Geometric modelling of kink banding in laminated structures

BY M. AHMER WADEE1,*, CHRISTINA VÖLMECKE2, JOSEPH F. HALEY1 AND STYLIANOS YIATROS3

1 Department of Civil and Environmental Engineering, Imperial College London, London SW7 2AZ, UK
2 LKM, Institut für Mechanik, Technische Universität Berlin, Germany
3 Department of Civil Engineering, Brunel University, Uxbridge UB8 3PH, UK

An analytical model founded on geometric and potential energy principles for kink band deformation in laminated composite struts is presented. It is adapted from an earlier successful study on confined layered structures that was formulated to model kink band formation in the folding of geological layers. This study’s principal aim was to explore the underlying mechanisms governing the kinking response of flat, laminated components comprising unidirectional composite laminae. A pilot parametric study indicates that the key features of the mechanical response are captured well and that quantitative comparisons with experiments presented in the literature are highly encouraging.

Keywords: kink banding; laminated materials; nonlinearity; energy methods; analytical modelling

1. Introduction

Kink banding is a phenomenon seen across many scales. It is a potential failure mode for any layered, laminated or fibrous material, held together by external pressure or some form of internal matrix, and subjected to compression parallel to the layers. Many examples can be found in the literature concerning the deformation of geological strata [1–3], wood and fibre composites [4–10] and internally in wire and fibre ropes [11,12]. There have been many attempts to reproduce kink banding theoretically, from early mechanical models [13,14] to more sophisticated formulations derived from continuum mechanics [15], finite elasticity theory [16] and numerical perspectives for more complex loading arrangements [17].

There has been much relevant work on composite materials, with significant problems being encountered as outlined thus. First, although two-dimensional models are commonly employed [18,19], modelling into the third dimension adds a significant extra component. It inevitably involves a smeared approach in the modelling of material properties because there is a mix of laminae and the

*Author for correspondence (a.wadee@imperial.ac.uk).

One contribution of 15 to a Theme Issue ‘Geometry and mechanics of layered structures and materials’.
matrix with the possibility of voids. Second, failure is likely to be governed by nonlinear material effects in shearing the matrix material [20], and this is considerably less easy to measure or control than the combination of overburden pressure and friction considered in work on kink banding during geological folding [21–23].

In the present paper, a pilot study is presented where the discrete model formulated for kink banding in geological layers is adapted such that it can be applied to unidirectional laminated composite struts that are compressed in a direction parallel to the laminae. This is achieved by releasing the assumption that the creation of voids between the layers is penalized by increasing the total system energy because, in the current case, no overburden pressure actively compresses the layers in the lateral direction. It is worth noting that the lateral direction is defined throughout as orthogonal to the layers. Therefore, the rotation of the laminae during the formation of the kink band causes a dilation, which is resisted by lateral tensile forces generated within the interlaminar region. The coincident shearing of this region also generates an additional resisting force, but, as mentioned already, this can be subject to nonlinearity—in particular, a reduced stiffness that may be either positive (hardening) or negative (softening perhaps leading to fracture), which is currently formulated with a piecewise linear constitutive law. Work done from dilation and shearing is evaluated; additional features from the original model (strain energies from bending and direct compression) and the work done from the external load can be incorporated without significant alterations. An advantage of the presented model is that the resulting equilibrium equations can be written and solved entirely in an analytical form, without having to resort to complex continuum models or numerical solvers.

The primary aim of the current work was to lay the foundations for future research. The geometric approach has yielded excellent comparisons with experiments for the model for kink banding in confined layers of paper that was used as an analogue for geological layers (termed the ‘geological model’ presently); the same is true currently with the present model being compared favourably with previously published experiments [5]. Moreover, the relative importance of the parameters governing the mechanical response is also identified in the current study. From this, conclusions are drawn about the possible further studies that would extend the current model to give meaningful comparisons with the actual structural response for a variety of practically significant scenarios.

2. Review of model for geological layers

A discrete formulation comprising springs, rigid links and Coulomb friction has been devised to model kink band deformation in geological layers that are held together by an overburden pressure [22]. It was formulated using energy principles, and key parts of the model are shown in figure 1a, b. It has been compared very favourably with simple laboratory experiments on layers of paper that were compressed laterally and then increasingly compressed axially to trigger the kink band formation process. The testing rig used in that study is shown schematically in figure 1c, and a typical test photograph is shown in figure 1d. Assuming that the layers were laterally compressible, a key assumption, the
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Figure 1. (a) Basic configuration of the discrete model for kink banding in n geological layers. Tectonic load on each layer is $P$ with a horizontal reaction force $H$, the axial stiffness of each layer is $k$, the lateral overburden pressure is $q$ and the kink band orientation angle is $\beta$. (b) Two internal layers of the geological model. The kink band width and angle is $b$ and $\alpha$, respectively, normal contact force between layers is $N$, friction coefficient is $\mu$, individual rotational springs stiffness is $c$ and stiffness of the surrounding elastic medium per unit layer is $k_f$. (c) Schematic of the experimental rig used for testing the geological model. (d) A typical deformation profile in a physical experiment showing a sequence of kink bands with corresponding orientation angles $\beta$.

Kink band orientation angle $\beta$, was predicted theoretically for the first time, it being related purely to the initially applied lateral strain derived from the overburden pressure $q$. Figure 2 shows the characteristic sequence of deformation with figure 2a showing the undeformed state with the applied overburden pressure and the lateral pre-compression defining $\beta$, and figure 2b showing the point where the interlayer friction is released when the internal lateral strain within the kink band is instantaneously zero and the band forms very quickly in the direction of $\beta$. It was later demonstrated that beyond the condition shown in figure 2c, where all the layers have the same thickness, whether internal or external to the kink band, lock-up begins to occur as shown in figure 2d, where the geometric constraint forces the layers within the kink band to compress laterally, causing restabilization. This marked the point where new kink bands formed and these could also be predicted by this approach after some modifications were made to

Phil. Trans. R. Soc. A (2012)
Figure 2. Sequence of kink band deformation in the geological folding model. (a) Initial state with \( \alpha = 0 \); (b) instantaneous release of contact and hence friction within the kink band when \( \alpha = \beta \); (c) layers inside and outside of the kink band all have equal thickness when \( \alpha = 2\beta \); (d) lock-up occurs when \( \alpha > 2\beta \) and a new band would form.

the model [23]; a detailed discussion of the mechanical response and comparison against experiments can be found in that article. Moreover, this model has also been demonstrated to be suitable for modelling internal kink band formation in individual composite fibres found commonly in fibre ropes under bending [12].

3. Pilot model for laminated composite struts

As discussed earlier, the system studied in Wadee et al. [22] had layers that were bound together by the mechanisms of overburden pressure and interlayer friction. The deformation was in fact admissible geometrically only if the layers were laterally compressible; the relationship between the kink band angle \( \alpha \), which could vary, and the orientation angle \( \beta \), which was fixed, being such that interlayer gaps, or voids, were not created. For a laminated strut in pure compression in the direction parallel to the laminae, most experimental evidence from the literature also suggests that the kink band orientation angle \( \beta \) is basically fixed for each laminate configuration [5]. It is noted, however, in a recent study on laminates under combined compression and shear that this angle can change as the kink propagates but the angle reaches a limit [24]; in the current work, \( \beta \) is taken as a constant equivalent to this limiting value from the beginning of the kink band deformation process, which is a simplifying assumption.
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Figure 3. Two internal laminae of the laminated composite model. The shaded region shows the interlaminar region, which is exaggerated in scale for clarity. Dilation and shearing forces with their corresponding displacements are given by $F_I$ and $F_{II}$ with $\delta_I$ and $\delta_{II}$, respectively. The highlighted section shows the lengths $AB$ and $BC$, which directly relate to $\delta_I$ and $\delta_{II}$, respectively.

Kink band deformation in laminates involves different mechanisms that incorporate the interlaminar region comprising the laminae and the matrix that binds the component together. Since the matrix is itself deformable and there is no overburden pressure to close any voids, the model needs significant modifications to account for the different characteristics of the laminated strut. It is worth noting that the assumption for the lay-up sequence of the composite in the present case is such that no twisting is generated from the applied compression. Figure 3 shows the adapted two-layer model, which omits the following features that are not relevant in the current case: the foundation stiffness and the overburden pressure, i.e. $q = k_f = 0$. The kink band formation is thus intrinsically linked to the deformation of the interlaminar region within the strut.

Shearing within the interlaminar region is the analogous process to sliding between the layers in the model for geological folding, the latter being modelled in the energy formulation as a work done overcoming the friction force. A piecewise linear model is used to simulate the force versus displacement relationship in terms of the shear resistance (figure 4), where fracture modes that are relevant for a linear-softening response (figure 4a) are defined in figure 4c–e.

Tensile expansion, or dilation, of the interlaminar region is modelled, however, with a purely linear elastic constitutive law. In the model described in §2, it was argued that when the interlayer contact force was released not only would the friction be released but also the overburden pressure would inhibit the formation of subsequent voids within the layered structure. Since in the current case there is no overburden or lateral pressure as such, potential dilation of the interlaminar region needs to be included. As in the previous model, however, the lamina deformation is assumed to lock up and potentially trigger a new band forming when $\alpha > 2\beta$; lateral compression in adjacent laminae would then be occurring and stiffening the response significantly. Hence, it is reasoned therefore that it could be energetically advantageous for the mechanical system to form a new kink band rather than to continue to deform the current one [23,25].

Phil. Trans. R. Soc. A (2012)
Figure 4. (a,b) Piecewise linear force versus displacement model for interlaminar shearing; (c–e) fracture modes. (a) Linear-softening response that is more representative of a fracture model, $\text{sgn}(\delta_C) = \text{sgn}(\delta_M)$; (b) linear-hardening response that is more appropriate for materials that show post-yield strength, $\text{sgn}(\delta_C) = -\text{sgn}(\delta_M)$. (c) Mode I is tearing; (d) mode II is shearing and (e) mode III is scissoring. In the current model, only mode II is relevant.

(a) Potential energy formulation

(i) Interlaminar dilation

The resistance to interlaminar dilation while the kink bands deform is modelled with a linear constitutive law with the dilation-resisting force $F_I$ relating to the dilation displacement $\delta_I$, thus

$$F_I(\alpha) = C_I \delta_I, \quad \delta_I(\alpha) = t \left[ \frac{\cos(\alpha - \beta)}{\cos \beta} - 1 \right],$$

with $C_I$ being the lateral stiffness of the laminate, related to the lateral Young’s modulus, and $t$ being the thickness of a single lamina. Since the area over which the interlaminar region dilates depends directly on the kink band width $b$, the stiffness $C_I = bdk_I$, where $k_I$ is the lateral stiffness per unit area of the laminate and $d$ is the breadth of the strut. Moreover, $k_I$ can be related to the lateral Young’s modulus $E_{22}$, where $E_{22} = k_I t$. However, with the lamina assumed to be laterally incompressible in the current model and the dilation displacement being assigned purely to the softer interlaminar matrix material, a clear departure from the geological model, the current lamina thickness is thus $t$ rather than $t \cos(\alpha - \beta)$. This is shown in figure 3 and is detailed in the highlighted area of that diagram. The relationship in equation (3.1) for $\delta_I$ is thus obtained from taking the length $AB$ from figure 3, where $\delta_I = AB - t$. Hence, there is a lateral...
tensile strain developed since the gap between the laminae grows as \( \alpha \) increases from zero to \( \beta \). The gap subsequently begins to reduce; when \( \alpha = 2\beta \) the gap returns to zero, marking the commencement of lock-up.

The work done in the dilation process is therefore given by

\[
U_D = \int_0^{\delta_1(\alpha)} F_1(\alpha') \frac{k_1 b^2 t^2 (1 - \cos(\alpha - \beta))^2}{\cos \beta} \, d\alpha',
\]

(3.2)

where \( \alpha' \) is a dummy variable to facilitate evaluation of the definite integral. It is assumed that the interlaminar region would not be damaged in the process of dilation and that the only nonlinearity in the constitutive law would be under shear. This is because the dilation displacement is relatively smaller than the shearing displacement that is discussed next (see later text); this has the additional advantage of maintaining model simplicity such that any mixed-mode fracture considerations can be left for future work.

(ii) Interlaminar shearing

Interlaminar shearing or the laminae sliding relative to one another is modelled with a piecewise linear constitutive law with the force-resisting shear \( F_{II} \) relating to the shearing displacement \( \delta_{II} \), thus

\[
F_{II}(\alpha) = C_{II} \delta_{II}, \quad \delta_{II}(\alpha) = \frac{t}{\cos \beta} [\sin(\alpha - \beta) + \sin \beta],
\]

(3.3)

with \( C_{II} \) being the shearing stiffness of the combination of the matrix and laminae sliding relative to one another. The relationship for \( \delta_{II} \) in terms of \( \alpha \) and \( \beta \) in equation (3.3) is given by examining the length \( BC \) in figure 3. However, since the band is basically assumed to form instantaneously before any rotation occurs, it is implied that \( F_{II}(0) = \delta_{II}(0) = 0 \). Moreover, as \( \beta \neq 0 \), the expression \( \delta_{II} = BC + t \tan \beta \) is obtained, such that the force and displacement conditions are satisfied. Taking the limit \( \alpha \to 0 \) gives \( \delta_1/\delta_{II} \to \tan \beta \), which shows that, for values of \( \beta < 45^\circ \), the shearing displacement \( \delta_{II} \) is greater than the dilation displacement \( \delta_1 \) for practical values of \( \beta \) that tend to be below \( 35^\circ \) [5,24].

When the shearing displacement reaches the initial proportionality limit, i.e. when \( \delta_{II} = \delta_C \) (figure 4), the relationship between \( F_{II} \) and \( \delta_{II} \) changes to

\[
F_{II} = C_{II} \delta_C \left( \frac{\delta_{II} - \delta_M}{\delta_C - \delta_M} \right),
\]

(3.4)

where \( \delta_M \) is the shearing displacement when the corresponding resistance force reduces to zero. Now, if \( \delta_{II}(\alpha_C) = \delta_C \) and \( \delta_{II}(\alpha_M) = \delta_M \), the expressions for the resisting force can be written as

\[
F_{II} = \begin{cases} 
\frac{C_{II} t [\sin(\alpha - \beta) + \sin \beta]}{\cos \beta} & \text{for } \alpha = [0, \alpha_C], \\
\frac{C_{II} t \left[ \sin(\alpha - \beta) - \sin(\alpha_M - \beta) \right]}{\cos \beta \left[ \sin(\alpha_C - \beta) - \sin(\alpha_M - \beta) \right]} [\sin(\alpha_C - \beta) + \sin \beta] & \text{for } \alpha > \alpha_C \text{ and } \alpha = [\alpha_C, \alpha_M] \text{ if } \text{sgn}(\delta_C) = \text{sgn}(\delta_M) > 0, \\
0 & \text{for } \alpha \geq \alpha_M \text{ and } \text{sgn}(\delta_C) = \text{sgn}(\delta_M) > 0.
\end{cases}
\]

(3.5)
Moreover, since the contact area under shear depends on the kink band width \( b \), the stiffness \( C_{II} = bdk_{II} \), where \( k_{II} \) is the shear stiffness per unit area of the lamina. Hence, the effective shear stress \( \tau = k_{II}b \) and therefore \( k_{II} \) can be related to the material shear modulus \( G_{12} \), where \( G_{12} = k_{II}t \). The work done in the shearing process is given by

\[
U_S = \int_0^{\delta_{II}} F_{II}(\alpha') \, d\left\{ \frac{t}{\cos \beta} [\sin(\alpha' - \beta) + \sin \beta] \right\} = \frac{k_{II}bdt^2}{2} \mathcal{L}(\alpha),
\]

where

\[
\mathcal{L}(\alpha) = \left[ \frac{\sin(\alpha - \beta) + \sin \beta}{\cos \beta} \right]^2
\]

for \( \alpha \leq \alpha_C \), or

\[
U_S = \frac{k_{II}bdt^2}{2} \left\{ \mathcal{L}(\alpha_C) + \int_{\alpha_C}^{\alpha} \left[ \frac{\sin(\alpha' - \beta) - \sin(\alpha_M - \beta)}{\sin(\alpha_C - \beta) - \sin(\alpha_M - \beta)} \right] \times \left[ \frac{\sin(\alpha_C - \beta) + \sin \beta}{\cos \beta} \right] \cos(\alpha' - \beta) \, d\alpha' \right\}
\]

\[
= \frac{k_{II}bdt^2}{2} S(\alpha),
\]

where

\[
S(\alpha) = \frac{1}{\cos^2 \beta} \left\{ \frac{\sin(\alpha_C - \beta) + \sin \beta}{\sin(\alpha_C - \beta) - \sin(\alpha_M - \beta)} \right\} \left[ \sin^2(\alpha - \beta) - \sin^2(\alpha_C - \beta) \right.

+ 2 \sin(\alpha_M - \beta)[\sin(\alpha_C - \beta) - \sin(\alpha - \beta)] + [\sin(\alpha_C - \beta) + \sin \beta]^2 \right\},
\]

beyond the proportionality limit where \( \alpha = [\alpha_C, \alpha_M] \). However, if \( \alpha > \alpha_M \) and \( \text{sgn}(\delta_C) = \text{sgn}(\delta_M) > 0 \), the shear resistance force vanishes and the expression for \( U_S \) becomes

\[
U_S = \frac{k_{II}bdt^2}{2} S(\alpha_M).
\]

There would still be the potential for frictional forces to resist shear even though \( \delta > \delta_M \), and the interlaminar region has lost all shear strength, particularly when \( \alpha > 2\beta \) and the adjacent laminae are laterally compressed. However, when the laminae are in lateral tension, it is assumed that the resistance to dilation is independent of shear, such that the dilation resistance remains linearly elastic even though the shear resistance may be zero owing to mode II fracture (figure 4d). This, again, is a simplifying assumption relying on the geometric conditions which dictate that dilation displacements are tending to reduce when those from shear in the matrix are approaching their maximum, where \( \beta \leq \alpha \leq 2\beta \).

(iii) Remaining energy contributions

As in the geological model, the strain energy stored in bending can be taken from a pair of rotational springs of stiffness \( c \),

\[
U_B = c\alpha^2.
\]
Figure 5. Bending of a lamina within a kink band: (a) definition of $x$; (b) idealized case; (c) actual case. Curvature $\kappa$ is defined as the total angle change $2\alpha$ over the assumed effective bending length of the band $2b$, hence $\kappa \approx \alpha/b$.

The stiffness of the rotational springs is related differently from the geological model as the expression for that model contained the overburden pressure $q$ [22]. The bending energy should strictly relate to curvature $\kappa$, where

$$U_b = 2\int_{-b/2}^{b/2} \frac{1}{2}EI\kappa^2 \, dx,$$  \hspace{1cm} (3.12)

with $x$ defining the domain of one bending corner comprising two lengths of $b/2$ that are external ($x = -b/2$) and internal ($x = +b/2$) to the kink band, respectively, as represented in figure 5. The quantity $\kappa$ is defined as the rate of change of the kink band angle $\alpha$ over the kink band width $b$, thus

$$\kappa \approx \frac{\alpha}{b} \Rightarrow c \approx \frac{EI}{b}. \hspace{1cm} (3.13)$$

The key point is that curvature changes sign at the midpoint of the kink band. Hence, the rotational stiffness $c$ is related to the flexural rigidity $EI$ of a lamina, with $E$ being its Young’s modulus in the axial direction (denoted as $E_{11}$ henceforth) and its second moment of area $I = dt^3/12$. The strain energy per layer associated with the in-line spring of stiffness $k$ is hence given by

$$U_m = \frac{1}{2}k\delta_a^2,$$  \hspace{1cm} (3.14)

where $\delta_a$ is the axial displacement of the springs. The in-line spring stiffness is $k = E_{11}dt/L$ for a single lamina with $L$ being the length of the strut. The work done by the external load can be taken simply as the sum of the displacement of the in-line springs $\delta_a$ and from the band deforming multiplied by the axial load $P$, which can be defined as the axial pressure $p$ multiplied by the cross-sectional area of a lamina $dt$,

$$P\Delta = pdt[\delta_a + b(1 - \cos \alpha)]. \hspace{1cm} (3.15)$$

(iv) Total potential energy functions

The total potential energy $V$ is given by the sum of the strain energies from bending $U_b$, the in-line springs $U_m$, interlaminar dilation $U_D$ and shearing $U_S$, minus the work done $P\Delta$, thus

$$V = U_b + U_m + U_D + U_S - P\Delta. \hspace{1cm} (3.16)$$
Since the dilation terms are assumed to be linearly elastic throughout their loading history, the total potential energy per axially loaded lamina takes three forms:

— The case where \( \alpha = [0, \alpha_C] \); so \( V = V^L \), i.e. linearly elastic in shear.
— The case where \( \alpha > \alpha_C \), so \( V = V^S \), i.e. the secondary shear stiffness is either a smaller positive value than the primary shear stiffness or a negative value.
— The case where \( \alpha > \alpha_M \) and \( \text{sgn}(\delta_C) = \text{sgn}(\delta_M) > 0 \); so \( V = V^Z \), i.e. no shear stiffness, which occurs only if the secondary shear stiffness is negative.

These forms of the total potential energy are given by the expressions

\[
V^L = V^I + \frac{k_{II} b dt^2}{2} \mathcal{L}(\alpha), \quad V^S = V^I + \frac{k_{II} b dt^2}{2} \mathcal{S}(\alpha) \quad \text{and} \quad V^Z = V^I + \frac{k_{II} b dt^2}{2} \mathcal{S}(\alpha_M),
\]

where \( V^I \) is given by

\[
V^I = \frac{k \delta_a^2}{2} + \frac{E_{11} dt^3 \alpha^2}{12 b} + \frac{k_1 b dt^2}{2} \left[ 1 - \frac{\cos(\alpha - \beta)}{\cos \beta} \right]^2 - pdt[\delta_a + b(1 - \cos \alpha)].
\]

The total potential energy functions are non-dimensionalized by dividing through by \( kt^2 \) and can be re-expressed in terms of rescaled parameters,

\[
\tilde{V}^L = \tilde{V}^I + \frac{\tilde{k}_{II} \tilde{b}}{2} \mathcal{L}(\alpha), \quad \tilde{V}^S = \tilde{V}^I + \frac{\tilde{k}_{II} \tilde{b}}{2} \mathcal{S}(\alpha) \quad \text{and} \quad \tilde{V}^Z = \tilde{V}^I + \frac{\tilde{k}_{II} \tilde{b}}{2} \mathcal{S}(\alpha_M),
\]

where

\[
\tilde{V}^I = \frac{V^I}{kt^2}, \quad \tilde{V}^L = \frac{V^L}{kt^2}, \quad \tilde{V}^S = \frac{V^S}{kt^2}, \quad \tilde{V}^Z = \frac{V^Z}{kt^2}, \quad \tilde{\delta} = \frac{\delta_a}{t}, \quad \tilde{\Delta} = \frac{\Delta}{t}, \quad \tilde{b} = \frac{b}{t},
\]

\[
\tilde{\rho} = \frac{pd}{k} = \frac{pL}{E_{11} t}, \quad \tilde{D} = \frac{E_{11} d}{12 k} = \frac{L}{12 t}, \quad \tilde{k}_1 = \frac{k_1 dt}{k} = \frac{E_{22} L}{E_{11} t} \quad \text{and} \quad \tilde{k}_{II} = \frac{k_{II} dt}{k} = \frac{G_{12} L}{E_{11} t}.
\]

(b) Equilibrium equations

The equilibrium equations are defined by the condition of stationary potential energy with respect to the end-shortening \( \delta_a \), the kink band angle \( \alpha \) and the kink band width \( b \); these can be written in non-dimensional terms, thus

\[
\tilde{\rho} = \tilde{\delta}, \quad (3.21)
\]

\[
\tilde{\rho} = \tilde{k}_1 I_a + \tilde{k}_{II} J_a + \frac{2 \tilde{D} \alpha}{\tilde{b}^2 \sin \alpha}, \quad (3.22)
\]

and

\[
\tilde{\rho} = \tilde{k}_1 I_b + \tilde{k}_{II} J_b - \frac{\tilde{D} \alpha^2}{\tilde{b}^2(1 - \cos \alpha)}.
\]

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Equation (3.21) defines the pre-kinking fundamental equilibrium path that accounts for pure compression of the in-line springs of stiffness $k$. Equations (3.22) and (3.23) define the post-instability states for the non-trivial kink band deformations; equating them allows the kink band width $b$ to be evaluated analytically,

$$\tilde{b} = \left\{ \frac{\partial \alpha [2/\sin \alpha + \alpha/(1 - \cos \alpha)]}{\tilde{k}_I(I_b - I_a) + \tilde{k}_II(J_b - J_a)} \right\}^{1/2}. \quad (3.24)$$

The expressions for $I_a$ and $I_b$ are given in detail as follows:

$$I_a = \left[ 1 - \frac{\cos(\alpha - \beta)}{\cos \beta} \right] \frac{\sin(\alpha - \beta)}{\sin \alpha \cos \beta} \quad \text{and} \quad I_b = \frac{1}{2(1 - \cos \alpha)} \left[ 1 - \frac{\cos(\alpha - \beta)}{\cos \beta} \right]^2 \quad (3.25)$$

where these expressions apply for the entire range of $\alpha$. However, the expressions for $J_a$ and $J_b$ change for each form of the total potential energy function; for $\tilde{V} = \tilde{V}_L$, the expressions are

$$J_a = \frac{\cos(\alpha - \beta)[\sin(\alpha - \beta) + \sin \beta]}{\sin \alpha \cos^2 \beta} \quad \text{and} \quad J_b = \frac{[\sin(\alpha - \beta) + \sin \beta]^2}{2(1 - \cos \alpha) \cos^2 \beta}; \quad (3.26)$$

for $\tilde{V} = \tilde{V}_S$,

$$J_a = \frac{\cos(\alpha - \beta)}{\sin \alpha \cos^2 \beta} \left[ \frac{\sin(\alpha_C - \beta) + \sin \beta}{\sin(\alpha_C - \beta) - \sin(\alpha_M - \beta)} \right] [\sin(\alpha - \beta) - \sin(\alpha_M - \beta)] \quad (3.27)$$

and

$$J_b = \left[ \frac{\sin(\alpha_C - \beta) + \sin \beta}{2(1 - \cos \alpha) \cos^2 \beta} \right] \left\{ \frac{\sin(\alpha_C - \beta) + \sin \beta}{\sin(\alpha_C - \beta) - \sin(\alpha_M - \beta)} \right\} + \frac{\sin^2(\alpha - \beta) - \sin^2(\alpha_C - \beta) + 2 \sin(\alpha_M - \beta)[\sin(\alpha_C - \beta) - \sin(\alpha - \beta)]}{\sin(\alpha_C - \beta) - \sin(\alpha_M - \beta)} \right\} \quad (3.28)$$

and for $\tilde{V} = \tilde{V}_Z$, $J_a = 0$ and

$$J_b = \frac{\sin \beta[\sin \beta + \sin(\alpha_C - \beta) + \sin(\alpha_M - \beta)] + \sin(\alpha_C - \beta) \sin(\alpha_M - \beta)}{2 \cos^2 \beta(1 - \cos \alpha)}. \quad (3.29)$$

The initial limiting case where $\alpha \to 0$ gives $b \to \infty$ and $\tilde{p} \to \tilde{k}_I \tan^2 \beta + \tilde{k}_II$. The result for $b$ suggests that the kink band is initially prevalent throughout the structure and the result for $p$ shows that the critical load depends primarily on the shear stiffness with a smaller contribution from the dilation stiffness that relates to $\beta$. This reproduces similar results from the literature where the critical stress is related to the shear modulus [4,5,13]; it also reflects a significant difference from the geological model, which has an infinite critical load and where the kink band width grows from zero length [22].
Figure 6. Representation of the experimental sample in Kyriakides et al. [5]. The rod comprised ICI APC-2/AS4 composite fibres, with properties as given in Table 1. The sample was confined such that there was negligible lateral compression but also that global buckling was not an issue.

Table 1. Properties used in the validation study to compare the current model with experiments presented in Kyriakides et al. [5]. The shear modulus range is over 4% shear strain.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rod fibre diameter</td>
<td>( t = 7 \times 10^{-3} ) mm</td>
</tr>
<tr>
<td>Overall rod length</td>
<td>( L = 76 ) mm</td>
</tr>
<tr>
<td>Longitudinal Young’s modulus</td>
<td>( E = E_{11} = 130.76 \text{kN mm}^{-2} )</td>
</tr>
<tr>
<td>Lateral Young’s modulus</td>
<td>( E_{22} = 10.40 \text{kN mm}^{-2} )</td>
</tr>
<tr>
<td>Shear modulus (initial to final)</td>
<td>( G_{12} = 6.03 \rightarrow 0.68 \text{kN mm}^{-2} )</td>
</tr>
<tr>
<td>Fibre volume fraction</td>
<td>( v_f = 60% )</td>
</tr>
</tbody>
</table>

4. Numerical investigations

(a) Validation against experimental results

Results from the current model are initially compared with published experiments on circular cylindrical composite rods with confined ends that exhibited kink bands under axial compression [5]. This aids the comparison between the current model and the experiments such that both loading levels and the kink band width can be compared; a similar approach was used in Edmunds & Wadee [12].

The dimensions of the overall sample had a diameter of 8.255 mm with the relevant properties given in Table 1. Note that the breadth \( d \) is not given because it cancels in all the relevant non-dimensional quantities. The sample comprised ICI APC-2/AS4 composite fibres. Since the sample was cylindrical, the system in Kyriakides et al. [5] was presented in terms of a cylindrical polar coordinate system with \( x_1 \) and \( x_2 \) being the longitudinal and the radial coordinates, respectively, as shown in Figure 6. Although the current model is formulated for flat rectangular laminae, the lamina thickness \( t \), which includes the composite action of the lamina and the matrix, can be perceived to be equivalent to the diameter of an individual fibre rod. This is due to the tight packing of the composite that has a relatively high volume fraction; any departures from this assumed value of \( t \) are likely therefore to be relatively small. For the tests presented to measure the change in shear modulus \( G_{12} \), there was no plateau shown in the test data (figs A3 and A5 in Kyriakides et al. [5]). In the current study, it is therefore assumed that the piecewise linear model for the shear stiffness reflects the initial and final values found in the experiments; hence \( \text{sgn}(\delta_M) = -\text{sgn}(\delta_C) \), i.e. a linear-hardening model is implemented as represented in Figure 4b.

The critical shear angle \( \gamma_C \) for the piecewise linear idealization is the angle beyond which the shear stiffness is replaced by a secondary smaller value; this is estimated from the earlier mentioned graphs in figs A3 and A5 in Kyriakides
et al. [5] to be 0.012 rad (or 0.69°). The shear angle can be expressed in terms of the kink band orientation angle $\beta$ and the kink band angle $\alpha$, such that

$$\tan \gamma_C = \frac{\delta_{II}(\alpha_C)}{\delta_I(\alpha_C) + t} = \frac{\sin(\alpha_C - \beta) + \sin \beta}{\cos(\alpha_C - \beta)}.$$  \hspace{1cm} (4.1)

Given that $\beta$ is assumed to remain constant during deformation, the critical kink band angle $\alpha_C$ can be found by rearranging (4.1), thus

$$\tan \gamma_C \cos(\alpha_C - \beta) - \sin(\alpha_C - \beta) - \sin \beta = 0,$$  \hspace{1cm} (4.2)

and solving for $\alpha_C$. This is achieved by substituting the critical shear angle $\gamma_C$ from above and the kink band orientation angle from Kyriakides et al. [5], where $\beta$ was reported to lie between 12° and 16° (0.2094 and 0.2793 rad) to the $x_2$ direction; for the specified values, $\alpha_C$ is approximately equal to $\gamma_C$ ($\alpha_C = 0.0120 - 0.0121$ rad). To obtain the correct final shear modulus (table 1), the value of $\delta_M/t = -0.094$ is used such that the ratios between the initial and final values of the shear stiffness reflect the reported experimental data (figure 4b).

Figure 7 shows numerical results from the current model, using the properties defined in table 1 with $\beta$ values as found in the published results. Note that the non-dimensional total end-shortening $\tilde{\Delta}$ is defined as

$$\tilde{\Delta} = \tilde{\delta} + \tilde{b}(1 - \cos \alpha).$$  \hspace{1cm} (4.3)

The actual kink band widths in the five tests were reported to range from 11 to 36 fibre diameters (directly corresponding to $\tilde{b}$ in the current model) and the compressive strengths were found to average 1.119 kN mm$^{-2}$, with a standard deviation of 0.043 kN mm$^{-2}$ (directly corresponding to $p$ in the current model). The results from the current model show highly unstable snap-back and hence the critical load would never be reached realistically (figure 7a, b), a well-established feature for systems of this type [4]. For comparison purposes, the pressure $p$ is taken at the point at which the structure stabilizes and reaches a plateau; in the current model, this occurs at the geometric lock-up condition $\alpha = 2\beta$. For the range of the $\beta$ angles considered, the non-dimensional stabilization pressure $\tilde{p}$ ranges from 80.5 to 82.3, which converts to an actual stabilization pressure $p$ ranging from 0.969 to 0.992 kN mm$^{-2}$: an error against the average from the experimental results of 11–13%, which is sufficiently small to offer encouragement.

Of further interest is the comparison for the kink band width between the tests and the current model. Observing the graph shown in figure 7c, as the kink band angle $\alpha$ increases, initially the non-dimensional kink band width $\tilde{b}$ falls from a large value to a small value, approximately 5.6 when $\alpha = 0.024$ rad ($\approx 1.4^\circ$). As $\alpha$ increases further, the kink band width begins to increase slowly; see table 2 for details of some key points. According to the sequence described in figure 2b–d, the kink band itself maximizes dilation when $\alpha = \beta$, minimizes it when $\alpha = 2\beta$ and locks up when $\alpha > 2\beta$. The results of the current model, particularly when $\alpha = 2\beta$, lie at the lower end of the range of observed values of the band widths from the published experiments. This seems sensible, given that the lock-up condition used, where $\alpha = 2\beta$, represents a lower bound [22], which implies that the current model...
Figure 7. Non-dimensional plots of load $\tilde{p}$ versus (a) total end-shortening $\tilde{\Delta}$ and (b) kink band angle $\alpha$ (rad); (c) kink band width $\tilde{b}$ versus kink band angle $\alpha$ (rad). Range of $\beta = 12^\circ - 16^\circ$. (d) Piecewise linear-hardening relationship of the effective shear stress $\tau$ (Nmm$^{-2}$) versus the normalized shearing displacement $\delta_{II}/t$ for $\beta = 16^\circ$. Properties of ICI APC-2/AS4 composite fibres and configuration and the range for $\beta$ were taken from Kyriakides et al. [5].

Table 2. Non-dimensional kink band width values from the current model at different stages of deformation. The conditions $\alpha = \beta$ and $\alpha = 2\beta$ are the points where the dilation within the band are effectively maximized and minimized, respectively; experiments in Kyriakides et al. [5] reported $\tilde{b} = 11 - 36$.

<table>
<thead>
<tr>
<th>Kink Band Angle</th>
<th>Case: $\beta = 12^\circ$</th>
<th>Case: $\beta = 16^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \beta$</td>
<td>$\tilde{b} = 10.4$</td>
<td>$\tilde{b} = 10.2$</td>
</tr>
<tr>
<td>$\alpha = 2\beta$</td>
<td>$\tilde{b} = 17.0$</td>
<td>$\tilde{b} = 19.5$</td>
</tr>
</tbody>
</table>

would also tend to predict lower bound kink band widths. Hence, the results from the comparisons between the current model and the published experiments [5] are highly encouraging; they offer very good quantitative agreement for the loading and the geometric deformation—key quantities that define the kink band phenomenon.

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(b) Parametric studies and discussion

The favourable comparisons between the current model and the published experiments in Kyriakides et al. [5] imply that the fundamental physics of the system are captured by the current approach. The study is therefore extended to present a series of model parametric variations to establish their relative effects. The basic geometric and material configuration is identical to that used in the validation study presented in table 1. Material and geometric parameters are varied individually, while maintaining the remaining ones at their original values. The parameters that are varied are the kink band orientation angle $b$, the critical kink band angle $a_C$, the composite direct and shear moduli, $E_{11}$, $E_{22}$ and $G_{12}$, and the shape of the piecewise linear relationship for shear. Finally, it is worth emphasizing that, where a range of a particular parameter is given, the graphs show curves in equal increments that are inclusive of the limits stated of the parameter.

(i) Orientation angle

In the current model, the orientation angle $b$ needs to be fixed a priori; hence the effects of different starting conditions for the model need to be established. Increasing $b$ from $10^\circ$ (0.1745 rad) to $30^\circ$ (0.5236 rad), a range that is representative of laminate experiments in the literature [5,24], leads to a stiffer response for increasing $a$, as shown in figure 8a,b, with the pressure capacity for $b = 30^\circ$ being more than double the capacity for $b = 10^\circ$ for values of $a < b$. The graphs in figure 8c,d raise an interesting point about the response, particularly when $b \geq 22.5^\circ$ (0.3927 rad), which seems to define a boundary where the kink band width $b$ loses its monotonically increasing property after it initially troughs for a small value of $a$, which was identified as approximately $1.4^\circ$ in §4a. Both graphs show that the kink band width at the lower bound lock-up condition $a = 2b$ temporarily peaks when $b = 22.5^\circ$. For higher orientation angles, the kink band width $b$ in fact peaks beyond $a = 1.4^\circ$, then troughs and then resumes the monotonic rise as seen for $b \leq 22.5^\circ$. Moreover, this also explains the reason why the stabilization pressure increases for $b > 22.5^\circ$, as shown in figure 8a, because the pressure has an inverse square relationship with the kink band width, as shown in the equilibrium equations (3.22) and (3.23). The graphs presented in figure 9 attribute this loss of monotonicity in $b$ (beyond $a = 1.4^\circ$) to the dominating influence of the dilation terms for larger $b$, particularly in the region of maximum dilation where $a \approx b$. In the first instance, it should be recalled that when $b$ is larger the potential maximum dilation displacement $\delta_1$ is also larger relative to $\delta_\parallel$ when $a = b$. Figure 9a shows that the maximum of the dilation term $\tilde{k}_1(I_b - I_a)$ from the expression for $b$, i.e. equation (3.24), increases substantially with $b$, whereas figure 9b shows only very marginal changes in the respective shear term $\tilde{k}_\parallel(J_b - J_a)$. The numerator in equation (3.24), $\tilde{b}_{num}$, which represents the influence of bending, is independent of $b$, as shown in figure 9c, but the respective denominator, $\tilde{b}_{den}$, shows that the dilation term influences the values significantly for the higher $b$ values, as shown in figure 9d. Once $a$ gradually increases above $b$, the dilation displacement progressively reduces and the shear term begins to dominate with the result that the kink band width resumes growth and lock-up occurs. This effect is similar to that found in the geological model
Figure 8. (a,b) Equilibrium paths for different $\beta = 10–30^\circ$ through non-dimensional plots of load $\tilde{p}$ versus (a) total end-shortening $\tilde{\Delta}$ and (b) kink band angle $\alpha$ (rad). (c) Non-dimensional kink band widths $\tilde{b}$ versus the kink band angle $\alpha$ (rad) for a range of orientation angles $\beta = 10–30^\circ$; circles mark the lower bound lock-up condition $\alpha = 2\beta$. (d) Values of non-dimensional kink band widths $\tilde{b}$ and applied axial pressure $\tilde{p}$ at the lower bound lock-up condition.

with the introduction of the foundation spring of stiffness $k_f$ [22], as shown in figure 1b; the kink band width was also found to plateau with higher foundation stiffnesses. It is worth noting that if the stiffness loss in the constitutive law for dilation was introduced, then the effect found in the present case would be generally less pronounced.

(ii) Critical shear angle and modulus

Increasing the critical kink band angle from $\alpha_C = 0.69^\circ$ to $0.96^\circ$ (0.0120 to 0.0168 rad), with a fixed limiting displacement $\delta_M/t = -0.094$ as before, shows an increase in the critical shear stress before the loss in stiffness occurs (see figure 10c) and leads to a monotonic increase in the axial pressure $p$ and the minimum kink band width $b$ (figure 10a,c). A subtly different pattern is observed in figure 10b,d, where trends for increasing the initial shear modulus...
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Figure 9. Graphs of various terms from the expression for the non-dimensional kink band width $\tilde{b}$, equation (3.24), for $\beta = 10–30^\circ$ versus the kink band angle $\alpha$ (rad). (a,b) Plots of dilation and shear terms, respectively. (c,d) Plots of the numerator and denominator of the $\tilde{b}$ expression.

$(G_{ij}^{\text{init}} = 6.03–8.45 \text{kNmm}^{-2})$ lead to higher stabilization pressures but smaller minimum kink band widths. These are logical results because the effect of increasing the critical kink band angle will lead to a later destabilization in shear and hence increase the load and band width; the increase in the initial shear modulus increases the resistance against shearing—the process of kink banding therefore requires more axial pressure to overcome the increased stiffness. However, the increased shear stiffness reduces the kink band width because there is a greater resistance to that type of deformation.

(iii) Hardening and softening in shear

The variation in the piecewise linear model for the shearing response is now discussed. The constitutive behaviour $F_{II}$ versus $\delta_{II}$ has been hitherto assumed to be a linear-hardening law, which corresponded with the data from the literature used in the validation exercise. Figure 11 shows the results for different secondary slopes while they remain positive (a linear-hardening law). The results exhibit fairly progressive behaviour; the reduced secondary slopes reduce the load-carrying capacity but make only marginal changes to the kink band widths. Figure 12 shows results for reducing the secondary slope further such that it becomes negative (a linear-softening law). In this case, the negative secondary slope mimics the behaviour of a fracture process in which the shear stiffness and strength have vanished and mode II fracture and crack propagation would
occur. However, a pattern is seen with the strength reducing and the band widths increasing for weaker properties in shear, which appears to be entirely logical. The softening of the internal structure gives less resistance to the kink banding process, allowing for larger rotations and gross deformations. The detailed effects of crack propagation have been left for future work, although recent work on buckling-driven delamination [26] has suggested that an analytical treatment of such effects may indeed be tractable.

(iv) Young’s moduli

Results for a twofold increase in the axial modulus $E_{11}$ suggest that this has only a marginal effect on the normalized stabilization pressure $\tilde{p}$, with an approximately 1.5 per cent increase. However, this result does imply an
Figure 11. Non-dimensional plots of load $\tilde{p}$ versus (a) total end-shortening $\Delta$ and (b) kink band angle $\alpha$ (rad); (c) kink band width $\tilde{b}$ versus kink band angle $\alpha$ (rad); (d) piecewise linear-hardening relationship of the effective shear stress $\tau$ (N mm$^{-2}$) versus the normalized shearing displacement $\delta_{II}/t$. Range of $\delta_M/t = -0.435$ to $-0.094$ and $\beta = 12^\circ$.

approximate doubling of the actual stabilization pressure $p$; hence, the effect is quantitative without significantly affecting the kink band deformation nor the load-displacement response qualitatively. However, increasing the lateral modulus $E_{22}$ results in a significantly stiffer response, the system stabilizing to a smaller kink band width (figure 13). These, again, are logical results because the effect of increasing the lateral modulus $E_{22}$ increases the resistance against dilation; the process of kink banding therefore requires more axial pressure to overcome this. Increasing the axial modulus increases the axial stiffness $k$, which in turn effectively reduces the relative dilation and shear stiffnesses without affecting the relative bending stiffness—see the scaling relationships in equation (3.20). Since bending is currently assumed to be purely linear, its relative effect becomes progressively more pronounced and then outweighs the reduced dilation and shear effects at large rotations. Obviously, if the bending was assumed to plateau owing to plasticity, this effect would be limited.

(v) Summary

The parametric studies have shown that the orientation angle $\beta$ of a laminate affects the interplay between dilation and shear within the matrix. For smaller values of $\beta$, shear tends to dominate throughout. For larger values of $\beta$, dilation dominates in the first part of the deformation $\alpha \leq \beta$ with its effect diminishing beyond this point and shear dominating once again. It has also
been shown that changes in the shear and lateral Young’s moduli alongside the constitutive law used to model mode II fracture within the matrix have significant effects. However, although varying the axial modulus has a significant effect quantitatively, it has only a minor effect qualitatively in terms of changing the normalized axial pressure and kink band widths and angles.

5. Concluding remarks

An analytical, nonlinear, potential-energy-based model for kink banding in compressed unidirectional laminated composite panels has been presented. Comparisons of results with published experiments suggest that very good agreement can be achieved from this relatively simple mechanical approach, provided that certain important characteristics are incorporated:

— Interlaminar dilation and shearing: the kink band process naturally causes shearing and changes the gap between the laminae. The matrix within the composite needs to resist both these displacements for the laminate to have integrity and significant structural strength.

— Bending energy: the resistance to rotation sets a length scale, which—in this case—is the kink band width $b$. 

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Figure 13. Non-dimensional plots for the range of axial direct modulus $E_{11} = 130.7 \sim 261.5$ kN mm$^{-2}$ in $(a,c)$ and lateral direct modulus $E_{22} = 10.4 \sim 52$ kN mm$^{-2}$ in $(b,d)$. $(a,b)$ Load $\tilde{p}$ versus $\Delta$; $(c,d)$ kink band width $\tilde{b}$ versus the kink band angle $\alpha$ (rad). Note that $\beta = 12^\circ$ throughout.

Linear constitutive relationships for the mechanisms of bending and dilation, and a piecewise linear relationship for the process of shearing, together with geometrically nonlinear relationships, have been implemented. The approach has been successful such that the mechanical response captures the fundamental physics of kink banding and agrees with the experiments from Kyriakides et al. [5] in terms of kink band widths and loading levels, without having to resort to sophisticated numerical or continuum formulations. Unlike the geological model [22], where a relationship was derived for the band orientation $\beta$ that was related to the overburden pressure $q$, in the current case the angle $\beta$ has to be assumed a priori because, as far as the authors are aware, no satisfactory procedure for predicting $\beta$ for composite laminates exists. For laminates, the magnitude of the orientation angle $\beta$ has been largely attributed to the precision and tolerances involved within the manufacturing process, where fibre waviness and misalignments can be introduced [5,15]. However, if the overburden pressure is considered to be the controlling parameter for the equivalent ‘manufacturing process’ that keeps the geological layers behaving together, then future work on modelling the process of manufacturing composite laminates may bear fruit; an indication of the parameters that govern the orientation angle for the current case.
may be established. Although this is a shortcoming for the present model, the results from the parametric study are very encouraging, with the trends appearing to be entirely logical.

The current model can of course be used as a basis for further work. In the first instance, the piecwise linear formulation applied currently for shear could also be extended to include dilation, giving the possibility of mixed-mode fracture [27] in the kink band. It would appear from the current study that this would be more prevalent in laminates that have a naturally larger orientation angle $\beta$, where the dilation process is relatively more significant. Moreover, loading cases that are more complex than uniform compression could be investigated; for example, numerical approaches have been developed in Vogler et al. [17] and Gutkin et al. [28] with varying degrees of success to investigate the formation of kink bands where there is a combination of shear and compression. An additional complication in the combined loading case is that the kink band propagation across the sample tends to occur more gradually in contrast to the present case, where the formation process is found to be fast and has been assumed to be so currently.

References
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