Pure Gaussian state generation via dissipation: a quantum stochastic differential equation approach

BY NAOKI YAMAMOTO*

Department of Applied Physics and Physico-Informatics, Keio University, Hiyoshi 3-1-14, Kohoku, Yokohama, Japan

Recently, the complete characterization of a general Gaussian dissipative system having a unique pure steady state was obtained. This result provides a clear guideline for engineering an environment such that the dissipative system has a desired pure steady state such as a cluster state. In this paper, we describe the system in terms of a quantum stochastic differential equation (QSDE) so that the environment channels can be explicitly dealt with. Then, a physical meaning of that characterization, which cannot be seen without the QSDE representation, is clarified; more specifically, the nullifier dynamics of any Gaussian system generating a unique pure steady state is passive. In addition, again based on the QSDE framework, we provide a general and practical method to implement a desired dissipative Gaussian system, which has a structure of quantum state transfer.

Keywords: dissipation; Gaussian state; pure state; cluster state; state transfer

1. Introduction

Towards quantum state preparation, which clearly plays a key part in quantum information processing, recently several dissipation-based approaches have been proposed. The basic idea of those approaches originates from the trivial fact that a thermal environment drives any state to the stable ground state. However, it has been shown that we can sometimes engineer a desired dissipative environment such that the corresponding stable state is a non-trivial and useful one, for example, a highly entangled pure state [1–14]. More specifically, under some conditions, we are allowed to synthesize an open quantum system described by the Markovian master equation

\[
\frac{d\hat{\rho}_t}{dt} = -i[\hat{H}, \hat{\rho}_t] + \sum_{k=1}^{m} \left( \hat{L}_k \hat{\rho}_t \hat{L}_k^\dagger - \frac{1}{2} \hat{L}_k^\dagger \hat{L}_k \hat{\rho}_t - \frac{1}{2} \hat{\rho}_t \hat{L}_k^\dagger \hat{L}_k \right),
\]

such that \( \hat{\rho}_t \) must converge into a given desired pure state \( \hat{\rho}_\infty \); that is, the Hamiltonian \( \hat{H} \) and the dissipative channel \( \hat{L}_k \) (\( k = 1, \ldots, m \)) are appropriately

* yamamoto@appi.keio.ac.jp

One contribution of 15 to a Theo Murphy Meeting Issue ‘Principles and applications of quantum control engineering’.
synthesized to achieve this goal. One of the main advantages of this approach is that the target state $\hat{\rho}_\infty$ is clearly robust against any perturbation to the state $\hat{\rho}_t$ during the dynamical process. In particular, it is independent of the initial state preparation.

In the finite-dimensional case, a necessary and sufficient condition for equation (1.1) to have a pure steady state was obtained in earlier studies [2,4], and especially in Kraus et al. [4] the authors provided a sufficient condition for $\hat{\rho}_\infty$ to be unique. The uniqueness characterization is of particular importance, because without such condition the desired convergence into the target state cannot be guaranteed. For infinite-dimensional systems, on the other hand, in Koga & Yamamoto [13] the authors particularly focused on a general Gaussian dissipative system and provided a complete parametrization of the system having a unique pure steady state. The merit of focusing on the class of Gaussian systems lies not only in its importance in quantum information technologies [15,16] but also in the fact that the parametrization is obtained in an easily tractable manner in the phase space; actually the uniqueness of $\hat{\rho}_\infty$ can be readily checked by simply calculating the rank of a specific matrix, while in the finite-dimensional case [4], we are required to verify that there is no specific subspace in the Hilbert space.

In this paper, we study a Gaussian system having a unique pure steady state in terms of a quantum stochastic differential equation (QSDE) [17–20]. The use of a QSDE allows us to describe the dynamics of an open system in a form where the stochastic environment channels appear explicitly. The master equation (1.1) is obtained as a result of averaging out all such stochastic effects brought from the environment.

The contribution of this paper is twofold. The first one is that we clarify the physical meaning of the conditions for the Gaussian system to have a unique pure steady state, which cannot be clearly seen when dealing with only the master equation. This result is obtained through investigating the QSDE of the corresponding (complex) nullifier. In general, it is known that any pure Gaussian state can be characterized as the common zero eigenstate of the corresponding nullifier operators [21]; this is the reason why investigating the full behaviour of the nullifier provides new information about the dynamic process towards the target pure state. Actually, we show that the nullifier dynamics of any Gaussian system generating a unique pure steady state is passive. As a by-product, the result is used to show a certain trade-off between the closeness of the steady state to a target Gaussian cluster state [22–25] and the convergence time into that steady state.

In the previous result [13], although the mathematical characterization of the desired dissipative channel $\hat{L}_k$ was obtained, its actual implementation was not discussed. Actually, the resulting desired dissipative channels usually have to non-locally act on the system, and no general method to effectively implement such dissipative channels is known. The second contribution of this paper is to give a partial answer to the question of how to practically construct a desired dissipative system. The proposed scheme has a structure of quantum state transfer from light to matter [26–28]; more specifically, a desired state of light is first generated and then that light field interacts with the oscillator system (memory), which, as a result, acquires the desired state by dissipation. This scheme is indeed practical, because, as shown in Menicucci et al. [25], any pure Gaussian cluster state of light
can be effectively generated using some beam splitters and optical parametric oscillators (OPOs). Note that the QSDE approach actually has to be taken in order to explicitly describe the input light field.

We use the following notations. For a matrix $A = (a_{ij})$, the symbols $A^\dagger$, $A^\top$ and $A^\sharp$ represent its Hermitian conjugate, transpose and elementwise complex conjugate of $A$, i.e. $A^\dagger = (a^*_p)_j^i$, $A^\top = (a_{ji})$ and $A^\sharp = (a^*_v)_i^j = (A^\dagger)^\top$, respectively. For a matrix of operators, $\hat{A} = (\hat{a}_{ij})$, we use the same notation, in which case $\hat{a}^*_x$ denotes the adjoint to $\hat{a}_{ij}$. $I_n$ denotes the $n \times n$ identity matrix. $\Re$ and $\Im$ denote the real and imaginary parts, respectively.

2. Preliminaries

In this section, a brief introduction to a Gaussian system and its QSDE representation is given. Then, we review the result of Koga & Yamamoto [13].

(a) Gaussian dissipative systems

A general $n$-mode bosonic system consists of $n$ subsystems with canonical conjugate pairs $(\hat{q}_i, \hat{p}_i)$. Denote the vector of total system variables by $\hat{x} := (\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n)^\top$. The canonical commutation relation (CCR) $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$ then leads to

$$\hat{x}^\dagger\hat{x} - (\hat{x}^\dagger\hat{x})^\top = i\Sigma, \quad \Sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. $$

Now let $\hat{\rho}$ be the density operator of this system and write the mean vector by $\langle \hat{x} \rangle$ and the covariance matrix by $V = (\Delta\hat{x}\Delta\hat{x}^\dagger + (\Delta\hat{x}\Delta\hat{x}^\dagger)^\top)/2$, $\Delta\hat{x} = \hat{x} - \langle \hat{x} \rangle$, where the mean $\langle \hat{X} \rangle = \text{Tr}(\hat{X}\hat{\rho})$ is taken elementwise. Note that the uncertainty relation $V + i\Sigma/2 \geq 0$ holds. A Gaussian state can be characterized by only the mean vector and the covariance matrix. A particularly important fact is that the covariance matrix $V$ corresponding to a pure Gaussian state always has the following general representation [21,29]:

$$V = \frac{1}{2}SS^T, \quad S = \begin{pmatrix} Y^{-1/2} & 0 \\ XY^{-1/2} & Y^{1/2} \end{pmatrix}, \quad (2.1)$$

where $X$ and $Y$ are $n \times n$ real symmetric and real positive definite matrices (i.e. $Y = Y^\top > 0$), respectively. In other words, a pure Gaussian state is completely parametrized by $X$ and $Y$. An important merit of this representation is that the complex graph matrix $Z := X + iY$ can be used for a graphical calculus for several Gaussian pure states [21]. In particular, a pure Gaussian state $|\psi_Z\rangle$ having the covariance matrix (2.1) always satisfies

$$\hat{r}|\psi_Z\rangle = 0, \quad \hat{r} := (-Z, I_n)\hat{x} = \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} - Z \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix}, \quad (2.2)$$

where the equation means that each entry of $\hat{r}$ acts on $|\psi_Z\rangle$. Conversely, if a pure Gaussian state $|\psi\rangle$ satisfies $(-Z, I_n)\hat{x}|\psi\rangle = 0$, then we have $|\psi\rangle = |\psi_Z\rangle$. The vector of operators $\hat{r}$ is called the nullifier for the pure Gaussian state $|\psi_Z\rangle$.
A linear system is such that the Hamiltonian $\hat{H}$ and $k$th dissipative channel $\hat{L}_k$ in equation (1.1) are respectively characterized by

$$\hat{H} = \frac{1}{2} \hat{x}^\top G \hat{x}, \quad \hat{L}_k = c_k^\top \hat{x},$$

(2.3)

where $G = G^\top \in \mathbb{R}^{2n \times 2n}$ and $c_k \in \mathbb{C}^{2n}$. For this system, the time evolution of $\langle \hat{x}_t \rangle$ and $V_t$ with the state $\hat{\rho}_t$ obeying equation (1.1) are given by $d\langle \hat{x}_t \rangle/dt = A \langle \hat{x}_t \rangle$ and $dV_t/dt = AV_t + V_tA^\top + D$, respectively. Here, $A = \Sigma[G + \Im(C^\dagger C)]$ and $D = \Sigma\Re(C^\dagger C)\Sigma^\top$ with $C = (c_1, \ldots, c_m)^\top \in \mathbb{C}^{m \times 2n}$ (see Wiseman & Milburn [20] for a more detailed discussion). In this paper, we assume that the initial state of $A$ is a Hurwitz matrix, i.e. all the eigenvalues of $A$ have negative real parts. If it exists, the mean vector is $\langle \hat{x}_\infty \rangle = 0$ and the covariance matrix $V_\infty$ is given by the unique solution to the following matrix equation:

$$AV_\infty + V_\infty A^\top + D = 0. \quad (2.4)$$

(b) The quantum stochastic differential equation framework

The situation we have in mind is that the system interacts with a countable set of environment channels. The time evolution of an observable of this open system is described in terms of a QSDE. A most simple form of this equation is obtained when the environment channels are all independent vacuum fields with ideal Markovian approximation taken. Let $\hat{a}_i(t)$ be the annihilation operator of the $i$th vacuum field; then the Markovian approximation means that $\hat{a}_i(t)$ instantaneously interacts with the system and satisfies the CCR [\(\hat{a}_i(s), \hat{a}_i^\ast(t)\)] = \(\delta_{ij}\delta(t - s)\). Define the field annihilation process operator by $\hat{A}_i(t) = \int_0^t \hat{a}_i(s)ds$, then this CCR leads to the following quantum Ito rule:

$$d\hat{A}_i d\hat{A}_j^\ast = \delta_{ij} dt, \quad d\hat{A}_i d\hat{A}_j = d\hat{A}_i^\ast d\hat{A}_j^\ast = d\hat{A}_i d\hat{A}_j^\ast = 0. \quad (2.5)$$

The system–field coupling in the time interval $[t, t + dt)$ is described by the unitary operation $\hat{U}(t + dt, t) = \exp[\sum_i(\hat{L}_i d\hat{A}_i^\ast - \hat{L}_i^\ast d\hat{A}_i)]$, where $\hat{L}_i$ is the system operator representing the coupling with the $i$th vacuum field. Then, the system observable at time $t$, $j_i(\hat{X}) = \hat{U}_t^\ast \hat{X} \hat{U}_t$, obeys the Ito-type QSDE

$$d_j_i(\hat{X}) = j_i(i[\hat{H}, \hat{X}] + \sum_{i=1}^m \left(\hat{L}_i^\ast \hat{X} \hat{L}_i - \frac{1}{2} \hat{L}_i^\ast \hat{L}_i \hat{X} - \frac{1}{2} \hat{X} \hat{L}_i^\ast \hat{L}_i\right)) dt$$

$$+ \sum_{i=1}^m (j_i([\hat{X}, \hat{L}_i])d\hat{A}_i^\ast - j_i([\hat{X}, \hat{L}_i^\ast])d\hat{A}_i), \quad (2.6)$$

where an additional system Hamiltonian $\hat{H}$ has been added. Note that $\hat{U}_{t+dt} = \hat{U}(t, t + dt) \hat{U}_t$, the mean value $\langle j_i(\hat{X}) \rangle$ is represented using the (unconditional) density operator $\hat{\rho}_t$ by $\langle j_i(\hat{X}) \rangle \equiv \text{Tr}(\hat{X}\hat{\rho}_t)$, which leads to the master equation (1.1). The change of the field operator can also be dealt with explicitly; the
output field $\hat{A}_i' := j_i(\hat{A}_i)$ after the interaction satisfies
\[ d\hat{A}_i' = j_i(\hat{L}_i)\, dt + d\hat{A}_i. \tag{2.7} \]

We are interested in the QSDE whose system Hamiltonian and dissipative channels are given by equation (2.3). Let us define $\hat{A}_t = (\hat{A}_1, \ldots, \hat{A}_m)^\top$, then the vector of system quadratures $\hat{x}_t = (j_1(\hat{q}_1), \ldots, j_1(\hat{q}_n), j_1(\hat{p}_1), \ldots, j_1(\hat{p}_n))^\top$ satisfies the linear QSDE [20,30–35]
\[ d\hat{x}_t = A\hat{x}_t\, dt - i\Sigma C^\dagger d\hat{A}_t + i\Sigma C^\top d\hat{A}_t', \tag{2.8} \]
where the system matrices $A$ and $C$ were defined in §2a. It is easy to see that the linear QSDE (2.8) actually leads to the time evolutions of the mean and the covariance matrix: $d\langle \hat{x}_t \rangle / dt = A\langle \hat{x}_t \rangle$ and $dV_t / dt = AV_t + V_tA^\top + D$. Also the output field equation (2.7) of $\hat{A}_i' = (\hat{A}_1', \ldots, \hat{A}_m')^\top$ then becomes
\[ d\hat{A}_i' = C\hat{x}_t\, dt + d\hat{A}_i. \tag{2.9} \]

(c) The dissipative Gaussian system generating a pure steady state

In Koga & Yamamoto [13], some conditions for a dissipative Gaussian system to have a unique pure steady state were obtained. A particularly useful result from the environment engineering viewpoint is the following (recall $Z := X + iY$).

**Theorem 2.1 (Koga & Yamamoto [13]).** Let $V$ be a given covariance matrix of the form (2.1). Then, this is the unique solution of equation (2.4) if and only if the system matrices are represented by
\[ C = P^\top (Z, I_n) \tag{2.10} \]
and
\[ G = \begin{pmatrix} RX + YRY - \Gamma Y^{-1}X - XY^{-1}R^\top & -XR + \Gamma Y^{-1} \\ -RX + Y^{-1}R^\top & R \end{pmatrix}, \tag{2.11} \]
where $P$ is a complex $n \times m$ matrix, $R$ is a real $n \times n$ symmetric matrix and $\Gamma$ is a real $n \times n$ skew symmetric matrix (i.e. $\Gamma + \Gamma^\top = 0$), and moreover, $P$ and $Q := -iRY - Y^{-1}R^\top$ satisfy the following rank condition:
\[ \text{rank}(P, QP, \ldots, Q^{n-1}P) = n. \tag{2.12} \]

This theorem states that any dissipative linear system having a unique pure Gaussian steady state is completely parametrized by the three matrices $P, R$ and $\Gamma$, which further have to satisfy the rank condition (2.12). In Koga & Yamamoto [13], this result was obtained through a fully algebraic treatment of equation (2.4), and the physical meanings of the conditions were not discussed. As mentioned in §1, nevertheless, they will be clarified within the QSDE framework; for convenience of the later discussion, we note that $G$ satisfies
\[ G\Sigma^\top \begin{pmatrix} -Z \\ I_n \end{pmatrix} = \begin{pmatrix} -Z \\ I_n \end{pmatrix} Q. \tag{2.13} \]
3. Dynamics of the nullifier

As seen in equation (2.2), the pure state $|\psi_Z\rangle$ is the common zero-eigenstate of the
nullifier vector $\hat{r} = (-Z, I_n)\hat{x}$. Hence, it is worth seeing the time evolution of $\hat{r}_t$,
when the conditions shown in theorem 2.1 are satisfied. Noting equations (2.10) and (2.13), we have

$$(-Z, I)A = (-Z, I)\Sigma G + \frac{1}{2i}(-Z, I)\Sigma(C^\dagger C - C^T C^\dagger)$$

$$= Q^T(-Z, I) + \frac{1}{2i}(-I, -Z)$$

$$\times \left\{ \left( \begin{array}{c} -Z^2 \\ I \end{array} \right) P^2 P^T(-Z, I) - \left( \begin{array}{c} -Z \\ I \end{array} \right) P P^\dagger(-Z^2, I) \right\}$$

$$= Q^T(-Z, I) + \frac{Z^2 - Z}{2i} P^2 P^T(-Z, I) = (Q^T - Y P^2 P^T)(-Z, I),$$

$$(-Z, I)(-i\Sigma C^T) = -i(-Z, I)\Sigma\left( \begin{array}{c} -Z^2 \\ I \end{array} \right) P^2 = i(Z - Z^2) P^2 = -2YP^2,$$

$$(-Z, I)(i\Sigma C^T) = i(-Z, I)\Sigma\left( \begin{array}{c} -Z \\ I \end{array} \right) P = 0.$$ 

Therefore, multiplying both sides of equation (2.8) by $(-Z, I_n)$ from the left, we have

$$\mathrm{d}\hat{r}_t = (Q^T - Y P^2 P^T)\hat{r}_t \mathrm{d}t - 2YP^2 \mathrm{d}\hat{\mathbf{A}}_t.$$  \hspace{1cm} (3.1)

Regarding the output process (2.9), as $C\hat{\mathbf{x}}_t = P^T(-Z, I_n)\hat{x}_t = P^T\hat{r}_t$, it is written by

$$\mathrm{d}\hat{\mathbf{A}}_t = P^T\hat{r}_t \mathrm{d}t + \mathrm{d}\hat{\mathbf{A}}_t.$$  \hspace{1cm} (3.2)

The coefficient matrix of the dynamics of $\hat{r}_t$ has the following property.

**Proposition 3.1.** The matrix $Q^T - Y P^2 P^T$ is Hurwitz if and only if the rank condition (2.12) is satisfied.

**Proof.** Let $b \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ be the eigenvector and the eigenvalue of $Q^T - Y P^2 P^T$, respectively; i.e. $(Q^T - Y P^2 P^T)b = \lambda b$. Then, multiplying this equation by $b^\dagger Y^{-1}$ from the left, we have

$$b^\dagger (Y^{-1}Q^T - P^2 P^T)b = \lambda \| Y^{-1/2}b \|^2,$$

where $\| \cdot \|$ denotes the standard Euclidean norm. This immediately yields $b^\dagger (Q^2 Y^{-1} - P^2 P^T)b = \lambda^* \| Y^{-1/2}b \|^2$. Recall now that $Q = -iRY - Y^{-1}I^T$; hence, $QY^{-1} = -iR - Y^{-1}I^TY^{-1}$ is skew Hermitian. Therefore, adding the above two equations yields $-2\| P^Tb \|^2 = (\lambda + \lambda^*) \| Y^{-1/2}b \|^2$, and we have $\Re(\lambda) = -\| P^Tb \|^2 / \| Y^{-1/2}b \|^2$. Let us here assume that $P^Tb = 0$. Then, we have $Q^n b = \lambda b$, and the matrix $C := (P, QP, \ldots, Q^{n-1} P)$ satisfies

$$C^\dagger b = \left( \begin{array}{c} P^T b \\ P^T Q^T b \\ \vdots \\ P^T (Q^T)^{n-1} b \end{array} \right) = \left( \begin{array}{c} \lambda b \\ \lambda P^T b \\ \vdots \\ \lambda^{n-1} P^T b \end{array} \right) = 0.$$
But this is contradiction to the assumption (2.12), and thus $P^T b \neq 0$. As a result, $\Re(\lambda)$ is strictly negative, implying that the matrix $Q^T - YP^z P^T$ is Hurwitz.

On the other hand, if $C$ is not of rank $n$, there exists an eigenvector of $Q^T - YP^z P^T$, say $b_0$, that satisfies $P^T b_0 = 0$. Then, from the above discussion, the corresponding eigenvalue $\lambda$ satisfies $\Re(\lambda) = 0$, and hence $Q^T - YP^z P^T$ is not Hurwitz.

On the basis of the above result, we can verify that the target pure Gaussian state is certainly generated. To see this, let us multiply all the entries of the nullifier dynamics (3.1) by the system–field composite state vector $|\Psi\rangle = |\psi\rangle \otimes |0\rangle$ from the right ($|0\rangle$ is the vacuum state). Then, owing to the relation $d\hat{A}_i|0\rangle = 0$, we have

$$
\frac{d}{dt} \hat{\rho}_i|\Psi\rangle = (Q^T - YP^z P^T)\hat{\rho}_i|\Psi\rangle.
$$

The Hurwitz property of the matrix $Q^T - YP^z P^T$ is equivalent to asymptotic stability of the dynamics; thus the nullifier vector $\hat{\rho}_i|\Psi\rangle = \hat{U}_\infty^* \hat{\rho}_i|\Psi\rangle$ converges to zero. Therefore, in the Schrödinger picture, $\hat{U}_\infty^* |\Psi\rangle$ is the common zero-eigenstate of $\hat{\rho}$, meaning that the system state becomes the target pure Gaussian state $|\psi_Z\rangle$ as $t \to \infty$.

The Hurwitz property obtained above allows us to obtain a specific input–output relation from the incoming field $\hat{A}_t$ to the outgoing field $\hat{A}_t'$, through equations (3.1) and (3.2). For this purpose, it is convenient to move into the frequency domain where both of these field operators as well as the internal system variable $\hat{\rho}_i$ are all Fourier transformed. The Fourier transformation of equations (3.1) and (3.2) are given by

$$
\begin{align*}
&i\omega \hat{\rho}(\omega) = (Q^T - YP^z P^T)\hat{\rho}(\omega) - 2YP^z \hat{A}(\omega), \\
&\hat{A}'(\omega) = P^T \hat{\rho}(\omega) + \hat{A}(\omega),
\end{align*}
$$

where the tilde notation denotes the Fourier-transformed operator. Note that for instance $\hat{A}_i(\omega)$ is not the Fourier transformation of $\hat{A}_i(t)$ but $\hat{a}_i(t)$. Also, more precisely, we should take Laplace transformation $\hat{a}_i(t) \to \hat{A}_i(s)$ and set $s = +0 + i\omega$ to obtain the Fourier transformation; for the rigorous treatment, see Gough and co-workers [30–32]. As a result, we have the input–output map from $\hat{A}(\omega)$ to $\hat{A}'(\omega)$:

$$
\hat{A}'(\omega) = F(\omega)\hat{A}(\omega), \quad F(\omega) := I_m - 2P^T (i\omega - Q^T + YP^z P^T)^{-1} YP^z. \quad (3.3)
$$

Here, we have used the Hurwitz property to justify that the initial contribution of the system was ignored in equation (3.3). The $m \times m$ matrix $F(\omega)$, called the transfer function matrix, has a striking property as shown below.

**Proposition 3.2.** The transfer function matrix $F(\omega)$ is unitary for all $\omega$.

**Proof.** The proof directly follows from lemma 2 of Gough et al. [30], but here it is given for convenience of readers. First, to simplify the calculation, let us define $\hat{P} = Y^{1/2} P$ and $\hat{Q} = Y^{1/2} Q Y^{-1/2}$. As $Q = -iRY - Y^{-1} \Gamma^T$, $\hat{Q}$ is skew Hermitian;
\[ \ddot{Q} + \dot{Q}^i = 0. \] With this notation, the transfer function matrix is represented by
\[ F(\omega) = I_m - 2\dot{P}^T(\iota\omega - \dot{Q}^T + \dot{P}^\sharp\dot{P}^\top)^{-1}\dot{P}^\sharp. \] Therefore, we have
\[ F(\omega)^\dagger F(\omega) = I_m - 2\dot{P}^T(-\iota\omega - \dot{Q}^T + \dot{P}^\sharp\dot{P}^\top)^{-1}\dot{P}^\sharp - 2\dot{P}^T(\iota\omega - \dot{Q}^T + \dot{P}^\sharp\dot{P}^\top)^{-1}\dot{P}^\sharp \\
+ 2\dot{P}^T(-\iota\omega - \dot{Q}^T + \dot{P}^\sharp\dot{P}^\top)^{-1}(2\dot{P}^\sharp\dot{P}^\top)(\iota\omega - \dot{Q}^T + \dot{P}^\sharp\dot{P}^\top)^{-1}\dot{P}^\sharp. \]

(3.4)

Here, it follows from \( \ddot{Q} + \dot{Q}^i = 0 \)
\[ 2\dot{P}^\sharp\dot{P}^\top = (-\iota\omega - \dot{Q}^T + \dot{P}^\sharp\dot{P}^\top) + (\iota\omega - \dot{Q}^T + \dot{P}^\sharp\dot{P}^\top). \]

Substituting this expression for the last term of equation (3.4), we end up with the relation
\[ F(\omega)^\dagger F(\omega) = I_m; \] hence \( F(\omega) \) is unitary for all \( \omega \).

This result states that the output power spectrum is flat in all the frequency domain, i.e. \( \langle \tilde{\mathbf{A}}'(\omega)\tilde{\mathbf{A}}'(\omega)^\dagger \rangle = F(\omega)\langle \tilde{\mathbf{A}}(\omega)\tilde{\mathbf{A}}(\omega)^\dagger \rangle F(\omega)^\dagger = I_m \). This means that, at steady state, it is impossible to extract any information about the internal system, as long as the matrix \( Q^T - YP^\sharp P^\top \) is Hurwitz and the input fields are in vacuum or coherent states.

Now we arrive at the stage where the physical meanings of the conditions given in theorem 2.1 can be clarified. First of all, the structure of equations (3.1) and (3.2) as well as the above two propositions remind us that the dynamics of the nullifier is a generalization of that for the simple single-mode optical damped cavity whose QSDE is described by
\[ \text{d}\hat{a}_t = \left( i\Delta - \frac{\kappa}{2} \right) \hat{a}_t \text{dt} - \sqrt{\kappa} \text{d}\hat{A}_t, \quad \text{d}\hat{A}_t = \sqrt{\kappa} \hat{a}_t \text{dt} + \text{d}\hat{A}_t, \]

(3.5)

where \( \hat{a}_t \) and \( \hat{A}_t \) denote the intra-cavity mode and the incoming vacuum field mode, respectively. \( \Delta \) and \( \kappa \) denote the detuning and the damping rate, respectively. Clearly, the state evolves into the vacuum, and also we have \( \langle \hat{A}'(\omega)^*\hat{A}'(\omega) \rangle = \langle \hat{A}(\omega)^*\hat{A}(\omega) \rangle = 1 \) for all \( \omega \), as in the nullifier case. These properties arise owing to (i) that energy is not supplied through the Hamiltonian, (ii) that the field does not supply energy but simply brings about the damping of the system and (iii) that the system is asymptotically stable. Mathematically, the first two statements mean that the dynamics does not contain the creation operators \( \hat{a}_t^* \) and \( \hat{A}_t^* \). Note that if the cavity contains a degenerate parametric amplifier, which is described by the Hamiltonian \( \hat{H}_{\text{DPA}} = i(\hat{a}_t^*^2 - \hat{a}_t^2) \), then the QSDE needs to be described in terms of both \( \hat{a} \) and \( \hat{a}^* \). The last condition (iii) guarantees that the state uniquely converges into the vacuum as well as that the output field does not contain any information about the system at steady state. Systems having the properties (i)–(iii) are in general called passive systems [30,33–35].

The above discussion implies that the nullifier dynamics is passive; more precisely, we obtain the physical meanings of the conditions (2.10)–(2.12) as follows:

— the matrix \( C \) has the form given in equation (2.10) so that the creation process \( \hat{A}_t^* \) does not appear in the QSDE of \( \hat{r}_t \); as mentioned earlier, this

_Phil. Trans. R. Soc. A_ (2012)
is equivalent to that there is no energy supply from the environment to the nullifier;
— the matrix $G$ has the form given in equation (2.11) so that the corresponding Hamiltonian $\hat{H} = \hat{x}^\top G \hat{x}/2$ does not supply energy for the nullifier; and
— the rank condition (2.12) implies that the coefficient matrix $Q^\top - Y P^\times P^\top$ is Hurwitz, or equivalently the asymptotic stability of the dynamics of the nullifier. This guarantees that the output power spectrum is flat in all frequencies, meaning that the output field does not contain any information about the system at steady state.

The last statement can be understood by studying the filtering equation [18,36], which enables us to update the conditional state based on the measurement result of the output field. In general, when the state of the master equation reaches the steady state and it is pure, then the corresponding filtering equation is identical to the master equation, meaning that we do not obtain any new information through measuring the output field for updating our knowledge. Note that this does not mean that the system is not controllable.

4. Quantum state transfer for dissipative system engineering

We have seen in theorem 2.1 how the dissipative channel $\hat{L}_k = c_k^\top \hat{x}$ with $C = (c_1^\top, \ldots, c_m^\top)$ should be chosen to engineer a desired Gaussian dissipative system. When we aim to generate a certain (useful) Gaussian state, however, it often turns out that the resulting $\hat{L}_k$ has to possess a specific structure that is hard to actually implement. For instance, a dissipative channel interacting with all the nodes, i.e. $\hat{L} = \ell_1 \hat{q}_1 + \ell_2 \hat{q}_2 + \cdots + \ell_n \hat{q}_n$, will be hard to implement. In this section, for the specific case where the system is subjected to $n$ independent input optical fields (i.e. $m = n$), we provide a practical procedure for implementing desired dissipative channels.

Let us first introduce the field quadratures $(k = 1, \ldots, n)$

$$
\hat{Q}_k = \frac{(\hat{A}_k + \hat{A}_k^\times)}{\sqrt{2}}, \quad \hat{P}_k = \frac{(\hat{A}_k - \hat{A}_k^\times)}{\sqrt{2i}},
$$

(4.1)

which satisfy the CCR $[d\hat{Q}_i, d\hat{P}_j] = \delta_{ij} dt$. Then, defining $\hat{Q} = (\hat{Q}_1, \ldots, \hat{Q}_n)^\top$ and $\hat{P} = (\hat{P}_1, \ldots, \hat{P}_n)^\top$, we find that the QSDE (2.8) is rewritten by

$$
d\hat{x}_t = A \hat{x}_t dt + \sqrt{2} \Sigma (C_r^\top, C_i^\top) d\hat{W}_t, \quad \hat{W}_t = \begin{pmatrix} \hat{Q}_t \\ \hat{P}_t \end{pmatrix},
$$

(4.2)

where $C_r = \Re(C)$ and $C_i = \Im(C)$.

The situation we have in mind is that a desired pure Gaussian state of light is first generated, and then that state is transferred to the system through the system–field coupling (figure 1). This is the framework of the quantum state transfer [26–28]; in this case, the system is called the memory and it should be independent of the input state we will transfer. More specifically, the target mode $\hat{W}_t^S$ is obtained from the vacuum mode $\hat{W}_t$ through the transformation

Phil. Trans. R. Soc. A (2012)
Figure 1. Quantum state transfer from the input mode $\hat{\mathcal{W}}_t^S$ to the memory mode $\hat{x}_t$.

$\hat{\mathcal{W}}_t^S = S\hat{\mathcal{W}}_t$ where the symplectic matrix $S$ is given in equation (2.1); then the quantum Ito rule (2.5) gives

$$d\hat{\mathcal{W}}_t^S(d\hat{\mathcal{W}}_t^S)^T = Sd\hat{\mathcal{W}}_t d\hat{\mathcal{W}}_t^T S^T = \frac{S(I_n + i\Sigma)S^T dt}{2} = \left(\frac{SS^T}{2} + \frac{i\Sigma}{2}\right) dt,$$

implying that the covariance matrix of the input field $\hat{\mathcal{W}}_t^S$ is certainly $V = SS^T/2$ (in the rigorous sense, this statement should be given in terms of the power spectrum density; see Gough & Wildfeuer [31]). Note that $S$ can contain a squeezing process, which was not included in the original QSDE framework of Hudson & Parthasarathy [17]; see Gough et al. [32] for a detailed discussion. Now the system (4.2) is written as

$$d\hat{x}_t = A\hat{x}_t dt + B d\hat{\mathcal{W}}_t^S, \quad B := \sqrt{2}\Sigma(C_r^T, C_i^T)S^{-1}, \quad (4.3)$$

where, as shown earlier, the new input field $\hat{\mathcal{W}}_t^S$ carries information of the target Gaussian state. The system, which serves as a memory, should satisfy the following two requirements.

(R1) The memory system (4.3) should not possess any information about the input state; that is, the system’s coefficient matrices $A$ and $B$ should be independent on $Z = X + iY$.

(R2) The state of the memory system (4.3) should converge to the target Gaussian state with covariance matrix $V_\infty = SS^T/2$. That is, $C$ and $G$ should be of the form (2.10) and (2.11) with $P$ and $Q$ satisfying the rank condition (2.12).

Below, we give a characterization of the desired memory system.

**Proposition 4.1.** Assume that the system satisfies the requirements (R1) and (R2). Then, the system has to be of the form

$$d\hat{x}_t = -2\kappa^2\hat{x}_t dt - 2\kappa d\hat{\mathcal{W}}_t^S, \quad (4.4)$$

where $\kappa$ is a scalar constant.
Proof. Note that $S^{-1} = \Sigma S^\top \Sigma^\top$. Then, substituting $C = P^\top (-Z, I_n)$ for $B$ in equation (4.3), we have

$$B = \sqrt{2} \begin{pmatrix} -P_2 Y^{1/2} - P_1 Y^{-1/2} X & P_1 Y^{-1/2} \\ -(Y P_1 + X P_2) Y^{1/2} - (X P_1 - Y P_2) Y^{-1/2} X & (X P_1 - Y P_2) Y^{-1/2} \end{pmatrix},$$

where $P_1 = \Re(P)$ and $P_2 = \Im(P)$. First let us look at the $(2,2)$ block matrix; because $X$ can take any symmetric matrix, here we set $X = 0$, implying $Y P_2 Y^{-1/2}$ is independent on $Y$. This readily implies that $P_2$ has to be of the form $P_2 = \sqrt{2} \kappa Y^{-1/2}$ with $\kappa$ a constant. Then, $X P_1 Y^{-1/2}$ has to be independent on $X$ and $Y$. But as the $(1,2)$ block matrix $\Theta := P_1 Y^{-1/2}$ also has to be independent on $X$ and $Y$, thus this is the case for $X \Theta$ as well. Then, $\Theta = 0$ is only allowed, and hence we obtain $P_1 = 0$. With these selections of $P_1$ and $P_2$, the $(1,1)$ and $(2,1)$ block matrices of $B$ take $-2\kappa I_n$ and zero, respectively. As a result, $B = -2\kappa I_{2n}$.

Next let us consider the matrix $A = \Sigma (G + \Im(C^\top C))$. From the above discussion, now we have $C = \sqrt{2} \kappa Y^{-1/2} (-Z, I_n)$, which leads to $A = \Sigma G - 2\kappa^2 I_{2n}$. This means that the matrix $G$ given in equation (2.11) must be independent on $X$ and $Y$. Then, similar to the above discussion, by setting $X = 0$, we find that $G = (Y R Y, \Gamma Y^{-1}; Y^{-1} \Gamma^\top, R)$ has to be independent on $X$ and $Y$. But this requirement is satisfied only when $R = 0$ and $\Gamma = 0$; as a result, we have $G = 0$. \hfill \blacksquare

This proposition states that, in order to dissipatively generate a desired pure Gaussian state in the state transfer set-up, we are required to prepare identical and independent oscillators as memories. Note that any pure Gaussian cluster state (see §5) can be effectively generated from the vacuum fields by applying suitably combined two-mode squeezing Hamiltonians and beam splitters [25], and hence the proposed scheme is practical.

5. Examples

(a) Example 1: Gaussian cluster state generation

It was shown in Menicucci et al. [21] that the graph matrix $Z = X + i Y$ can be used to capture several Gaussian graph states in a convenient graphical manner. In particular, the so-called canonical Gaussian cluster state [22–25], which plays an essential role in continuous-variable one-way quantum computation, corresponds to

$$Z = X + i e^{-2\alpha} I_n,$$

where $X$ is the symmetric adjacency matrix representing the graph structure of the cluster state; for instance, the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

represent the chain, T-shape and square structures, respectively (figure 2). On the other hand, $Y = e^{-2\alpha} I$ corresponds to the approximation error of the state with covariance matrix (2.1) to the ideal cluster state; i.e. bigger $\alpha$ means that
pure Gaussian state generation via dissipation

(a)  (b)  (c)

1 2 3 4
3 1 2
1 2

Figure 2. Typical cluster state with (a) chain, (b) T-shape and (c) square structures.

the state well approximates the ideal cluster state having the graph structure assigned by $X$.

Now, the nullifier dynamics (3.1) is of the form

$$d\hat{\gamma} = (Q^\top - e^{-2a} P^x P^\top)\hat{\gamma} dt - 2e^{-2a} P^x d\hat{A}_t,$$

and the real part of the eigenvalue of the coefficient matrix $Q^\top - e^{-2a} P^x P^\top$ is

$$\Re(\lambda) = -e^{-2a} \| P^\top b \|,$$

where $b$ is the corresponding eigenvector. This means that making $a$ bigger, or equivalently making the state more close to the ideal cluster state, renders the stability of the nullifier dynamics worse. Another observation from a more practical viewpoint is as follows. Let us define the convergence time to the target by

$$T = \frac{1}{\min |\Re(\lambda)|},$$

and denote the approximation error of the state to the ideal cluster one by $\epsilon = e^{-2a}$. Then, it is straightforward to find $T \epsilon \geq c$ with $c$ a constant. Therefore, in order to dissipatively generate a pure Gaussian state that is very close to a desired cluster state, the convergence time has to be long.

More generally, as shown in Menicucci et al. [21], the matrices $X$ and $Y$ respectively correspond to the ideal and realistic parts of a Gaussian graph state in the sense that the covariance matrix of $(-X, I_n)\hat{x}$ is given by $Y/2$. That is, $Y$ can be regarded as the approximation error in approximating the ideal graph structure $X$. Therefore, the above-mentioned trade-off holds for a general Gaussian graph state.

(b) Example 2: two-mode squeezed state

There exist a number of proposals to generate a steady two-mode squeezed state in, for instance, atomic ensembles or nano-mechanical oscillators. The system matrices describing the two-mode squeezed state are given by $X = 0$ and

$$Y = \begin{pmatrix}
\cosh(2\alpha) & -\sinh(2\alpha) \\
-\sinh(2\alpha) & \cosh(2\alpha)
\end{pmatrix},$$

where $\alpha$ denotes the squeezing parameter representing the degree of entanglement. In [10,11], the dissipative channels achieving this goal were shown to be

$$\hat{L}_1 = \mu \hat{a}_1 + \nu \hat{a}_1^*, \quad \hat{L}_2 = \mu \hat{a}_2 + \nu \hat{a}_2^*,$$

where $\mu = \cosh(\alpha)$ and $\nu = -\sinh(\alpha)$, while $\hat{H} = 0$. In our formulation, this corresponds to setting $[13]$

$$P = \begin{pmatrix}
icosh(\alpha) & isinh(\alpha) \\
isin(\alpha) & icosh(\alpha)
\end{pmatrix}, \quad R = 0, \quad I = 0.$$

However, the dissipative channels (5.3) are not easy to implement, because they are global and non-trivial coupling between the systems and the environment.
On the other hand, this dissipative system can be more easily implemented within the state transfer framework provided in §4, because we are only required to generate a two-mode squeezed state of optical fields and prepare two identical and independent oscillators. Note that a two-mode squeezed state of light can be effectively generated using a non-degenerate OPO.

6. Conclusion

In this paper, the dissipation-based state preparation method for the general Gaussian case, which was originally formulated in [13], was reconsidered in terms of the QSDE. This approach clarified that the nullifier dynamics of any Gaussian system generating a unique pure steady state is passive. As a by-product, it was shown that there exists a trade-off between the closeness of the steady state to a given ideal graph state and the convergence time to that state. In addition, a convenient physical implementation method of a desired Gaussian dissipative system was provided; the scheme has the structure of quantum state transfer, which is a key ingredient in quantum information technologies.

References