REVIEW

A glimpse of quantum phenomena in optical lattices

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Optical lattices in cold atomic systems offer an excellent setting for realizing quantum condensed matter phenomena. Here, a glimpse of such a setting is provided for the non-specialist. Some basic aspects of cold atomic gases and optical lattices are reviewed. Quantum many-body physics is explored in the case of interacting bosons on a lattice. Quantum behaviour in the presence of a potential landscape is examined for three different cases: a hexagonal lattice potential, a quasi-periodic potential and a disorder potential.

Keywords: optical lattices; superfluid–Mott insulator physics; honeycomb lattice; quasi-periodicity; disorder

1. Introduction

As part of nature’s cornucopia, Vulcan created a panoply of crystalline forms at his forge. Form led to phenomenon; underlying composition and structure dictated the optical, chemical, mechanical and electrical properties of matter. In addition, at the heart of these structures lay symmetry, a notion that has captivated the human imagination. Over millennia, nature and her symmetries inspired not only mankind’s study of the world, but also the desire to create microcosms, as reflected in the temples of the ancients.

In the past century, the formulation of quantum mechanics has cast the symmetry of materials and the relationship between structure and phenomena in a new light. Decades of crystallographic studies, as reflected in the work of Alan Mackay, have led to perceptive insights and characterizations of lattice structures and symmetries in a breathtaking range of materials. Quantum many-body physics has accounted for various emergent phenomena stemming from an interplay of material structure, quantum statistics and interactions. As for the desire to create, various fields, such as those of nanoscience, photonics and ultracold trapped atomic physics, have enabled the crafting of mini-worlds at the quantum level. Two beautiful instances, the works of McGrath et al. [1] and

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One contribution of 14 to a Theme Issue ‘Beyond crystals: the dialectic of materials and information’.
Cademartiri et al. [2] in this Theme Issue, describe the fabrication of man-made structures at the nanoscale. The former views these structures as building blocks for self-assembly, emphasizing the role of shape, and the latter investigates the controlled epitaxial growth of material on quasi-crystals. In this paper, we explore the quantum physics that becomes manifest in engineered lattice structures at low temperatures in the cold atomic physics setting, focusing on one example of many-body physics and three different lattice structures.

The past two decades have witnessed several astounding advances in cold atomic physics, one being the ability to create crystal-like potential landscapes. Using the powerful method of interfering laser beams, one is able to go ‘beyond crystals’ found in nature by constructing specific optical lattice structures at will. The high degree of control in cold atomic experiments over lattice parameters, choice of quantum statistics and interactions between particles allow for isolation and focusing on particular quantum phenomena. Information on the phenomena is carried in the nature of the ground state of the system and its excitations. Here, we explore one instance of quantum many-body behaviour that involves interacting bosons in an optical lattice exhibiting two distinct phases of matter—a superfluid and an insulator. This instance offers a small peek into the scope of different systems and models realized in the cold atomic setting, some of them ultimately geared towards unravelling the mysteries that nature’s crystals continue to pose today.

At the quantum level, important links between structure, phenomenon and information lie in the nature of energy eigenstates of a particle in the presence of a potential energy landscape. The exposition here studies three potential landscapes of differing degrees of symmetry, of which two cases, namely the hexagonal lattice potential and quasi-periodic potentials, are in tribute to the pioneering explorations of Alan Mackay. The hexagonal lattice imposes severe constraints, which lead to beautiful band structure effects. Quasi-periodicity exhibits a dichotomy between disorder in real space and order in reciprocal space that has measurable consequences in cold atomic systems. We complete the progression from order to disorder in the third case, that of a random potential that breaks all translational symmetry. The choice of lattice structures and phenomena studied here are in no sense comprehensive nor are the details or list of references; the purpose of this paper is not to provide a full-fledged review of the topic and of recent developments. It is to take the non-specialist, assumed to have some working knowledge of quantum mechanics and minimal familiarity with solid-state physics, on a small journey through some fascinating phenomena that are familiar and exciting to the author as a means of encountering the quantum world in an optical lattice playground.

2. Cold atomic gases and optical lattices

One of the remarkable aspects of experiments on cold atomic trapped gases, as the name suggests, is that constantly evolving techniques are able to access temperatures that are orders of magnitude lower than hitherto observed on the Earth, providing a cold enough environment to access quantum behaviour [3,4]. But how cold is ‘cold’? The answer depends on two important factors among several others—quantum statistics (bosons versus fermions) and the density, \( \rho \), of
the gas. Given that bosons can multiply occupy quantum states, well below a characteristic temperature, they macroscopically occupy one (or a few) low-energy quantum state(s) to form a condensate. For fermions, given that Pauli’s exclusion principle allows only one particle per state, the characteristic temperature demarcates the regime where primarily the lowest energy levels are occupied up to a distinct Fermi energy, forming a degenerate Fermi gas. In three dimensions, a simple estimate of this temperature gives 

\[ kT \sim \frac{\hbar^{2} \rho^{2/3}}{m} \]

for both particles, where \( m \) is the mass of each constituent particle, \( \hbar = 2\pi \hbar \) is Planck’s constant and \( k \) is Boltzmann’s constant. For bosons, this scale is set by comparing the thermal de Broglie wavelength with the mean interparticle spacing, and for fermions, it is set by estimating the Fermi energy. In bosonic systems such as liquid helium-4, this temperature is of the order of a few kelvins. For the more ubiquitous case of fermions, in solid-state systems such as metallic copper, the Fermi temperature is of the order of 10^4 K, whereas in neutron stars, it can be as high as 10^{13} K. In contrast, cold atomic gases are dilute and the characteristic temperature is usually of the order of 0.1–10 \( \mu \)K or even lower. These temperatures are commonly achieved through a combination of cooling techniques, enabling the observation of Bose–Einstein condensates and degenerate Fermi gases.

Typically, the atomic gases are suspended by employing traps and artificially created potential landscapes, and are thus effectively isolated from the environment. Of prime interest here, lattice potentials can be created by interfering appropriate configurations of laser beams. The associated light couples with the dipole moment induced in the trapped atoms, resulting in a potential energy shift known as the AC Stark effect. For a two-level atom having an energy splitting \( \hbar \omega_0 \) and excited-state energy width \( \hbar \Gamma \), laser beams each of frequency \( \omega \) induce a dipole potential of the form [5,6]

\[ V_{\text{dip}}(r) \approx \frac{3\pi c^{2} I}{2\omega_{0}^{3} |\Delta|} I(r), \tag{2.1} \]

where \( \Delta = \omega - \omega_0 \), and \( I(r) = \epsilon_0 c |E(r, t)|^2 / 2 \) is the light field intensity due to the net electric field \( E \) at a point \( r \) and at a time \( t \). A blue detuning (\( \Delta > 0 \)) provides a repulsive potential and a red detuning (\( \Delta < 0 \)) an attractive one. For beam sources of linearly polarized light having an electric field of strength \( E_0 \), polarization \( \hat{e}_z \), wavevectors \( k_a \) and relative phase shifts \( \phi_a \), the net electric field takes the form

\[ E(r, t) = E_0 \sum_a e^{i(k_a \cdot r - \phi_a)} e^{-i\omega t} \hat{e}_z. \tag{2.2} \]

It can be seen that such a configuration of laser beams yields a potential \( V_{\text{dip}}(r) \) composed of a sum of functions \( \cos(b_a \cdot r) \), where \( b_a \) corresponds to differences in wavevectors for all possible pairs of laser beams. The optical lattice formed by such a potential can be characterized in crystallographic terms [7]. The characteristic energy scale for observing lattice effects is proportional to the tunnelling strength for atoms to hop between potential minima and, in these systems, is lower than the three-dimensional estimate given earlier.

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Several different elements may be used for trapping, quite commonly, the alkali elements such as lithium, sodium, potassium and rubidium. Depending on the isotopes used, these atoms can be fermions or bosons. A powerful feature of these trapped systems is that the nature and strength of interactions between atoms can be significantly varied. The popular technique of ‘Feshbach resonances’ involves alteration of atomic scattering properties via coupling to a low-energy bound state [3,8]. The atoms can also be endowed with effective spin degrees of freedom; for a given element, different hyperfine species can play the role of spin components [3]. Spin-dependent optical lattices can be realized by treating the polarization of light as a variable parameter. Depending on the underlying set-up, they can be subject to various spin-dependent Hamiltonians having a spectrum of magnetically ordered ground states.

As for measurement and imaging, trapped atomic systems differ so greatly from their solid-state counterparts in their physical set-up and their coupling to external probes that a completely new set of techniques has been called for. Time-of-flight imaging makes for a popular technique wherein the trap is switched off and the system is allowed to expand, reflecting the momentum distribution of particles. Radio-frequency spectroscopy, another oft-used technique, involves conversion of one hyperfine species of atoms to another by applying an oscillatory magnetic field and garnering information on the system based on resonances. A third technique involves displacing the system and observing subsequent dynamics upon release as a means of measuring damping and dissipation in the system. One technique that does derive its analogue from solid-state crystallography is Bragg spectroscopy: light scattering off atoms trapped in an optical lattice differs in momentum from the incident light by a reciprocal lattice vector.

Having briefly reviewed some basic concepts of cold atomic physics, we are now in a position to embark on sketches of select quantum phenomena that emerge in the presence of various optical lattice configurations.

3. The superfluid–Mott insulator system

An underlying lattice and interacting particles together form the building blocks for realizing rich quantum phases and phenomena. This section presents the instance of repulsively interacting lattice bosons, widely studied in optical lattice systems and known for exhibiting two distinct states of matter—the Mott insulator and the superfluid. The two possible phases emerge due to the competition between interactions, which favour pinning an optimal number of bosons to each site, and kinetic energy, which favours delocalizing the bosons throughout the lattice. In the insulation phase, an energetic cost prevents bosons from moving from site to site, impeding any information transfer across the system. In contrast, in the superfluid phase, bosons condense into a ground state that is spread throughout the system, and gapless excitations above this state are able to propagate information freely across large distances.

The two phases can be accessed by tuning any of the three characteristic parameters of the system: the strength for bosons to tunnel between sites, $w$; the interaction strength between bosons, $U$; and the chemical potential, $\mu$. These ingredients form the basis of the extensively studied ‘Bose–Hubbard model’, which

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clearly parametrizes the phases and their properties [9,10]. The Mott insulation phase and the pinning of bosons on each site can easily be understood when tunnelling between sites is negligible. The energy for having \( n \) bosons on each site \( i \) then becomes

\[
E_i^\text{n} = \frac{U}{2} n(n-1) - \mu n. \tag{3.1}
\]

Interactions favour fewer bosons, while a larger chemical potential favours a larger occupation number. Upon minimizing the energy and imposing the constraint that \( n \) be an integer, one finds that the optimal number of bosons \( n_0 \) respects the constraint \( n_0 - 1 < \mu / U < n_0 \). Moreover, adding or removing a boson from a site requires a finite amount of energy, making the system insulating. The presence of a small amount of tunnelling energy between sites does not alter these properties. The Mott insulating state can hence be characterized by zero compressibility, \( \partial n_0 / \partial \mu = 0 \); a small change in chemical potential cannot provide the energy required to change the average number of particles in the system. Upon increasing the tunnelling strength between lattice sites, however, beyond a critical value of \( w \), bosons can tunnel between sites for zero energy cost, and an infinitesimal change in the chemical potential can change the number of particles on each site; the system becomes compressible. At this point, the system forms a Bose–Einstein condensate (BEC) and acts as a superfluid [11]. Unlike in the Mott insulating state, spatially separated regions are correlated with one another, exhibiting ‘long-range order’. Figure 1a displays the favoured phases as a function of the characteristic parameters (\( Z \) denotes the number of neighbours for each site).

The optical lattice system offers scope for an exhaustive exploration of the superfluid–Mott insulator phase diagram as shown in figure 1a. The height of the lattice potential controls the tunnelling, \( w \), and the interactions between bosons, \( U \). Feshbach physics allows \( U \) to be tuned more drastically. The total number of particles loaded into the system, \( N \), determines the global chemical potential, \( \mu \). A prevalent consideration in these systems is that the particles are kept contained within a spatial region by means of a confining potential \( V(r_i) \), resulting in the possible coexistence of different phases. Roughly speaking, the effect of the confining potential is to modify the local chemical potential on each site by an offset, \( \mu_i \rightarrow \mu - V(r_i) \). At any given \( w \) and \( U \), this identification translates to following a spatially varying \( \mu_i \) and associated phases, as exemplified by the dotted arrow in figure 1a in the Bose–Hubbard phase diagram upon traversing the trap from its minimum to higher potentials. For a spherically symmetric trap, such a setting yields alternating spherical shells of superfluid and Mott insulating regions, as depicted in figure 1b. This exquisite arrangement of coexisting phases is known as the ‘wedding-cake’ structure, for its density distribution of flattened Mott regions at different heights interspersed by drops along superfluid regions.

The superfluid and Mott insulating phases, as well as the wedding-cake structure, have been keenly explored in experiments and imaged in several different ways [5,8]. In time-of-flight measurements, the BEC exhibits characteristic peaks at specific momenta, the zero-momentum peak being indicative of a condensed state having coherence across the entire sample. Spectroscopy measurements have probed the excitation spectra of the two phases, the superfluid having gapless excitations and the Mott states having gaps that depend on the number of particles on each site. The system provides fertile ground...
Figure 1. (a) The superfluid (BEC)–Mott insulator (MI) phase diagram, as derived from the Bose–Hubbard model, shows MI lobes of different occupation number, $n$, juxtaposing a single superfluid region. (b) An optical lattice, formed by interfering laser beams depicted by bold arrows, superimposing a trapping potential. (c) For the spherically symmetric trapping potential in (b), the local chemical potential changes on each site, going from high at the minimum to low towards the exterior. This variation can traverse multiple phases, as indicated by the dotted arrows in (a) and (c), leading to a shell structure composed of different phases. (Online version in colour.)

for exploring issues in many-body physics, such as: the critical behaviour around quantum phase boundaries; coupling between different phases of matter, including ‘Josephson tunnelling’ between superfluids mediated by a Mott insulating layer; non-equilibrium dynamics following a rapid change of parameters, for instance, ‘quenching’ from a Mott insulating phase to a region hosting a superfluid ground state; and exotic phases associated with more than a single species of lattice bosons.

Finally, it is to be remarked that the optical lattice structure enables the isolation and harnessing of information at the single-atomic quantum level. The Mott insulating state localizes atoms to each site; such atoms can act as qubits characterized by an internal degree of freedom such as the hyperfine level. The latest experiments have addressed the issue of probing and manipulating states at the level of individual lattice sites. These capabilities make the optical lattice structure highly attractive as a platform for quantum simulation and implementing quantum computational algorithms.

### 4. The hexagonal lattice

While §3 did not assume any particular geometry for the underlying lattice, the following sections are devoted to quantum effects brought about by specific lattice geometries and potential landscapes. We explore the nature
of single-particle eigenstates in these situations, applications to well-studied examples in solid-state situations as a starting point and the broad prospects in optical lattice settings.

The lattice described here, the hexagonal or the honeycomb lattice, forms the basis of a popular carbon-based solid-state material—graphene. The lattice consists of a unit cell having two points and two lattice unit vectors, such as shown in figure 2a. The reciprocal lattice in momentum space, denoted by the basis (k_x, k_y), is also hexagonal. The band structure for quantum particles hopping between sites on the hexagonal lattice can be described within a tight-binding approximation [12]. For a system described by purely nearest-neighbour hopping of strength \( w \), the underlying honeycomb lattice yields an energy dispersion [13,14]

\[
E_{\pm}(k) = \pm w \sqrt{3 + 2 \cos(\sqrt{3}k_y a) + 4 \cos\left(\frac{\sqrt{3}k_y a}{2}\right) \cos\left(\frac{3k_x a}{2}\right)},
\]  

(4.1)

where \( a \) denotes the lattice spacing. The schematic structure assumed by this dispersion, depicted in figure 2b, resembles a tent tethered at six points meeting its reflection.

For a system of fermions, such as the mobile electrons of graphene, several remarkable features of the honeycomb lattice dispersion become manifest when the number of fermions equals half the available states. At this half-filling, the Fermi energy lies where the ‘tent’ meets its reflection; gapless excitations occur at the ‘pegs’, known as Dirac points, of which only two are inequivalent. In terms of the original lattice, the eigenstates around these points correspond to specific linear superpositions of wave functions peaked at the two lattice points A and B of

\[\text{Phil. Trans. R. Soc. A (2012)}\]
the unit cell. For momentum deviations $q$ around the Dirac points, the dispersion assumes the conical form

$$E_{\pm}(q) \approx \pm v_F|q|, \quad v_F = \frac{3w_a}{2},$$

(4.2)

as illustrated in the inset of figure 2b. The highlighting feature of this dispersion is that it is identical to that of a relativistic, massless particle described by the Dirac equation, making the honeycomb lattice an arena for exploring high-energy physics. This Dirac physics also results in unusual fermionic properties in the presence of a magnetic field, such as a density-dependent cyclotron mass and an anomalous quantum Hall effect. Furthermore, in the presence of potential barriers that do not couple the two Dirac points, the system exhibits ‘Klein tunnelling’, which corresponds to quasi-particles propagating perfectly across the barrier devoid of reflection [14]. While these features have been much studied in the electronic properties of graphene, constructing hexagonal lattices in cold atomic system enables selective focus on their different aspects as well as observation of new physics.

In optical lattices, the hexagonal structure can be created by appropriate laser beam configurations. One possibility is to interfere three coplanar beams of the same frequency and polarization having wavevectors $k_i, i = 1, 2, 3$, oriented at an angle of $2\pi/3$ with respect to one another [15,16]. For wavevectors of equal magnitude $k_L$ and the configuration of figure 2b, the resultant optical lattice has reciprocal primitive vectors $b_1 = k_3 - k_1 = \sqrt{3}k_3(\hat{x} - \sqrt{3}\hat{y})/2$ and $b_2 = k_1 - k_2 = \sqrt{3}k_1(\hat{x} + \sqrt{3}\hat{y})/2$. For blue-detuned lasers, it can be shown that the potential landscape created by this lattice, described by equations (2.1) and (2.2), is that of the honeycomb [15]. Varying the polarization of the laser beams can result in lattices whose potential landscape is sensitive to the spin of the particles involved [17].

The graphene environment can be re-created in such optical lattices by loading fermions to around half-filling. Given that the optical lattice is not produced by an underlying crystal, it completely eliminates the effect of lattice vibrations, namely phonons. Modifications in laser beam configurations can lead to tuning band structure properties [18,19] and the opening up of energy gaps in the Dirac dispersion, and simulating different physical effects in graphene, such as spin–orbit coupling. While the nature and effects of interactions in graphene are still matters of debate, interactions can be completely tuned in the cold atomic set-up. The emergent Dirac fermions can even be subject to attractive interactions, resulting in Cooper pairs and superconductivity, or the formation of tightly bound pairs, effectively forming bosonic units, which can condense [20]. Under certain conditions, interactions can lead to true crystallization wherein constituent particles, in addition to being confined to the lattice, further break translational symmetry spontaneously as in a crystal [21]. But why be restricted to graphene and fermions? The lattice offers a marvellous playground for bosons and spin physics as well. As a beautiful instance, in a recent set of experiments, two hyperfine species of rubidium-87 atoms have been loaded into a spin-sensitive honeycomb lattice system [17]. One species forms a Mott insulating state on every other site, whereas the other species, although delocalized throughout the lattice, experiences the effect of the pinned species, thus exhibiting a novel realization of the superfluid–Mott insulator physics of §3.
5. Quasi-crystalline structure

As a natural topic for this Theme Issue and Alan Mackay’s studies, one might enquire about the fascinating prospect of generating lattices associated with quasi-crystals in the optical lattice setting. In §4, we saw that the laser beam configuration for the two-dimensional hexagonal lattice involved three beams generating two reciprocal lattice vectors. This scheme brings to mind a generalization to multiple laser beams that could lead to quasi-periodicity. Indeed, in \( N \) dimensions, where reciprocal space is spanned by \( N \) basis vectors \( \hat{b}_i, \ i = 1, \ldots, N \), if a lattice is generated by reciprocal vectors involving not only the basis vectors, but also another set whose expansion involves irrational numbers, the lattice has quasi-crystalline order \([22,23]\). The simplest case of quasi-crystalline order involves a one-dimensional lattice formed by laser beams of two colours and associated momenta \( k_1 \) and \( k_2 \) having an irrational ratio.

One could then ask about creating the celebrated instance of fivefold symmetry explored by Mackay \([24]\) and, in particular, the two-dimensional case of Penrose tilings \([25]\). Such quasi-order can be achieved by arranging five beams of equal intensity and frequency in a plane, each making an angle of 72° with its neighbours, and polarized in-plane \([26,27]\). The resultant potential landscape produced by equations (2.1) and (2.2) resembles Penrose tilings. In fact, the interference of five laser beams has also been used in a different setting, namely for creating a quasi-crystalline substrate interacting with a colloidal monolayer \([28]\). As noted by Mackay, the fivefold quasi-periodic structure is a higher-dimensional analogue of the one-dimensional quasi-periodic lattice in that it is a cross-modulation of two lattice patterns of irrational ratio, each having points at integral intervals \([29]\).

Quasi-periodic lattices have been realized in one-, two- and three-dimensional optical lattice settings. Such lattices show a curious dichotomy in that particles inhabiting quasi-periodic lattices show traits of being in both ordered and disordered media depending on what is measured. Clearly, the reciprocal lattice possesses ordering at specific momenta. This ordering has been measured as peaks in Bragg spectroscopy \([22]\) and would be manifest in the momentum distribution of a BEC formed by a system of bosons loaded into such a lattice \([27]\). On the other hand, in real space, particles experience a complete absence of translational symmetry; quasi-periodic lattices have been proposed as a means of simulating disorder \([30]\). The nature of the eigenstates \([30,31]\) and transport properties \([26,27,30]\) of particles in quasi-periodic lattices, be they bosons or fermions, are expected to resemble those of particles in a random medium, which is the topic of §6.

6. Disorder and localization

In the previous sections, we saw a progression from periodic lattice structures to quasi-periodicity. This section describes a complete violation of any spatial order, stemming from the presence of a random potential. The quantum mechanical eigenstates in the presence of such a potential can have two distinct forms resulting in insulating versus conducting states. These states differ from the insulator and the conductor in the superfluid–Mott insulator system in that they
are embedded in the nature of single-particle eigenstates as opposed to emerging from a competition between kinetic energy and interactions between particles. Given the commanding the extensive body of knowledge on this topic of disorder and randomness, we only touch upon the story presented by some of the key concepts in the solid-state context and then glance at its applications to cold atomic systems.

In a crystalline medium devoid of impurities and disorder, the quantum mechanical wave function of a particle can be described in terms of Bloch waves. In the presence of sparse but randomly distributed scattering sites, Bloch waves lose their coherence via scattering but the wave function remains extended throughout the sample. In 1958, Anderson proposed that strong randomness can result in a dramatic change in the form of the wave function $\psi(r)$, characterized by localization around a point $r_0$ and an associated exponentially decaying envelope over a length scale $\xi$ [32,33],

$$|\psi(r)| \sim \exp\left(\frac{|r - r_0|}{\xi}\right).$$  \hspace{1cm} (6.1)

A transition between these two vastly different kinds of wave functions is indicative of the Anderson metal–insulator transition, or, more generally, a delocalization–localization transition. Extended states enable transport of information across the system in the metallic phase, whereas localized states disallow such transport in the insulating phase. The transition between these phases can occur as a function of energy—given a random potential distribution centred at an energy $E_0$, states much lower or higher in energy tend to be localized to potential minima or maxima, but, around $E_0$, tend to be spread out through the system. The critical energy, $E_c$, at which the transition takes place is known as the mobility edge. The two phases can also be accessed by tuning disorder—a critical disorder strength, $V_c$, is required before states become localized. Figure 3 presents a qualitative picture of extended versus localized states based on simulations of a tight-binding two-dimensional lattice system having on-site disorder. Figure 3a shows an extended-type eigenstate close to zero energy whose probability density is on average spread throughout the system. Figure 3b shows a localized state at high energy; the probability density is mostly confined to an isolated region. These two ways of filling space can be described quantitatively, as introduced by Thouless [34], by evaluating the mean fourth power of the probability amplitude:

$$A = \frac{\int |\psi(r)|^4 \, dr}{(\int |\psi(r)|^2 \, dr)^2}. \hspace{1cm} (6.2)$$

For an extended state, within a volume $\Omega$, the amplitude would scale on average as $1/\sqrt{\Omega}$ and thus $A$ would scale as $1/\Omega$. However, for a localized state, the amplitude would be concentrated around a few isolated points of set measure zero, thus making $A$ a number that does not scale with volume. As a curiosity, it is to be noted that certain special wave functions can fill space in a fractal fashion interpolating between these two behaviours [35].

The analyses described above and related concepts formed the initial steps for understanding metallic versus insulating behaviour in terms of a ‘scaling theory for localization’ [33,34]. For a given disordered system, the idea here is to study the manner in which a particular measure of transport properties
scales with the system size. For instance, the conductivity of the material is one such natural measure in disordered electronic solid-state systems. A decrease in conductivity with the system size is indicative of an insulating phase, and an increase indicates a conducting or metallic phase. On the basis of scaling, it was argued that, in general, disorder always renders a system insulating in one or two dimensions in the thermodynamic limit, although only marginally so for weak disorder in two dimensions. (Thus, while simulations of disordered systems in two dimensions could show extended-type wave functions in finite-sized systems, such as in figure 3a, they would not exist in large enough systems.) Three dimensions, however, can host a second-order phase transition between the metallic and insulating phases. As a function of disorder strength $V$, the localization length $\xi$
diverges at the critical point in a universal fashion independent of the details of
the system, taking the form
\[ \xi \sim |V - V_c|^{-\nu}, \]
where \( \nu \) is a universal constant. While these localization studies have gone far
afield in solid-state systems over the decades, puzzles still remain.

Cold atomic systems offer environments in which several factors can be
controlled in studying the physics of disorder [30]—degree of randomness,
interactions (including the absence thereof), effective dimensionality, quantum
statistics and the presence of certain symmetries in the system that would alter
localization properties. Disorder can be created through various means [30].
Superimposing incommensurate periodic potentials, such as indicated in §5, has
been one method. But while this method breaks translational invariance locally,
ultimately the system is sensitive to ordering in reciprocal space. Another method
involves passing a laser beam through a diffuser to create a speckle pattern
reflecting a truly random potential. A third method is to randomly distribute
particles of one species in an optical lattice to create a disorder potential for
another species.

Over the years, several aspects of localization have been exquisitely captured
in cold atomic systems [30]. It must be mentioned that identifying a scaling
variable, such as the analogue of conductivity, a systematic study of its scaling
properties, and a full-fledged analysis of critical behaviour are still in order.
Typical probes include time-of-flight images, revealing the spatial and momentum
distributions of particles as well as dynamics in a random medium, and collective
mode behaviour, reflecting localization physics in the damping of oscillations.
Now, while the physics of Anderson localization itself is an attribute of single-
particle states, in an ensemble of particles, the particular states being probed
depend, among other things, on the quantum statistics of the particles. In bosonic
systems, given that each state can have multiple occupancy, at low enough
temperatures, a narrow range of states can be probed. Experiments in disordered
bosonic systems have revealed a progression of localization features as a function
of disorder, showing coherence across the entire system for weak disorder and
randomly distributed condensates for strong disorder [30]. In fermionic systems,
Pauli’s exclusion principle dictates that particles fill energy levels up to a Fermi
energy; this energy can be varied to scan through the mobility edge. Very recent
studies of Anderson localization in three-dimensional fermionic systems have
indeed revealed evidence for the presence of a distinct mobility edge based on
the differing transport properties of fermions occupying localized versus extended
states [36].

7. Concluding remarks

We have glimpsed the ways in which the lattice structure, interactions and
disorder each shape quantum phenomena. These ingredients are just the tip
of the iceberg in that their combined effects lead to a bedazzling range of rich
scenarios. For instance, disorder and interactions in lattice bosonic systems can
together give rise to an unusual phase, the Bose glass, which is compressible
but possesses no long-range superfluid order. In fermionic lattice systems, the
A disordered Fermi–Hubbard model is a prototype for studying complex solid-state materials, such as high-$T_c$ superconductors, which have eluded a full theoretical understanding.

Explorations of lattice systems in the cold atomic setting can go beyond those of crystals found in nature in realizing innovative and specifically chosen lattice geometries, controlled and isolated studies of different physical ingredients, and drastic but calibrated changes in the environment. They provide a playground for quantum simulation of key models that host exotic phases of matter and perhaps describe aspects of complex materials found in nature. The ability to perform sudden quenches in these systems opens new avenues for studying non-equilibrium dynamics. These systems are also candidates for forming the physical basis of quantum computational schemes. All this, and yet the building of this quantum microcosm is still at its infancy.

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Finally, it is a privilege and a joy to be contributing to this Theme Issue of the Philosophical Transactions of the Royal Society dedicated to Alan Mackay. My parents, sister and I have shared a warm friendship with Alan and Sheila Mackay over the decades. Alan’s work, visions and vibrant scientific curiosity continue to inspire us today.

References