Single photon quantum filtering using non-Markovian embeddings

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We determine quantum master and filter equations for continuous measurement of systems coupled to input fields in certain non-classical continuous-mode states, specifically single photon states. The quantum filters are shown to be derivable from an embedding into a larger non-Markovian system, and are given by a system of coupled stochastic differential equations.

1. Introduction

In recent years, single photon states of light and superpositions of coherent states have become increasingly important owing to applications in quantum technology, in particular quantum computing and quantum information systems [1–5]. For instance, the light may interact with a system, say an atom, quantum dot or cavity, and this system may be used as a quantum memory [3], or to control the pulse shape of the single photon state [4]. When light interacts with a quantum system, information about the system is contained in the scattered light (output) and this may be used to monitor or control the system. The problem of extracting information from continuous measurement of the scattered light is a problem of quantum filtering [6–13]; however, this has tended to consider inputs only in a vacuum or other Gaussian state, with quadrature or counting measurements. The purpose of this paper is to solve a quantum filtering problem for systems driven by fields in single photon states.

When the input field is in a single photon state, the master equation describing unconditional dynamics was shown to be a system of coupled equations in Gheri et al. [14], an apparently non-Markovian feature. Markovian embeddings were used in Breuer [15] to derive quantum trajectory equations (quantum filtering equations) for a class of non-Markovian master equations. In recent work, the authors have shown how to construct ancilla systems to combine with the

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Figure 1. A system initialized in a state $|\eta\rangle$ coupled to a field in a state $|\Psi\rangle$ (single photon or superposition of coherent states). The output field is continuously monitored by homodyne detection (assumed perfect) to produce a classical measurement signal $Y(t)$. The output $Y(t)$ is filtered to produce estimates $\hat{X}(t) = \pi_{t}(X)$ of system operators $X$ at time $t$. (Online version in colour.)

system of interest to form a Markovian-extended system driven by vacuum from which quantum filtering results may be obtained for single photon states and superpositions of coherent states from the standard filter for the extended system [16,17]. However, depending on the complexity of the non-classical state, it may be difficult to determine suitable ancilla systems. In this paper, we present an alternative approach where the extended system forms a non-Markovian system, with the ancilla, system and field initialized in a superposition state. While standard filtering results do not apply, the quantum stochastic methods can, nevertheless, be applied to determine the quantum filters. In this way, we expand the range of methods that may be applied to derive quantum filters for non-classical states. Full technical details, together with a treatment of superposition states, are provided in Gough et al. [18].

2. Quantum filtering: problem formulation

We consider a quantum system $S$ coupled to a quantum field $B$, as shown in figure 1. The field $B$ has two components: the input field ($B_{\text{in}}$) and the output field ($B_{\text{out}}$). In what follows, the system $S$ is assumed to be defined on a Hilbert space $\mathcal{H}_S$, with an initial state denoted $|\eta\rangle \in \mathcal{H}_S$. The input field ($B_{\text{in}}$) is described in terms of annihilation $B(x)$ and creation $B^{\dagger}(x)$ operators defined on a Fock space $\mathfrak{F}$ ([19, ch. II] and [10, §4]), with initial state $|\Psi\rangle$. The quantum expectation will be denoted by

$$\mathbb{E}[\cdot] = \langle \eta \otimes \Psi| \cdot |\eta \otimes \Psi \rangle,$$

and we sometimes write $\mathbb{E}_{\eta\Psi}$ for emphasis. As illustrated in figure 1, the field interacts with the quantum system $S$, and the results of this interaction provide information about the system that may be obtained through continuous measurement of an observable $Y(t)$ of the output field $B_{\text{out}}(t)$.

(a) The filtering problem

Given an observable $X$ of the system, determine the form of the best estimate $\pi_{t}(X)$ of the observable $X$ in the Heisenberg picture at time $t$ based on the readout $\{Y(s), 0 \leq s \leq t\}$. Or equivalently, in the Schrödinger picture, determine the conditional state from which the statistical properties of $\pi_{t}(X)$ may be computed.

The dynamics of the system will be described, using the quantum stochastic calculus [10,19–22]. Quantum stochastic integrals are defined in terms of
fundamental field operators $B(t), B^*(t)$ and $A(t)$ ([19, ch. II] and [10, §4]).\(^1\) The non-zero Itô products for the field operators are
\[
\begin{align*}
\mathrm{d}B(t)\mathrm{d}B^*(t) &= \mathrm{d}t, \quad \mathrm{d}B(t)\mathrm{d}A(t) = \mathrm{d}B(t) \\
\mathrm{d}A(t)\mathrm{d}A(t) &= \mathrm{d}A(t), \quad \mathrm{d}A(t)\mathrm{d}B^*(t) = \mathrm{d}B^*(t).
\end{align*}
\]
Equation (2.1)

The dynamics of the composite system is described by a unitary $U(t)$ solving the Schrödinger equation, or quantum stochastic differential equation,
\[
\mathrm{d}U(t) = \{(S - I)\mathrm{d}A(t) + L\mathrm{d}B^*(t) - L^*S\mathrm{d}B(t) - (\frac{1}{i}L^*L + iH)\mathrm{d}t\}U(t),
\]
with initial condition $U(0) = I$. Here, $H$ is a fixed self-adjoint operator representing the free Hamiltonian of the system, and $L$ and $S$ are system operators determining the coupling of the system to the field, with $S$ unitary. For the sake of simplicity, we assume that the parameters $S, L, H$ are bounded; however, we remark that under some suitable additional conditions the results should also apply to unbounded parameters [23,24].

(b) The plant dynamics

A system operator $X$ at time $t$ is given in the Heisenberg picture by $X(t) = j_t(X) = U(t)^*X(U(t))$ and it follows from the quantum Itô calculus that
\[
\begin{align*}
\mathrm{d}j_t(X) &= j_t(S^*XS - X)\mathrm{d}A(t) + j_t(S^*[X, L])\mathrm{d}B(t)^* \\
&\quad + j_t([L^*, X]S)\mathrm{d}B(t) + j_t(\mathcal{L}(X))\mathrm{d}t,
\end{align*}
\]
where $\mathcal{L}(X) = \frac{1}{2}L^*X + \frac{1}{2}[L^*, X]L - i[X, H]$ is known as the Lindblad generator, and the quartet of maps $X \mapsto L(X), S^*XS - X, S^*[X, L], [L^*, X]S$ are known as Evans–Hudson maps.

(c) The read-out process (field quadrature)

The output field is given by $B_{\text{out}}(t) = U(t)^*B(t)U(t)$.\(^2\) In this paper, we consider the output field observable $Y(t) = U(t)^*Z(t)U(t)$, where $Z(t) = B(t) + B^*(t)$ is a quadrature observable of the input field (the counting case $Z(t) = A(t)$ is discussed in Gough et al. [17]). Note that both $Z(t)$ and $Y(t)$ are self-adjoint and self-commutative: $[Z(t), Z(s)] = 0$ and $[Y(t), Y(s)] = 0$. We write $\mathcal{Z}_t$ and $\mathcal{Y}_t$ for the subspaces of commuting operators generated by the observables $Z(s), Y(s), 0 \leq s \leq t$, respectively.\(^3\) They are related by the unitary rotation $\mathcal{Y}_t = U(t)^*\mathcal{Z}_tU(t)$. Physically, $Y(t)$ may represent the integrated photocurrent arising in an idealized (perfect) homodyne photodetection scheme [6,13,25], as in figure 1.

\(^1\)In terms of annihilation and creation of white noise operators $b(t), b^*(t)$ that satisfy singular commutation relations $[b(s), b^*(t)] = \delta(t - s)$, the fundamental field operators are given by $B(t) = \int_0^t b(s)\mathrm{d}s$, $B^*(t) = \int_t^0 b^*(s)\mathrm{d}s$ and $A(t) = \int_0^t b^*(s)b(s)\mathrm{d}s$. Also, we may write $B(\xi) = \int_0^\infty \xi^*(s)\mathrm{d}B(s)$.

\(^2\)Recall $B(t) = B_{\text{in}}(t)$ is the input field.

\(^3\) $\mathcal{Z}_t$ and $\mathcal{Y}_t$ are commutative von Neumann algebras. They are also filtrations, e.g. $\mathcal{Z}_t \subset \mathcal{Z}_s$ whenever $t_1 < t_2$.

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Our goal is to derive the quantum filter for the quantum conditional expectation [10, definition 3.13]

\[ \pi_t(X) = \mathbb{E}_{\eta|\psi}[X(t)|\mathcal{B}_t]. \]  

(2.4)

This conditional expectation is well defined, because \( X(t) \) commutes with the subspace \( \mathcal{B}_t \) (non-demolition condition). The conditional estimate \( \pi_t(X) \) is affiliated to \( \mathcal{B}_t \) (written in abbreviated fashion as \( \pi_t(X) \in \mathcal{B}_t \) and is characterized by the requirement that

\[ \mathbb{E}_{\eta|\psi}[\pi_t(X) K] = \mathbb{E}_{\eta|\psi}[X(t) K], \quad \forall K \in \mathcal{B}_t. \]  

(2.5)

State of the field

In this paper, we consider the input field to be in the single photon state \( |\psi\rangle = |1_\xi\rangle \), where \( \xi \) is a complex-valued function such that \( \int_0^\infty |\xi(s)|^2 \, ds = 1 \) (representing the wave packet shape).

Vacuum input field

We recall that the quantum filter for the case of vacuum field input satisfies the differential equation

\[ d\pi_t(X) = \pi_t(\mathcal{L}(X)) \, dt + (\pi_t(XL + L^\dagger X) - \pi_t(L + L^\dagger) \pi_t(X)) \, dW(t), \]  

(2.6)

where \( W \) is a Wiener process, called the innovations process, and is given by \( dW(t) = dY(t) - \pi_t(L + L^\dagger) \, dt \). Alternatively, we may express this in the Schrödinger picture by introducing the conditional density \( \rho(t) \),

\[ d\rho(t) = \mathcal{L}'\rho(t) \, dt + \mathcal{H}[L] \rho(t) \, dW(t), \]  

where \( \mathcal{H}[A]B = AB + BA^\dagger - \text{tr}[(A+A^\dagger)B]B \) and \( \mathcal{L}' \) denotes the superoperator ‘conjugate’ of \( \mathcal{L} \). This is the way the stochastic master equation is usually written for the homodyne detection case [13]. Clearly, taking the expectation, that is, averaging over all possible read-out, gives the master equation \( d\rho(t) = \mathcal{L}'\rho(t) \, dt \) because \( \mathbb{E}[dW(t)] = 0 \).

3. Master equation

Before deriving the quantum filter, we recall the dynamical equations for the unconditioned single photon expectation [14,16]. We begin by defining the single photon states.

The continuous-mode single photon state \( |\psi\rangle = |1_\xi\rangle \) is ([26, §6.3] and [4, eqn (9)])

\[ |1_\xi\rangle = B^*(\xi)|0\rangle, \]  

(3.1)

where \( \xi \) is a complex-valued function such that \( \int_0^\infty |\xi(s)|^2 \, ds = 1 \), and \( |0\rangle \) is the vacuum state of the field. Expression (3.1) says that the single photon wavepacket

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with temporal shape $\xi$ is created from the vacuum using the field operator $B^*(\xi)$. We introduce the notation
\[ E_{jk}(X \otimes F) = \langle \Phi_j | (X \otimes F) | \Phi_k \rangle, \]
(3.2)
where $\Phi_0 = |\eta 0\rangle$ and $\Phi_1 = |\eta 1\rangle$, so that $E_{00}$ and $E_{11}$ are expectation with respect to the product states $|\eta 0\rangle$ and $|\eta 1\rangle$, respectively. By setting $B_t^- (\xi) = B(\xi \chi_{[0, t]}), B_t^+ (\xi) = B(\xi \chi_{[t, \infty)}), we have $E_{11}[K(t)] = E_{00}[B_t^-(\xi)K(t)B_t^+(\xi) + r(t)K(t)]$, with $r(t) = \int_t^\infty |\xi(s)|^2 \, ds$.

We express the master equation in Heisenberg form using the expectations
\[ \hat{\mu}_t^{jk}(X) = E_{jk}[X(t)]. \]
(3.3)
Note that for all $t \geq 0$ we have $\mu_t^{00}(I) = 1 = \mu_t^{11}(I), \mu_t^{01}(I) = 0 = \mu_t^{10}(I)$.

The master equation in Heisenberg form for the system when the field is in the single photon state $|1\rangle$ is given by the system of equations
\[ \dot{\mu}_t^{11}(X) = \mu_t^{11}(\mathcal{L}(X)) + \mu_t^{01}(S^*[X, L]X)\xi^*(t) + \mu_t^{10}([L^*, X]S)\xi(t) + \mu_t^{00}(S^*XS - S)|\xi(t)|^2, \]
(3.4)
\[ \dot{\mu}_t^{10}(X) = \mu_t^{10}(\mathcal{L}(X)) + \mu_t^{00}(S^*[X, L]X)\xi^*(t), \]
(3.5)
\[ \dot{\mu}_t^{01}(X) = \mu_t^{01}(\mathcal{L}(X)) + \mu_t^{00}([L^*, X]S)\xi(t) \]
(3.6)
and
\[ \dot{\mu}_t^{00}(X) = \mu_t^{00}(\mathcal{L}(X)). \]
(3.7)
The initial conditions are $\mu_0^{11}(X) = \mu_0^{00}(X) = \langle \eta, X \eta \rangle, \mu_0^{10}(X) = \mu_0^{01}(X) = 0$.

It is apparent that the single photon expectation $\mu_t^{11}(X) = E_{11}[X(t)]$ cannot be determined from a single differential equation, but instead by a system of coupled equations (3.4)–(3.7). Note that the unitary matrix $S$ appearing in the Schrödinger equation (2.2) does appear in the single photon master equations (3.4)–(3.7), in contrast to the vacuum case (which corresponds to (3.7)).

In order to obtain a Schrödinger form of the master equations, we define (generalized) density operators $\rho^{jk}(t)$ by
\[ \langle \rho^{jk}(t), X \rangle = \mu_t^{jk}(X). \]
(3.8)
The operators $\rho^{jk}(t)$ enjoy the symmetry properties
\[ \rho^{00*}(t) = \rho^{00}(t), \quad \rho^{01*}(t) = \rho^{10}(t), \quad \rho^{11*}(t) = \rho^{11}(t). \]
(3.9)
The master equation in Schrödinger form for the system $S$ when the field is in the single photon state $|\xi\rangle$ is given by the system of equations
\[ \dot{\rho}^{11}(t) = \mathcal{L}^*(\rho^{11}(t)) + [S\rho^{01}(t), L^*]\xi(t) + [L, \rho^{10}(t)S^*]\xi^*(t) \]
\[ + (S\rho^{00}(t)S^* - \rho^{00}(t))|\xi(t)|^2, \]
(3.10)
\[ \dot{\rho}^{10}(t) = \mathcal{L}^*(\rho^{10}(t)) + [S\rho^{00}(t), L^*]\xi(t), \]
(3.11)
\[ \dot{\rho}^{01}(t) = \mathcal{L}^*(\rho^{01}(t)) + [L, \rho^{00}(t)S^*]\xi^*(t) \]
(3.12)
and
\[ \dot{\rho}^{00}(t) = \mathcal{L}^*(\rho^{00}(t)). \]
(3.13)

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Figure 2. System embedded in the extended system. Although there is no coupling between the system and ancilla two-level system, the ancilla, system and field are assumed to be initialized in a superposition state \(|\Sigma\rangle\) defined in equation (4.3). (Online version in colour.)

The initial conditions are

\[ \rho^{11}(0) = \rho^{00}(0) = |\eta\rangle\langle\eta|, \quad \rho^{10}(0) = \rho^{01}(0) = 0. \]  

(3.14)

An example of the master equation is presented in §8.

4. Embedding

We construct a suitable embedding for the system and single photon field, and show how the system of master equations (3.4)–(3.7) can be compactly represented as a single equation for a larger system. We should emphasize, however, that our embedding is not the same as that used in recent studies [15–17]. The embedding is illustrated in figure 2. Recall that the system and field are defined on a Hilbert space \(H_S \otimes F\). We define an extended space

\[ \tilde{H} = C^2 \otimes H_S = H_S \oplus H, \]  

(4.1)

which includes the system, field and an ancilla two-level system. As orthonormal basis for \(C^2\), we take \(|e_0\rangle = [1 0], |e_1\rangle = [0 1]\), and for \(A = [a_{11} a_{10} a_{01} a_{00}]\), we may represent \(A \otimes X \otimes F\) on the extended space \(\tilde{H}\) as

\[ A \otimes (X \otimes F) = \begin{bmatrix} a_{11}(X \otimes F) & a_{10}(X \otimes F) \\ a_{01}(X \otimes F) & a_{00}(X \otimes F) \end{bmatrix}. \]  

(4.2)

We allow the extended system to evolve unitarily according to \(I \otimes U(t)\), where \(U(t)\) is the unitary operator for the system and field, given by the Schrödinger equation (2.2). Note, in particular, that the system is not coupled to the ancilla \(C^2\), and observables of this two-level system are static. We initialize the extended system in the superposition state

\[ |\Sigma\rangle = \alpha_1 |e_1\eta \rangle + \alpha_0 |e_0\eta \rangle, \]  

(4.3)

where \(|\alpha_0|^2 + |\alpha_1|^2 = 1\). This state evolves according to \(|\Sigma(t)\rangle = (I \otimes U(t))|\Sigma\rangle\).
For notational convenience, we write
\[ w_{jk} = \alpha^*_j \alpha_k \] (4.4)
and note that \[ w = \sum_{jk} w_{jk} |e_j\rangle \langle e_k| \] is a density matrix for \( \mathbb{C}^2 \).

The expectation with respect to the superposition state \( |\Sigma\rangle \) is given by
\[ \tilde{\mu}_t(A \otimes X) = \mathbb{E}_\psi[A \otimes X(t)] = \langle \Sigma | (A \otimes X(t)) | \Sigma \rangle = \sum_{jk} w_{jk} \mu_{jk}^t(X). \] (4.5)

This expectation is correctly normalized, \( \mu_t(I \otimes I) = 1 \), and the expectations \( \mu_{jk}^t(X) \) (3.3) are scaled components of \( \tilde{\mu}_t(A \otimes X) \):
\[ \mu_{jk}^t(X) = \frac{w_{11} \tilde{\mu}_t(|e_j\rangle \langle e_k| \otimes X)}{w_{jk} \tilde{\mu}_t(|e_i\rangle \langle e_i| \otimes I)}, \] (4.6)
for \( w_{jk} \neq 0 \); otherwise, it can be set to, say, 0. We also have
\[ \mu_{jk}^t(X) = \frac{w_{11} \tilde{\mu}_t(|e_j\rangle \langle e_k| \otimes X)}{w_{jk} \tilde{\mu}_t(|e_1\rangle \langle e_1| \otimes I)}. \] (4.7)

5. Master equation in the extended system

In the extended space, the Schrödinger and Heisenberg pictures are related by
\[ \mathbb{E}_{\Sigma(t)}[A \otimes X \otimes F] = \mathbb{E}_\Sigma[A \otimes U^*(t)(X \otimes F) U(t)]. \] (5.1)

On the ancilla, we set \( \sigma_+ = |e_1\rangle \langle e_0| = [0 1 \ 1 0]^T \), \( \sigma_- = |e_0\rangle \langle e_1| = [1 0 \ 0 1]^T \).

Assume \( \alpha_0 \neq 0 \), then the expectation \( \tilde{\mu}_t(A \otimes X) \) (defined by (4.5)) evolves according to
\[ \dot{\tilde{\mu}}_t(A \otimes X) = \tilde{\mu}_t(\mathcal{G}_t(A \otimes X)), \] (5.2)
where, with \( \nu = \alpha_1 / \alpha_0 \),
\[ \mathcal{G}_t(A \otimes X) = A \otimes \mathcal{L}(X) + (A \sigma_+ \otimes [L^*, X] S \nu \xi(t) + (\sigma_- A) \otimes S^*[X, L] \nu^* \xi^*(t) + (\sigma_- A \sigma_+) \otimes (S^* XS - X) |\nu \xi(t)|^2. \] (5.3)

The reader may easily verify that the system of master equations (3.4)–(3.7) for \( \mu_{jk}^t(X) \), \( j, k = 1, 0 \), follows from equation (5.2) by setting \( A = |e_j\rangle \langle e_k| \).

6. Quantum filter for the extended system

The extended system provides a convenient framework for quantum filtering, because all expectations can be expressed in terms of \( |\Sigma\rangle \). Our immediate goal in this section is to determine the equation for the quantum conditional expectation
\[ \tilde{\pi}_t(A \otimes X) = \mathbb{E}_\Sigma[A \otimes X(t) | I \otimes \mathcal{B}_t], \] (6.1)
and we will explain how the quantum filter for the single photon field may be obtained from equation (6.1). The continuously monitored field observable that
corresponds to the conditional expectation (6.1) is \( I \otimes Y(t) = I \otimes U(t)^*ZU(t) \), and we have the corresponding output equation for the extended system:

\[
d(I \otimes Y(t)) = I \otimes (L(t) + L^*(t)) \, dt + I \otimes (S(t) \, dB(t) + S^*(t) \, dB^*(t)).
\] (6.2)

Note that an operator \( K \) in the unital commutative algebra \( I \otimes \mathcal{Y}_t \) has the form \( K = I \otimes \tilde{K} \), where \( \tilde{K} \in \mathcal{Y}_t \). By the spectral theorem [10, theorem 3.3], we may identify \( K \) and \( \tilde{K} \), both of which are equivalent to a classical stochastic process \( K_t(s), 0 \leq s \leq t \). The quantum conditional expectation \( \tilde{\pi}_t(A \otimes X) \in I \otimes \mathcal{Y}_t \) is well defined, because \( A \otimes X(t) \) is in the commutant \( I \otimes \mathcal{Y}_t' \) of the algebra \( I \otimes \mathcal{Y}_t \), and is characterized by the requirement that

\[
\mathbb{E}_\Sigma[\tilde{\pi}_t(A \otimes X)I \otimes K] = \mathbb{E}_\Sigma[(A \otimes X(t))(I \otimes K)]
\] (6.3)

for all \( K \in \mathcal{Y}_t \) [10, definition 3.13].

Assume \( \alpha_0 \neq 0 \). The conditional expectation \( \tilde{\pi}_t(A \otimes X) \) defined by (6.1) satisfies

\[
d\tilde{\pi}_t(A \otimes X) = \tilde{\pi}_t(\mathcal{G}_t(A \otimes X)) \, dt + \mathcal{H}_t(A \otimes X) \, dW(t),
\]

where

\[
\mathcal{H}_t(A \otimes X) = \tilde{\pi}_t(A \otimes (XL + L^*X)) - \tilde{\pi}_t(A \otimes X)\pi_t(I \otimes (L + L^*))
\]

\[
+ \tilde{\pi}_t((A\sigma_+) \otimes XS)\nu_\xi(t) + \tilde{\pi}_t((\sigma_- A) \otimes S^*X)\nu^*\xi^*(t)
\]

\[
- \tilde{\pi}_t(A \otimes X)\tilde{\pi}_t((\sigma_+ \otimes S)\nu_\xi(t) + (\sigma_- \otimes S^*)\nu^*\xi^*(t))
\] (6.4)

and

\[
dW(t) = dY(t) - \tilde{\pi}_t(I \otimes (L + L^*) + (\sigma_+ \otimes S)\nu_\xi(t) + (\sigma_- \otimes S^*)\nu^*\xi^*(t)) \, dt.
\] (6.5)

The process \( W(t) \) defined by (6.5) can be shown to be a \( I \otimes \mathcal{Y}_t \) Wiener process with respect to \( |\Sigma| \) [18, theorem 3.6] and is called the innovations process.

Notice the terms involving \( \sigma_{\pm} \) in the filter (equation (6.4)) and in the innovations process (equation (6.5)). These terms arise from expectations involving the single photon state. Note that, owing to the martingale property of the innovations process \( W(t) \), we see that if we take the expected value of equation (6.4) we recover equation (5.2), consistent with \( \mathbb{E}_\Sigma[\tilde{\pi}_t(A \otimes X)] = \tilde{\mu}_t(A \otimes X) \) and the definition of conditional expectation.

### 7. Single photon quantum filter

We now determine the quantum filter for the conditional state when the field is in the single photon state, as stated in equation (2.4). Define the conditional quantities \( \pi_t^{jk}(X) \) by

\[
\pi_t^{jk}(X) = \frac{w_{11} \tilde{\pi}_t(|e_j\rangle\langle e_k| \otimes X)}{w_{jk} \tilde{\pi}_t(|e_1\rangle\langle e_1| \otimes I)},
\] (7.1)

where \( \tilde{\pi}_t(A \otimes X) \) is the conditional state for the extended system defined by (6.1). Then, it is easy to see that for all \( K \in \mathcal{Y}_t \) we have \( \mathbb{E}_{11}[\pi_t^{jk}(X)K] = \mathbb{E}_{jk}[j_t(X)K] \). We can now present our main theorem for the quantum filter for the single photon field state [18, theorem 3.8].
Theorem 7.1. The quantum filter for the conditional expectation with respect to the single photon field is given in the Heisenberg picture by

\[ \pi_t(X) = \mathbb{E}_{\mathcal{Y}_t}[X(t) | \mathcal{Y}_t] = \pi^{11}_t(X), \quad (7.2) \]

where \( \pi^{11}_t(X) \) is defined by (7.1), and is given by the system of equations

\[ \text{d}\pi^{11}_t(X) = (\pi^{11}_t(L(X)) + \pi^{01}_t([S^*X, L])\xi^*(t) + \pi^{10}_t([L^*, X]S)\xi(t) + \pi^{00}_t(S^*XS - X)|\xi(t)|^2) \text{d}t \]
\[ + (\pi^{11}_t(XL + L^*X) + \pi^{01}_t(S^*X)\xi^*(t) + \pi^{10}_t(XS)\xi(t)) \text{d}W(t), \]
\[ - \pi^{11}_t(X)(\pi^{11}_t(L + L^*) + \pi^{01}_t(S)\xi(t) + \pi^{10}_t(S^*)\xi^*(t)) \text{d}W(t), \]
\[ \text{d}\pi^{10}_t(X) = (\pi^{10}_t(L(X)) + \pi^{01}_t(S^*[X, L])\xi^*(t)) \text{d}t \]
\[ + (\pi^{10}_t(XL + L^*X) + \pi^{00}_t(S^*X)\xi(t)) \text{d}t \]
\[ - \pi^{10}_t(X)(\pi^{11}_t(L + L^*) + \pi^{01}_t(S)\xi(t) + \pi^{10}_t(S^*)\xi^*(t)) \text{d}W(t), \]
\[ \text{d}\pi^{01}_t(X) = (\pi^{01}_t(L(X)) + \pi^{00}_t([L^*, X]S)\xi(t)) \text{d}t \]
\[ + (\pi^{01}_t(XL + L^*X) + \pi^{00}_t(XS)\xi(t)) \text{d}W(t) \]
\[ - \pi^{01}_t(X)(\pi^{11}_t(L + L^*) + \pi^{01}_t(S)\xi(t) + \pi^{10}_t(S^*)\xi^*(t)) \text{d}W(t) \]
\[ \text{and } \quad \text{d}\pi^{00}_t(X) = \pi^{00}_t(L(X)) \text{d}t + (\pi^{01}_t(XL + L^*X) \]
\[ - \pi^{00}_t(X)(\pi^{11}_t(L + L^*) + \pi^{01}_t(S)\xi(t) + \pi^{10}_t(S^*)\xi^*(t)) \text{d}W(t). \]

Here, the innovations process \( W(t) \) is a \( \mathcal{Y}_t \) Wiener process with respect to the single photon state and is defined by

\[ \text{d}W(t) = \text{d}Y(t) - (\pi^{11}_t(L + L^*) + \pi^{01}_t(S)\xi(t) + \pi^{00}_t(S^*)\xi^*(t)) \text{d}t. \quad (7.3) \]

The initial conditions are \( \pi^{11}_0(X) = \pi^{01}_0(X) = \langle \eta, X\eta \rangle, \pi^{00}_0(X) = \pi^{01}_0(X) = 0. \)

To obtain a stochastic master equation in Schrödinger form, we define (generalized) conditional density operators \( \hat{\rho}^{jk}(t) \) by

\[ \langle \hat{\rho}^{jk}(t), X \rangle = \pi^{jk}_t(X). \quad (7.4) \]

The operators \( \hat{\rho}^{jk}(t) \) enjoy the symmetry properties

\[ \hat{\rho}^{00*}(t) = \hat{\rho}^{00}(t), \quad \hat{\rho}^{01*}(t) = \hat{\rho}^{10}(t), \quad \hat{\rho}^{11*}(t) = \hat{\rho}^{11}(t). \quad (7.5) \]
The quantum filter for the conditional expectation with respect to the single photon field is given in the Schrödinger picture by the system of stochastic master equations

\[
\begin{align*}
d\hat{\rho}_{11}(t) &= (\mathcal{L}^*\hat{\rho}_{11}(t)) + [S\hat{\rho}_{01}(t), L^*]\hat{\xi}(t) + [L, \hat{\rho}_{10}(t)S^*]\hat{\xi}^*(t) \nonumber \\
&+ (S\hat{\rho}_{00}(t)S^* - \hat{\rho}_{00}(t))|\hat{\xi}(t)|^2 \, dt \\
&+ (L\hat{\rho}_{11}(t) + \hat{\rho}_{11}(t)L^* + \hat{\rho}_{10}(t)S^*\hat{\xi}^*(t) + S\hat{\rho}_{01}(t)\hat{\xi}(t) \\
&- (\pi_{11}^1(L + L^*) + \pi_{11}^0(S)\hat{\xi}(t) + \pi_{10}^0(S^*)\hat{\xi}^*(t))\hat{\rho}_{11}(t)) \, dW(t), \tag{7.6}
\end{align*}
\]

\[
\begin{align*}
d\hat{\rho}_{10}(t) &= (\mathcal{L}^*\hat{\rho}_{10}(t)) + [S\hat{\rho}_{00}(t), L^*]\hat{\xi}(t) \nonumber \\
&+ (L\hat{\rho}_{10}(t) + \hat{\rho}_{10}(t)L^* + S\hat{\rho}_{00}(t)\hat{\xi}(t) \\
&- (\pi_{11}^1(L + L^*) + \pi_{11}^0(S)\hat{\xi}(t) + \pi_{10}^0(S^*)\hat{\xi}^*(t))\hat{\rho}_{10}(t)) \, dW(t), \tag{7.7}
\end{align*}
\]

\[
\begin{align*}
d\hat{\rho}_{01}(t) &= (\mathcal{L}^*\hat{\rho}_{01}(t)) + [L, \hat{\rho}_{00}(t)S^*]\hat{\xi}^*(t) \nonumber \\
&+ (L\hat{\rho}_{01}(t) + \hat{\rho}_{01}(t)L^* + \hat{\rho}_{00}(t)S^*\hat{\xi}^*(t) \\
&- (\pi_{11}^1(L + L^*) + \pi_{11}^0(S)\hat{\xi}^*(t) + \pi_{10}^0(S^*)\hat{\xi}^*(t))\hat{\rho}_{01}(t)) \, dW(t), \tag{7.8}
\end{align*}
\]

and

\[
\begin{align*}
d\hat{\rho}_{00}(t) &= \mathcal{L}^*(\hat{\rho}_{00}(t)) \, dt + (L\hat{\rho}_{00}(t) + \hat{\rho}_{00}(t)L^* \\
&- (\pi_{11}^1(L + L^*) + \pi_{11}^0(S)\hat{\xi}^*(t) + \pi_{10}^0(S^*)\hat{\xi}^*(t))\hat{\rho}_{00}(t)) \, dW(t), \tag{7.9}
\end{align*}
\]

where \( W(t) \) is the innovations process given by (7.3). The initial conditions are

\[
\hat{\rho}_{11}(0) = \hat{\rho}_{00}(0) = |\eta\rangle\langle\eta|, \quad \hat{\rho}_{10}(0) = \hat{\rho}_{01}(0) = 0. \tag{7.10}
\]

8. Example

When the system is a two-level system or qubit, the filtering equations reduce to a finite set of stochastic differential equations. In this case, we have \( \mathcal{H}_S = \mathbb{C}^2 \), and the algebra of system operators is generated by the Pauli matrices

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.1}
\]

The system is specified by the parameters \( S = I, \ L = \sqrt{\kappa}\sigma_- \) and \( H = \omega\sigma_z \). Here, \( \kappa > 0 \) is a scalar parameter.

We begin with the master equations (3.10)–(3.13), and write

\[
\begin{align*}
\rho_{00} &= \frac{1}{2}(I + x^{00}\sigma_x + y^{00}\sigma_y + z^{00}\sigma_z), \tag{8.2}
\end{align*}
\]

\[
\begin{align*}
\rho_{01} &= \frac{1}{2}(x^{01}\sigma_x + y^{01}\sigma_y + z^{01}\sigma_z) = \rho^{10*} \tag{8.3}
\end{align*}
\]

and

\[
\begin{align*}
\rho_{11} &= \frac{1}{2}(I + x^{11}\sigma_x + y^{11}\sigma_y + z^{11}\sigma_z). \tag{8.4}
\end{align*}
\]

Note that \( x^{00}, y^{00}, z^{00} \) and \( x^{11}, y^{11}, z^{11} \) are real, whereas \( x^{01}, y^{01}, z^{01} \) may be complex. Also note, for example, \( \rho^{00}(\sigma_x) = x^{00}, \rho^{01}(\sigma_x) = x^{01*} \), etc. Then, we obtain
nine coupled equations for the nine coefficients:

\[
\begin{align*}
\dot{x}^{00} &= -2\omega y^{00} - \frac{\kappa}{2} x^{00}, \\
\dot{y}^{00} &= 2\omega x^{00} - \frac{\kappa}{2} y^{00}, \\
\dot{z}^{00} &= -\kappa (1 + z^{11}), \\
\dot{x}^{01} &= -\frac{\kappa}{2} x^{01} - 2\omega y^{01} - \sqrt{\kappa} \xi(t)^* z^{00}, \\
\dot{y}^{01} &= 2\omega x^{01} - \frac{\kappa}{2} y^{01} - i\sqrt{\kappa} \xi(t)^* z^{00}, \\
\dot{z}^{01} &= -\kappa z^{01} - \sqrt{\kappa} x^{00} \xi(t)^* + i\sqrt{\kappa} y^{00} \xi(t)^*, \\
\dot{x}^{11} &= -\frac{\kappa}{2} x^{11} - 2\omega y^{11} + \sqrt{\kappa} z^{10} \xi(t) + \sqrt{\kappa} z^{01*} \xi(t)^*, \\
\dot{y}^{11} &= 2\omega x^{11} - \frac{\kappa}{2} y^{11} + i\sqrt{\kappa} z^{01} \xi(t) - i\sqrt{\kappa} z^{01*} \xi(t)^* \\
\text{and} \quad \dot{z}^{11} &= -\kappa - \kappa z^{11} - \sqrt{\kappa} x^{10} \xi(t) - i\sqrt{\kappa} y^{10} \xi(t) - \sqrt{\kappa} x^{01*} \xi(t)^* + i\sqrt{\kappa} y^{01*} \xi(t)^*.
\end{align*}
\]

For the quantum filter (7.6)–(7.9), we use a slightly more general representation for \(\hat{\rho}^{jk}\) given by:

\[
\hat{\rho}^{jk} = \frac{1}{2}(\hat{c}^{jk} I + \hat{x}^{jk} \sigma_x + \hat{y}^{jk} \sigma_y + \hat{z}^{jk} \sigma_z),
\]

for \(j, k = 0, 1\). Because \(\langle \hat{\rho}^{11}, I \rangle = 1\) (i.e. \(\hat{\rho}^{11}\) is a normalized conditional density operator), we always have that \(\hat{c}^{11} = 1\) at all times. However, unlike the master equation, this will not be so for \(\hat{c}^{01}, \hat{c}^{10}, \hat{c}^{00}\), as these coefficients will evolve in time. This is the reason we need to consider the more general representation for \(\hat{\rho}^{jk}\). The quantum filter for the two-level system is given by the finite set of coupled equations

\[
\begin{align*}
\text{d}\hat{c}^{00} &= \left(\sqrt{\kappa} \hat{x}^{00} - \left(\sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \hat{c}^{01} \xi(t) + \frac{1}{2} \hat{c}^{01*} \xi(t)^*\right) \hat{c}^{00}\right) \text{d}W(t), \\
\text{d}\hat{x}^{00} &= \left(-2\omega y^{00} - \frac{\kappa}{2} x^{00}\right) \text{d}t \\
&\quad + \left(\sqrt{\kappa} \hat{x}^{00} - \left(\sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \hat{c}^{01} \xi(t) + \frac{1}{2} \hat{c}^{01*} \xi(t)^*\right) \hat{x}^{00}\right) \text{d}W(t), \\
\text{d}\hat{y}^{00} &= \left(2\omega x^{00} - \frac{\kappa}{2} y^{00}\right) \text{d}t - \left(\sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \hat{c}^{01} \xi(t) + \frac{1}{2} \hat{c}^{01*} \xi(t)^* \hat{y}^{00}\right) \text{d}W(t), \\
\text{d}\hat{z}^{00} &= \left(-\kappa (1 + z^{00})\right) \text{d}t \\
&\quad + \left(\sqrt{\kappa} \hat{x}^{00} - \left(\sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \hat{c}^{01} \xi(t) + \frac{1}{2} \hat{c}^{01*} \xi(t)^*\right) \hat{z}^{00}\right) \text{d}W(t), \\
\text{d}\hat{c}^{01} &= \left(\sqrt{\kappa} \hat{x}^{01} + \hat{c}^{00} \xi(t)^* - \left(\sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \hat{c}^{01} \xi(t) + \frac{1}{2} \hat{c}^{01*} \xi(t)^*\right) \hat{c}^{01}\right) \text{d}W(t),
\end{align*}
\]
The innovations process is given by

\[ d\hat{x}^{01} = \left( -\frac{K}{2} \hat{x}^{01} - 2\omega y^{01} - \sqrt{\kappa} \xi(t)^* z^{00} \right) dt \]

\[ + \left( \hat{x}^{00} \xi(t)^* + \sqrt{\kappa} c^{01} \left( \sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} c^{01} \xi(t) + \frac{1}{2} \xi^{01} \xi(t)^* \right) \right) dW(t), \]

\[ d\hat{y}^{01} = \left( 2\omega x^{01} - \frac{K}{2} y^{01} - i\sqrt{\kappa} y^{00} \xi(t)^* \right) dt \]

\[ + \left( \hat{y}^{00} \xi(t)^* - \left( \sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \xi^{01} \xi(t) + \frac{1}{2} c^{01} \xi(t)^* \right) \right) dW(t), \]

\[ d\hat{z}^{01} = (-\kappa z^{01} - \sqrt{\kappa} x^{00} \xi(t)^* + i\sqrt{\kappa} y^{00} \xi(t)^*) dt \]

\[ + \left( \sqrt{\kappa} \hat{x}^{01} + \hat{z}^{00} \xi(t)^* - \left( \sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \xi^{01} \xi(t) + \frac{1}{2} c^{01} \xi(t)^* \right) \right) dW(t), \]

\[ d\hat{x}^{11} = \left( -\frac{K}{2} x^{11} - 2\omega y^{11} + \sqrt{\kappa} z^{10} \xi(t) + \sqrt{\kappa} z^{01} \xi(t)^* \right) dt + \left( \sqrt{\kappa} + \hat{x}^{01} \xi(t)^* \right) \]

\[ + \hat{x}^{01} \xi(t) - \left( \sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \xi^{01} \xi(t) + \frac{1}{2} c^{01} \xi(t)^* \right) \hat{z}^{11} dW(t), \]

\[ d\hat{y}^{11} = \left( 2\omega x^{11} - \frac{K}{2} y^{11} + i\sqrt{\kappa} z^{01} \xi(t) - i\sqrt{\kappa} z^{01} \xi(t)^* \right) dt \]

\[ + \left( \hat{y}^{01} \xi(t)^* + \hat{y}^{01} \xi(t) - \left( \sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \xi^{01} \xi(t) + \frac{1}{2} c^{01} \xi(t)^* \right) \right) dW(t) \]

and

\[ d\hat{z}^{11} = (-\kappa - \kappa z^{11} - \sqrt{\kappa} x^{10} \xi(t) - i\sqrt{\kappa} y^{10} \xi(t) - \sqrt{\kappa} x^{01} \xi(t)^* + i\sqrt{\kappa} y^{01} \xi(t)^* \right) dt + \left( \sqrt{\kappa} \hat{x}^{11} + \hat{z}^{01} \xi(t)^* + \hat{z}^{01} \xi(t) \right) \]

\[ - \left( \sqrt{\kappa} \hat{x}^{11} + \frac{1}{2} \xi^{01} \xi(t) + \frac{1}{2} c^{01} \xi(t)^* \right) \hat{z}^{11} dW(t). \]

The innovations process is given by

\[ dW(t) = dY(t) - \left( \sqrt{\kappa} \hat{x}^{11}(t) + \xi^{01}(t) \xi(t) + \xi^{10}(t) \xi(t)^* \right) dt. \] (8.5)

9. Discussion and conclusion

We have derived the master equation and quantum filter for a class of open quantum systems that are coupled to continuous-mode fields in single photon states. The quantum filter consists of coupled equations that determine the evolution of the conditional state of the system under continuous measurement performed on the output field, in contrast to the familiar single filtering equation for open Markov quantum systems coupled to coherent boson fields. This coupled equation structure of the master and filter equations is a reflection of the non-Markov nature of systems coupled to the non-classical fields. Indeed, a key feature of our approach is the embedding of the system into a larger extended system, a technique often used in the analysis of non-Markov systems, providing an elegant framework within which to study the dynamics, both unconditional and conditional, of the system. In contrast to Markovian embeddings [15–17], the
extended system is initialized in a superposition state. This embedding provides
a framework within which the tools of the quantum stochastic calculus may
be efficiently applied to determine quantum filtering equations. We expect that
the use of suitable embeddings, both Markovian and non-Markovian, could be
adapted to study quantum systems coupled to other types of non-classical fields,
superposition states [18].

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References
2 Knill, E., Laflamme, R. & Milburn, G. 2001 A scheme for efficient quantum computation with
linear optics. Nature 409, 46–52. (doi:10.1038/35051009)
3 Maitre, X., Hagley, E., Nogues, G., Wunderlich, C., Goy, P., Brune, M., Raimond, J. & Haroche,
(doi:10.1103/PhysRevLett.79.769)
113–117. (doi:10.1140/epjst/e2008-00699-5)
Weinfurter, H. 2006 Observation of entanglement of a single photon with a trapped atom.
6 Barchielli, A. 2006 Continual measurements in quantum mechanics and quantum stochastic
calculus. In Open quantum systems III: recent developments (eds S. Attal, A. Joye & C. A.
Verlag.
7 Barchielli, A. & Belavkin, V. 1991 Measurements continuous in time and a posteriori states in
8 Belavkin, V. 1992 Quantum continual measurements and a posteriori collapse on CCR.
9 Belavkin, V. 1992 Quantum stochastic calculus and quantum nonlinear filtering. J. Multivariate
10 Bouten, L., van Handel, R. & James, M. 2007 An introduction to quantum filtering. SIAM J.
Control Optim. 46, 2199–2241. (doi:10.1137/060651239)
13 Wiseman, H. & Milburn, G. 2010 Quantum measurement and control. Cambridge, UK: Cambridge
University Press.
14 Gheri, K., Ellinger, K., Pellizzari, T. & Zoller, P. 1998 Photon-wavepackets as flying quantum
012106. (doi:10.1103/PhysRevA.70.012106)
16 Gough, J., James, M. & Nurdin, H. 2011 Quantum master equation and filter for systems driven
by fields in a single photon state. In Proc. 50th IEEE Conf. on Decision and Control. Orlando,

Phil. Trans. R. Soc. A (2012)