There are many situations arising both in nature and in the world of manufacturing where a material or a structure has been built up in layers. Three immediately obvious examples are: sedimentary rock structures in the geological setting, carbon-fibre composite constructions as found in the aerospace and automotive industries, and laminated paperboard used for packaging. As has often been the case in parallel but apparently disconnected areas of research work, progress has not necessarily transferred smoothly and immediately from one field to the other. Work often takes place along similar but unconnected lines, and the wheel can be reinvented many times over. It is most certainly a definitive role of a major interdisciplinary journal like *Philosophical Transactions of the Royal Society A* to help bridge some of these sizeable gaps.

So, with this in mind, we embark on this issue, drawing attention to similarities and differences between behaviour of layered structures as they appear in these three quite different areas of study. We characterize such systems as being strongly influenced by geometrical constraints that act to keep the layers fitting snugly together, held by different kinds of constrictive force. In structural geology for example, overburden pressure arising from other layers laid on top provides such forces; for layered composites, it can be the bond provided by the infilling matrix material. Such constraints oblige the layers to interact, often strongly, and can severely restrict how they are able to deflect. The interactive effects can be fully dominant (see, for example, the paper here by Peletier & Veneroni [1]), strongly influential [2–4], or brought in gradually as in some parametric studies [5]. In many cases, interest naturally falls on situations where work is done against such bonds, so that they stretch or break to allow voiding [2], or delamination [6]. In tension [7], cracking between layers of differing fibre orientation (delamination) is a critical form of failure where significant interlaminar stresses occur between these layers; these may be due to discontinuities such as free edges or intralaminar matrix cracks parallel to fibres. In compression, the resulting patterns of deformation can be particularly complex and striking in their form. An example is found on the front cover of this issue, which shows a configuration adopted by a layered stack of paper under compression while being constrained by clamps. The constraint of the layers to stay in contact leads to folding patterns with singularities, kink bands and many other interesting features, all of which pose analytical challenges. While these patterns come from a laboratory experiment, similar forms are seen in the large scale in sedimentary rocks, in the medium
scale in composite materials and in the small scale in crystal martensites. The commonality of such features across the scales is a recurring theme of this issue.

One limiting circumstance, common to both structural geology and layered composite materials, comes about when a single layer or a block of bonded layers is so stiff in relation to others that it they tend to dominate the deformation process. In the terminology of structural geology, a *competent* layer (or layers) deforms in such a way that its ultimate appearance is largely uninfluenced by the remaining weaker layers, while the latter are obliged to *accommodate* their deformation to whatever shape is thus imposed. So-called *thin film* models in composite technology adopt a similar philosophy, in assuming that under compression, a competent substrate will remain completely flat while a thin accommodating layer accepts the same end-shortening by delamination-induced buckling [3]; in reality, buckling of one layer would influence and bend the other, but this effect is often taken to be negligible.

We will illustrate the ideas and motivations behind this Theme Issue by considering the simple compressive structural model as shown in figure 1. This model will draw out many of the points covered in this issue, illustrating a range of subtle features without the need for excessive analysis, and yet it has direct applications to models of the folding of sedimentary rock layers and of delaminated materials. The competent layer in this model comprises two rigid bars of length $L/2$, hinged by a plastic rotational spring with the static response characteristic shown at the bottom left; no rotation $\theta$ can take place at the hinge until the moment in the spring reaches the *plastic moment* $M_p$, whereupon it rotates against this constant resistive moment. A weaker accommodating beam of bending stiffness $EI$ is pinned to it at each end, where inline springs of stiffness $k_2$, which are unstressed in the flat state, undergo deflections $Q$ representing relative slip between the components. The accommodating structure is taken as inextensional but is allowed to buckle, as seen at the bottom right, with displacements described by the angle $\psi(s)$, where $s$ represents a measure along the buckled length as shown. The system is loaded horizontally by a load $P$, displacing by an amount $\Delta$, and there is a further spring $k_1$ at the point of loading to give an initial, finite (pre-buckled) stiffness. The entire system is placed in a bath of pressure $q$, against which work needs to be done to create voids.

There is no immediate need for a plastic spring; an elastic spring could have served the same purpose. However, the plastic version resonates strongly with a number of systems in this issue, notably models of kink banding [6,9] and inflated membranes [10], in that it suggests that the flat ‘fundamental’ state is always stable and hence the critical load is infinite. This is an uneasy state of affairs: in reality, this state would only be metastable, and high enough loads would make the system highly sensitive to small perturbations or imperfections to trigger a dynamic instability.

Two modes of static response for this simple structure are seen in figure 2, found by using shooting methods on a system of ordinary differential equations derived from variational arguments [8]; higher modes may also exist but will be ignored here. At the top we show, on a plot of load $P$ against its corresponding deflection $\Delta$, the equilibrium paths for the specific shape we shall refer to as the *closing mode*; here, the form of the accommodating structure reflects that of the competent layer but rounds off the sharp corner that would otherwise be imposed.
Two equilibrium states exist, the \textit{fundamental path} seen as a solid line where the arms are completely aligned ($\theta = 0$), and \textit{buckled path} in the bent configuration ($\theta \neq 0$), shown as a dashed line. With increasing $P$, $\theta \to 0$ in the buckled state and these paths approach each other asymptotically, but never meet. As the load falls from this position, $\theta$ grows, until at some positive value of load, the path reverses direction and follows a route with $P$ increasing and $\theta$ continuing to grow. While falling, the buckled path is unstable, but under conditions of controlled end-shortening it would restabilize at the point where $\Delta$ is a minimum. Thus, the possibility arises of a dynamic jump at constant $\Delta$, from the fundamental state to the now stable rising post-buckled state. To initiate this, an energy hump needs to be overcome, but at high loads this would be expected to be relatively small. As an elastic layer can only bend to finite curvature, to adopt this deflected state, there must be significant slip between the layers taken up by the springs $k_2$.

If the stiffness of springs $k_2$ is large, there is a clear energy penalty associated with this closing configuration, and the system may want to adopt the alternative \textit{opening mode} configuration shown in figure 2b. Now the beam at the centre bends in the opposite sense to the corner and can accommodate with more ease the differences in length, but at the expense of doing work against overburden pressure in creating void space. The buckled equilibrium path has similarities
with that of figure 2a but with subtle differences. In particular, although it gets close to the fundamental path over some of its length, this is only over a finite load range. The solution reaches a limiting load value at a small value of $\theta$, where it separates into two different up-buckled shapes as shown. The smaller up-buckle restabilizes with increasing $\theta$ much like the closing mode, whereas the larger buckled shape would eventually develop into a teardrop shape and contact with itself; for the present system this point is never reached however, as before it can occur the delamination has grown to its maximum allowable length.

Neither mode can be found without negotiating an energy hump. However, since each is able to overcome its corresponding energy barrier by adopting an imperfection of a sympathetic shape, there are clearly circumstances, over a finite range of loads, where the opening form may be of equal or greater practical significance than its closing counterpart. This is certainly the case for controlled creasing and folding of laminated paperboard, as described here by Beex & Peerlings [11] and Mullineux et al. [12]. At the creasing stage, a constant radius ‘creasing rule’ is applied to the laminate to induce an initial imperfection in the opposing sense to the subsequent fold, such that the inside layers buckle inwards and the opening form is induced. The length and depth of

Figure 2. Output from numerical-based shooting methods derived from a calculus of variations formulation of the model of figure 1, plotted for $EI = 0.5$, $k_1 = 30$, $k_2 = 200$, $M_p = 40$, $q = 0.1$. (a) Closing mode: the buckled (blue/dashed) solution and the fundamental (red/solid) paths approach each other asymptotically as $P \to \infty$ but never meet. (b) Opening mode: the buckled solution reaches a limit point (seen here as a cusp) at a finite load value where the response separates into shapes with growing and diminishing void sizes. (Online version in colour.)
this imposed imperfection needs to be carefully controlled: large enough to induce the required subsequent buckle but without tearing or otherwise damaging the layers. Although the buckled form is similar to that of figure 2b, it differs from it in one very significant way in that the length of the buckled region is determined \emph{a priori} by the size of the creasing tools, rather than being free to choose itself.

The same is largely true for all delamination models presented here relevant to carbon-fibre composites. The model for delamination-induced buckling in a beam under pure bending of Kinawy \textit{et al.} [4], for example, is also of pre-determined length. For the system of figure 2, the freely chosen length of buckled beam implies that there is zero bending moment at the point of lift-off, but if a delamination length is preset, this is no longer likely to be true; normally, a bending moment would be found at the end of a buckled length of thin sub-laminate, matched with a counterpart in the thick sub-laminate. This interaction usually implies that the buckled layer and substratum initially move in the same direction, with a shape similar to the closing mode (figure 2). Yet, the focus naturally falls on the situation where they grow in opposing senses, as in the opening mode. This interest is for two reasons: first, the opening shape is the most immediately dangerous from a damage and propagation perspective; secondly, although a finite size opening imperfection is required to trigger the instability, in the experimental environment, this appears naturally with the addition of PTFE tape used to induce delaminations artificially.

The simple model of figure 1 is useful from yet another viewpoint, to facilitate discussion on fracture and propagation. The absence of curvature in each component at the point of separation means that a dynamical jump from one equilibrium state to another is in this case likely to be governed by slip between the layers, and hence heavily influenced by frictional resistance. In contrast, if the buckled length is predetermined, the resulting mismatch in curvatures produces bending action tending to peel the layers apart in tension; this form of fracture lies behind the contribution of, for example, Davidson & Waas [13]. The difference between the two actions—direct tension/compression and shear—mirrors the traditional distinction between mode I and mode II failures in linear elastic fracture mechanics, and is central to many of the contributions here. Wadee \textit{et al.} [6] acknowledge that tension in the mode I sense does occur during kink-banding in carbon-fibre composites, but conclude that shearing action is of most significance in the subsequent fracture. In contrast, Butler \textit{et al.} [3] acknowledge that shearing action may be involved in the propagation of delamination in plates, but nevertheless conclude that the complex interactive process can be represented by an equivalent tensile or mode I failure. Each system therefore needs to be assessed separately, according to its allowable kinematic freedoms.

To conclude, the system of figure 1 has nominally been introduced in the context of structural geology [8], but up to now this has barely been mentioned; we have used the model largely to build the connection with the folding of layered paperboard and delamination problems in layered composite structures and materials. Two substantial contributions to this Theme Issue have come directly from the geological community, and it is useful at this stage to draw them into our present perspective. Mainly owing to the works of Biot [14], much past and present modelling of the evolution of geological folding has been based on linear viscous rather than elastic constitutive laws. In particular, the concept of dominant wavelength, that with the most rapidly growing amplitude, has
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had a huge influence; this work is well summarized here in the contribution by Schmalholz & Schmid [5]. Moreover, even if layers are elastic rather than viscous, multi-layered structures are liable to slip relative to one another, and a realistic material representation should therefore perhaps be more like a fluid, with less resistance to shear than direct compression or tension. There is thus a great deal of sense in the traditional view that geological structures evolve over long time periods like viscous systems. Yet, such modelling tends to lead to regular periodic behaviour and it is clear from observations in the field, and recent nonlinear formulations, that this can only be part of the story. Other effects, such as the localization found in kink banding [6] and elsewhere [2], for example, are not available within this restricted framework: they are inherently nonlinear phenomena. This is the main point to be made by Hobbs & Ord [15]. Similarly, Schmalholz & Schmid [5], while starting with the conventional view, go to considerable lengths to embrace such nonlinear concepts. It is our considered opinion as Guest Editors of this Theme Issue that the combined effect of these two thoughtful pieces of work will lift the modelling of geological folding to a new level of understanding.

Finally, we must mention two papers which, while not directly focussed on layered structures, carry many of the same common features. The contribution of Fu & Xie [10] on bulging localizations in inflated tubes is effectively concerned with a single layer. Nevertheless, its bifurcation behaviour is very similar to that found in the model of figure 1. The system has a fundamental equilibrium solution that approaches its post-buckling counterpart asymptotically, providing an infinite critical load that is significantly eroded by sympathetic imperfections, in this case in the form of a bulge. Similarly, the paper of Seffen [16] is again largely centred on single-layer structures, but as stated in his introduction: ‘although our shells are essentially single-layered structures that can stretch and bend, we focus on capturing the hierarchy between the local, discrete nature of shell and its overall shape. To this end, we define meta-surfaces that enforce compatibility requirements and afford a homogenized view of the global deformation. Thus, the ‘layered’ aspect underpins the analytical approach rather than being entirely based on physical properties’. In this context, both single-layer papers are covered by the description of a thin shell as two surfaces or layers, representing separately bending and membrane actions [17], which are constrained to act together during deformation much like the separate layers of interest here.

In conclusion, the papers in this issue demonstrate the diversity of phenomena associated with layered structures, found in a variety of physical contexts. Some of these phenomena can be described by a (small deviation) linear theory. However most, such as localized states, voiding, delamination, kink bands and fracture, require the full paraphernalia of a nonlinear theory that combines both the effects of layer geometry, and the physical effects of compressive, shear and frictional forces, to describe fully the richness of the observed patterns. A major, and difficult, open problem remains, namely what is a complete description of all observable patterns of folding in (composite) layered structures and which combination of effects leads to which pattern? We hope that through this issue, we have managed to convince the reader both of the richness of the patterns observable in layered materials, and the need for careful (nonlinear) theories to deal with the many open problems yet to be resolved.
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References