongoing original wave and subtracts power. If the WEC and its components do not move, then no power can be absorbed but the incoming wave will still be scattered or ‘diffracted’. The wave generated in this case is often called the ‘diffracted wave’, for which $f(0)$ is mostly imaginary. It is the motion of the WEC that absorbs energy: it does this by producing another component of the generated wave, often called the ‘radiated wave’, for which $f(0)$ has a significant real component and subtracts power from the ongoing wave. So, on the whole, the radiated and diffracted waves correspond, respectively, to the real and imaginary components of $f(\theta)$ as defined here. But this correlation is not exact because diffraction implies some reduction of the ongoing wave amplitude: for pure diffraction, $f(0)$ must have a real component, usually small, but large in the case of the Salter Duck to be discussed below.

Thus, one can split the generated wave into diffracted and radiated components, but the distinction between the real and imaginary parts of the generated amplitude is more fundamental. The sea is unaware of our distinction

between diffracted and radiated waves; however, it is sensitive to the phase of the total generated wave. Therefore, the phase of $f(0)$ is all important. Only the real part enters into the numerator of (4.7). Using the real part, we exclude the ‘diffracted’ wave. But the total generated amplitude is in the denominator of (4.7).

Formulae for the CW resembling (4.7) have been derived in other studies [1–6,9], but the symbols do not always have the same meaning. In some versions of the formula, the total generated wave is used. Some have $|f(0)|^2$ in the numerator, while we have $|\Re f(0)|^2$. If diffraction is small, this difference is minor.

Mei [13, eqn (9.17)] derives a formula virtually identical to our (4.6) with $f(\theta)$ corresponding to the total generated wave amplitude, and like us he selects the real part of $f(0)$. From this, our (4.7) would follow exactly; but Mei does not appear to take this step, at least in this publication. Rainey [11] also gives a formula similar to our (4.6) again selecting the real part of $f(0)$.

So according to the present analysis and equation (4.7), the CW is completely specified by the angular distribution of the combined total of all waves generated by the system. For a small WEC in the open sea, $|\Re f(0)|^2$ is required in the numerator; the backward-travelling waves specified by $f(\pi)$ cannot interfere usefully with the incident waves which travel in the forward direction.

For a system of generators which emit isotropically, the best obtainable CW is $\lambda/2\pi$. This result was first published, without proof, by Budal & Falnes [14] in 1975. But if the generated polar diagram is strongly peaked forwards, as it is for wave energy machines with forward-travelling oscillations [7,10,12], then the CW can be much higher. With the formula (4.7) one can calculate the capture by all sorts of complex systems including arrays of separate machines. One needs to know only the overall polar function $f(\theta)$.

6. Salter Duck

The Salter Duck merits special discussion because it is often thought that (4.7) cannot apply. The Duck is an asymmetrical cam which oscillates in pitch, more or less flat on the side facing the waves (the back) but round on the forward facing side (the front). As a result it can absorb energy and match the waves arriving at the back, but its motion radiates nothing forwards. Therefore, we are told that $f(0)$ is zero, so (4.7) cannot apply; but this misses the essence of the theory.

Consider an idealized Duck, several wavelengths wide, pitching about an axis somehow fixed in the sea. If the Duck does not move, then it presents a rigid barrier to the waves; most of the power incident on its frontage is reflected and there is a large wave shadow on the forward side. The total power in the sea is unchanged. This is normally understood by saying that the stationary Duck generates a ‘diffracted wave’ which is superposed on the original sea wave, so that the combination satisfies the boundary conditions, zero velocity normal to the Duck surface. The diffracted wave travelling away from the Duck is the same on each side. In the forward direction, it has a large real part that cancels the original wave creating the wave shadow. In the backward direction, there is a strong reflection.

The wave generated by the Duck motion, the ‘radiated wave’, must be added linearly. In the ideal case, the backward-radiated wave cancels the backward part of the diffracted wave, so the total wave generated backwards is zero: there is

no reflection. The Duck motion radiates nothing forwards, so in this direction, the diffracted wave continues unchanged: we have seen that it produces the wave shadow. The total wave generated by the Duck is the sum of the diffracted and the radiated waves. We see that the total wave generated backwards is zero, whereas there is a strong real component generated forwards.

In (4.7), $f(0)$ must include the diffracted and the radiated wave and in this case $\Re f(0)$ is large so that the equation implies a substantial CW. The wider the Duck, the narrower the forward polar diagram and the larger the CW; all as expected. Equation (4.7), correctly understood, applies perfectly to the Duck.

This equation is valid for any reasonably compact WEC or combination of WECs in the open sea, provided $f(\theta)$ is the angular distribution of the total wave generated by the system and its phase and amplitude have been optimized. In most cases, this is achieved by adjusting the resonant frequency and internal damping of the system.

7. Wave energy converters on sea walls

Now consider a situation in which the incident wave is strongly reflected by a sea wall which has WECs mounted upon it [15]. Virtually, no waves continue in the original forward direction. In this case, forward emission by the WEC is irrelevant; the component of the external wave carrying power away from the WEC is in the backward (reflected) direction and it is this part of the sea wave that the wave generated by the WEC can usefully modify by interference. The calculation is the same as the above, but relative now to the backward direction, and the CW will again be given by (4.7) but with $f(0)$ replaced with $f(\pi)$.

8. Other cases

More complex situations can arise if the situation in the sea is more complicated. A sea wall with gaps could reflect half the incident wave power and allow half to be transmitted. If a WEC were located in one of the gaps, then the forward emission would interfere with the transmitted wave, whereas the backward emission would interfere with the reflected wave. So, both $f(0)$ and $f(\pi)$ would be relevant and equations (4.6) and (4.7) should be modified accordingly. Oscillating water columns located on the corners of harbour walls are considered by Mei [15]. If the walls are inclined at 45° to the incident wave direction on each side, then reflected waves would travel away from the WEC at 90° and 270° : in this case, both $f(\pi/2)$ and $f(3\pi/2)$ would become relevant. It appears that while equation (4.7) applies in many cases, there is no general formula that fits all situations.

9. Summary

The CW of a WEC can be obtained from the energy deficit in the far wave field. This is related to the polar diagram of the combined waves generated by the WEC. The same calculation gives the detailed pattern of waves in the lee of the device.

Finally, it should be clear that for a system in the open sea, the generated amplitude in the forward direction enters into the numerator of (4.7) and not the backward amplitude as has been asserted and is still claimed by some protagonists. But for WECs mounted on sea walls, it is the wave emitted backwards that determines the CW. In other cases, emission at other angles may be relevant.

I pay tribute to my pioneering friend Johannes Falnes and thank him for the discussions that have stimulated this paper. I also thank Rod Rainey, Chiang Mei, Nick Newman and David Evans for their many illuminating comments.

References

- 1 Newman, J. N. 1976 The interaction of stationary vessels with regular waves. In *Proc. 11th Symp. on Naval Hydrodynamics, London, UK*, pp. 491–501.
- 2 Newman, J. N. 1979 Absorption of wave energy by elongated bodies. *Appl. Ocean Res.* **1**, 189–196. (doi:10.1016/0141-1187(79)90026-9)
- 3 Mei, C. C. & Newman, J. N. 1979 Wave power extraction by floating bodies. In *Proc. 1st Symp. on Wave Energy Utilisation, 30 October–1 November 1979*. Göteborg, Sweden: Chalmers University of Technology.
- 4 Mei, C. C. 1976 Power extraction from water waves. *J. Ship Res.* **20**, 63–66.
- 5 Evans, D. V. 1979 Some theoretical aspects of three-dimensional wave-energy absorbers. In *Proc. 1st Symp. on Wave Energy Utilisation, 30 October–1 November 1979*. Göteborg, Sweden: Chalmers University of Technology.
- 6 Evans, D. V. 1980 Some analytic results for two and three dimensional wave-energy absorbers. In *Power from sea waves* (ed. B. Count), pp. 213–249. London, UK: Academic Press.
- 7 Farley, F. J. M. 1982 Wave energy conversion by flexible resonant rafts [see appendix]. *Appl. Ocean Res.* **4**, 57–63. (doi:10.1016/S0141-1187(82)80022-9)
- 8 Farley, F. J. M. 2004 Optical theorem in atomic physics and wave power. In *19th Int. Workshop on Water Waves and Floating Bodies, Cortona, Italy, 28–31 March 2004*, paper 14.
- 9 Falnes, J. & Hals, J. 2012 Heaving buoys, point absorbers and arrays. *Phil. Trans. R. Soc. A* **370**, 246–277. (doi:10.1098/rsta.2011.0249)
- 10 Rainey, R. C. T. 2001 The Pelamis wave energy converter: it may be jolly good in practice, but will it work in theory? In *16th Int. Workshop on Water Waves and Floating Bodies, Hiroshima, Japan, 22–25 April 2001*, paper 34.
- 11 Rainey, R. C. T. 2012 Key features of wave energy. *Phil. Trans. R. Soc. A* **370**, 425–438. (doi:10.1098/rsta.2011.0251)
- 12 Yemm, R., Pizer, D., Retzler, C. & Henderson, R. 2012 Pelamis: experience from concept to connection. *Phil. Trans. R. Soc. A* **370**, 365–380. (doi:10.1098/rsta.2011.0312)
- 13 Mei, C. C. 1989 *The applied dynamics of ocean surface waves*, p. 353. Singapore: World Scientific.
- 14 Budal, K. & Falnes, J. 1975 Corrigendum to ‘A resonant point absorber of ocean-wave power’. *Nature* **257**, 626. (doi:10.1038/257626e0)
- 15 Mei, C. C. 2012 Hydrodynamic principles of wave power extraction. *Phil. Trans. R. Soc. A* **370**, 208–234. (doi:10.1098/rsta.2011.0178)