On structure-preserving transformations of the Itô generator matrix for model reduction of quantum feedback networks

BY HENDRA I. NURDIN1,* AND JOHN E. GOUGH2

1Research School of Engineering, Australian National University, Canberra, Australian Capital Territory, Australia
2Institute of Mathematics and Physics, Aberystwyth University, Aberystwyth, Ceredigion SY23 3BZ, UK

Two standard operations of model reduction for quantum feedback networks, elimination of internal connections under the instantaneous feedback limit and adiabatic elimination of fast degrees of freedom, are cast as structure-preserving transformations of Itô generator matrices. It is shown that the order in which they are applied is inconsequential.

Keywords: quantum networks; model reduction; Schur complement; singular perturbation

1. Introduction

The last two decades have seen the emergence and explosion of global research activities in quantum information science that promise to deliver quantum technologies, a class of technologies that rely on and exploit the laws of quantum mechanics, which can beat the best-known capabilities of current technological systems in sensing, communication and computation. Most of the envisioned quantum technologies are quantum information processing systems that process quantum information [1,2]. Typical proposals are realized as quantum networks: linear quantum optical computing [3], the quantum internet [4] and quantum error correction [5,6]. Quantum networks have also been experimentally realized in proof-of-principle demonstrations of quantum information processing [7,8]. Besides quantum information processing, quantum networks have also been proposed for new ultra low power photonic devices that perform classical information processing. In particular, photonic devices that act as photonic analogues of classical electronic circuits and logic devices [9–12].

Even relatively simple quantum networks may be difficult to simulate owing to the large number of variables that need to be propagated. It is therefore necessary to look at model reduction. For instance, this has been used to obtain a

*Author and address for correspondence: School of Electrical Engineering and Telecommunications, The University of New South Wales, Sydney, New South Wales 2052, Australia (h.nurdin@unsw.edu.au).

One contribution of 15 to a Theo Murphy Meeting Issue ‘Principles and applications of quantum control engineering’.
tractable network model of a coherent-feedback system implementing a quantum-error-correction scheme for quantum memory [5]. In particular, this involved reduced quantum stochastic differential equation (QSDE) models for several components that make up the nodes of the network. In fact, the process led to a simple and intuitive quantum master equation that describes the evolution of the composite state of the three qubits of the quantum memory and the two atom-based optical switches that jointly act as a coherent-feedback controller. The idea for this coherent-feedback realization of a three-qubit bit (phase)-flip quantum-error-correction code, which can correct only for single-qubit bit (phase)-flip errors, was subsequently extended to a coherent-feedback realization of a nine-qubit Bacon–Shor subsystem code that can correct for arbitrary single-qubit errors [6] (figure 1). Again, here QSDE model reduction played a crucial role in justifying the intuitive quantum master equation that describes the operation of the coherent-feedback QEC circuit.

This paper considers a class of dynamical quantum networks with open Markov quantum systems as nodes and in which nodes are interconnected by bosonic optical fields (such as coherent laser beams). Here, the optical fields serve as quantum links or ‘wires’ between nodes in the network. Time delays in the propagation of the optical fields mean that the network as a whole is no longer Markov, but fortunately, an effective Markov model may be recovered in the zero time delay limit [13–16]. The effective Markov model can then be viewed as a large single-node network, as illustrated in figure 2. This kind of limit will be referred to as an instantaneous feedback limit.

Figure 1. A coherent-feedback quantum network that implements a nine-qubit Bacon–Shor subsystem quantum-error-correction code from Kerckhoff et al. [6]. The four relays \((R_1, R_2, R_3, R_4)\) act jointly as a coherent-feedback controller. (Online version in colour.)
Another commonly used approximation is adiabatic elimination (or singular perturbation) of quantum systems that have fast and slow sub-dynamics with well-separated time scales [17–19]. Besides model simplification, adiabatic elimination has also proved to be a powerful tool for the approximate engineering of ‘exotic’ two or more body couplings [5,9,20,21].

The instantaneous feedback and adiabatic elimination operations can be applied together on a quantum feedback network, but this may lead to ambiguous results because they can be applied in two different orders that could potentially lead to two different reduced models; see figure 2. That is, the two operations may not commute. In Gough et al. [22], it was established, for a special class of quantum networks containing fast oscillating quantum harmonic oscillators, that the instantaneous feedback and adiabatic elimination limits are interchangeable. The main contribution of this paper is to extend the results of Gough et al. [22] to general classes of quantum networks with Markovian components.

2. Quantum stochastic differential equations and the Itô generator matrix

We work in the category of the Hudson and Parthasarathy (bosonic) quantum stochastic models [16,23–25]. Here, we fix a separable Hilbert space $\mathcal{H}$, called the initial or system (Hilbert) space, describing the joint state space of the systems at the nodes of the network, and a finite-dimensional multiplicity space $\mathcal{K}$ labelling the input fields. The open quantum system and the quantum boson fields jointly evolve in a unitary manner according to the solution of a right

Phil. Trans. R. Soc. A (2012)
Hudson–Parthasarathy QSDE, using the Einstein summation convention,
\[ U(t) = I + \int_0^t U(s) \, G_{\alpha\beta} \, dA^{\alpha\beta}(s), \]

with \( \alpha, \beta = 0, 1, 2, \ldots, n \) (\( n \) denotes the dimension of \( \mathcal{H} \)) and \( G = [G_{\alpha\beta}] \) a right Itô generator matrix in the set \( \mathcal{G}(\mathcal{H}, \mathcal{K}) \) of all right Itô generator matrices on systems with initial space \( \mathcal{H} \) and multiplicity space \( \mathcal{K} \); see Gough & James [15] and Evans et al. [26] for conventions and notation. Here, right (left) QSDE means that the generator \( G_{\alpha\beta} \, dA^{\alpha\beta}(s) \) appears to the right (left) of the unitary \( U(t) \). Following Bouten et al. [17], we work with right unitary processes for technical reasons. The solution \( U(t) \) of the QSDE, when it exists, is an adapted quantum stochastic process. The right Itô generator matrix is written as
\[ G = \begin{bmatrix} K & L \\ M & N - I \end{bmatrix}, \]

with respect to the standard decomposition of the coefficient space \( \mathcal{C} = \mathcal{H} \otimes (\mathbb{C} \oplus \mathcal{K}) \), that is, as \( \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{K}) \). Here, \( K = G_{00}, \ L = [G_{0j}]_{j=1,2,\ldots,n}, \ M = [G_{j0}]_{j=1,2,\ldots,n}, \ N = [G_{ij}]_{i,j=1,2,\ldots,n}. \) Throughout this paper, we shall assume that all the components of \( K, K^*, L, L^*, M, M^*, N \) and \( N^* \) have a common invariant domain \( \mathcal{D} \) in \( \mathcal{H} \) (here, \( * \) denotes the adjoint of a Hilbert space operator). We further require that the Hudson–Parthasarathy conditions are satisfied: \( N \) is unitary, \( K + K^* = -LL^* \) and \( M = -NL^* \). Note that if the coefficients are bounded, then these conditions are necessary and sufficient for \( U(t) \) to be a unitary co-cycle (if they are unbounded, then the solution may not extend to a unitary co-cycle). In the general case, if \( U(t) \) is a well-defined unitary and \( |\psi_0\rangle \) is the initial pure state of the composite system consisting of the system and the fields at time 0, then this state vector evolves in time in the Schrödinger picture as \( |\psi(t)\rangle = U(t)^*|\psi_0\rangle \). We assume throughout that the operator coefficients of the QSDE satisfy sufficient conditions that guarantee a unique solution that extends to a unitary co-cycle on \( \mathcal{H} \otimes \Gamma(L^2_{\mathcal{K}}[0,\infty)) \) (in particular, this will always be the case when the coefficients are bounded); see Fagnola and co-workers [27,28] for the unbounded case.

Note that \( G \) is simply the adjoint of the corresponding left Itô generator matrices introduced for left QSDEs in Gough & James [15], and plays a similar role to the latter for right QSDEs. Because we will be working exclusively with right QSDEs, from this point on, when we say Itô generator matrix, we will mean the right Itô generator matrix.

We use the notation \( X^- \) for a generalized inverse of an operator \( X \in \mathcal{L}(\mathcal{H}) \), that is, \( XX^-X = X \). Throughout, we require that \( X, X^*, X^-, X^{-*} \) have \( \mathcal{D} \) as the invariant domain. Note then that \( X^{-*} = (X^*)^- \).

**Definition 2.1.** Given a non-trivial decomposition of the coefficient space \( \mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 \), we define the generalized Schur complement operation of Itô matrices as
\[ S_{\mathcal{C}_1 \rightarrow \mathcal{C}_2} G = G_{11} - G_{12} G_{22}^{-1} G_{21}, \]

where
\[ G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \]
is the partition of $G$ with respect to the decomposition. The domain of $S_{\mathcal{C} \rightarrow \mathcal{C}_1}$ is the set of $G \in \mathcal{L}(\mathcal{C}_1 \oplus \mathcal{C}_2)$ for which we have the image and kernel space inclusions $\text{im}(G_{21}) \subseteq \text{im}(G_{22})$ and $\text{ker}(G_{22}) \subseteq \text{ker}(G_{12})$ (this ensures that the choice of generalized inverse is unimportant; see Gough et al. [22] and references therein). $S_{\mathcal{C} \rightarrow \mathcal{C}_1}$ maps into the reduced space $\mathcal{L}(\mathcal{C}_1)$. We shall often use the shorthand $G/ G_{22}$ for the generalized Schur complement.

Of course, if $G_{22} \mid_D$ is invertible, then the generalized Schur complement reduces to the ordinary Schur complement with the generalized inverse $G_{22}^{-1}$ replaced by $(G_{22} \mid_D)^{-1}$.

### 3. Eliminating internal connections

The total multiplicity space $\mathcal{K}$ may be decomposed into external and internal elements as follows:

$$\mathcal{K} = \mathcal{K}_e \oplus \mathcal{K}_i,$$

leading to decomposition $\mathcal{C} = \mathcal{C}_e \oplus \mathcal{C}_i$, where $\mathcal{C}_e = \mathfrak{h} \otimes (\mathfrak{C} \oplus \mathcal{K}_e)$. It was shown in Gough & James [15] that in the instantaneous feedback limit for the internal connections, the reduced Itô generator matrix is the Schur complement of the pre-interconnection network Itô generator matrix, $S_{\mathcal{C} \rightarrow \mathcal{C}_e} G$. With respect to the decomposition $\mathcal{C} = \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathcal{K}_e) \oplus (\mathfrak{h} \otimes \mathcal{K}_i)$, we have, with $L = [L_e \ L_i]$, $N_a = [N_{ae} \ N_{ai}]$,

$$\begin{bmatrix} K & L_e & L_i \\ M_e & N_{ee} - I & N_{ei} \\ M_i & N_{ie} & N_{ii} - I \end{bmatrix} / (N_{ii} - I) = \begin{bmatrix} K & L_e \\ M_e & N_{ee} - I \end{bmatrix} - \begin{bmatrix} L_i \\ N_{ei} \end{bmatrix} (N_{ii} - I)^{-1} [M_i \ N_{ie}],$$

where it is a condition that $N_{ii} - I$ be invertible for the network connections to be well posed. We denote the operation $S_{\mathcal{C} \rightarrow \mathcal{C}_e}$ of instantaneous feedback reduction by $\mathcal{F}$ whenever the context is clear, and for well-posed connections, it maps between the categories of Itô generator matrices in $\mathfrak{G}(\mathfrak{h}, \mathcal{K})$ to $\mathfrak{G}(\mathfrak{h}, \mathcal{K}_e)$ [15].

### 4. Adiabatic elimination of quantum stochastic differential equations: structural assumptions

The following section reviews the adiabatic elimination results of Bouten et al. [17]. We consider a QSDE of the form

$$U^{(k)}(t) = I + \int_0^t U^{(k)}(s) G^{(k)} \alpha^{(k)}(s) dA^{(k)}(s),$$

where as before $\alpha = 0, 1, \ldots, n$ and $G^{(k)} = [G^{(k)}_{a\beta}]$ is an Itô generator matrix $G^{(k)} \in \mathfrak{G}(\mathfrak{h}, \mathcal{K})$ that can be expressed as

$$G^{(k)} = \begin{bmatrix} K^{(k)} & L^{(k)} \\ M^{(k)} & N^{(k)} - I \end{bmatrix}.$$
with $K^{(k)} = G^{(k)}_{00} = k^2 Y + kA + B$, $L^{(k)} = [G^{(k)}_{ij}]_{j=1,2,...,n} = kF + G$, $M^{(k)} = [G^{(k)}_{jl}]_{l=1,2,...,n}$ and $N^{(k)} = [G^{(k)}_{ik}]_{i=1,2,...,n}$, where $k$ is a positive parameter representing coupling strength. The operators $Y$, $A$, $B$, $F$, $G$, $N$, and their respective adjoints, have the coefficient space:

Denote the orthogonal projections onto $G$ with $B$ as a common invariant domain, and the coefficients satisfy the Hudson–Parthasarathy conditions $L^{(k)} = L^{(k)*}$, $M^{(k)} = -N^{(k)*} L^{(k)}$ and $N^{(k)} N^{(k)*} = N^{(k)*} N^{(k)} = I$. In particular, this implies that $B + B^* = -GG^*$, $A + A^* = -(FG^* + GF^*)$ and $Y + Y^* = -FF^*$. The general situation is that there is a decomposition of the initial/system space $\mathfrak{h}_s$ into slow and fast subspaces (the subscripts $s$ and $f$ denote fast and slow, respectively),

$$\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{h}_f.$$  

Denote the orthogonal projections onto $\mathfrak{h}_s, \mathfrak{h}_f$ by $P_s, P_f$, respectively. With an obvious abuse of notation, we use the same partition for the decomposition of the coefficient space: $\mathfrak{C} = \mathfrak{C}_s \oplus \mathfrak{C}_f$, where $\mathfrak{C}_s = \mathfrak{h}_s \otimes (\mathfrak{C} \oplus \mathfrak{K})$. With respect to the decomposition $\mathfrak{h}_s \oplus \mathfrak{h}_f$, one requires [17] the following.

1. $P_s \mathcal{D} \subset \mathcal{D}$.
2. $N^{(k)} = N$ is $k$ independent.
3. $P_s F = 0$. That is, $F$ has the structure

$$F = \begin{bmatrix}
0 & 0 \\
F_{fs} & F_{ff}
\end{bmatrix}.$$  

4. The Hamiltonian $H^{(k)} = (1/2i)(K^{(k)} - K^{(k)*})$, where $i = \sqrt{-1}$, takes the form $H(k) = H^{(0)} + kH^{(1)} + k^2 H^{(2)}$, where $P_s H^{(1)} P_s = 0$ and $P_s H^{(2)} = H^{(2)} P_s = 0$, that is,

$$H = \begin{bmatrix}
H_{ss}^{(0)}, & H_{sf}^{(0)} + kH_{sf}^{(1)} \\
H_{fs}^{(0)} + kH_{fs}^{(1)}, & H_{ff}^{(0)} + kH_{ff}^{(1)} + k^2 H_{ff}^{(2)}
\end{bmatrix}.$$  

Conditions 3 and 4 are equivalent to $Y$ having the structure

$$Y = \begin{bmatrix}
0 & 0 \\
0 & P_f Y P_f
\end{bmatrix}.$$  

5. In the expansion

$$K^{(k)} = -L^{(k)} \frac{1}{2} L^{(k)*} - iH^{(k)} \equiv k^2 Y + kA + B,$$

we require that the operator $Y_{ff} = -\frac{1}{2} \sum_{a=s,f} F_{fa} F_{fa}^* - iH_{ff}^{(2)}$ is invertible. In particular, conditions 3–5 are equivalent to $Y$ having a generalized inverse $Y^-$ with the diagonal structure

$$Y^- = \begin{bmatrix}
P_s Y^{-1} P_s & 0 \\
0 & Y_{ff}^{-1}
\end{bmatrix}.$$  

By using a repeated index summation convention over the index range \{s, f\} from now on, we find that the operator $B$ has components $B_{ab} = -\frac{1}{2} G_{ca} G_{cb}^{*}$.
\[ iH^{(0)}_{ab} \] with respect to the slow–fast block decomposition. Likewise,
\[
A \equiv \begin{bmatrix} 0 & A_{sf} \\ A_{fs} & A_{ff} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}G_{sc}F_{fc}^* - iH_{sf}^{(1)} \\ -\frac{1}{2}F_{fc}G_{sc}^* - iH_{fs}^{(1)} & -\frac{1}{2}F_{fc}G_{fc}^* - \frac{1}{2}G_{fc}F_{fc}^* - iH_{ff}^{(1)} \end{bmatrix}
\]
and \[ Y \equiv \begin{bmatrix} 0 & 0 \\ 0 & Y_{ff} \end{bmatrix}. \] (4.1)

With respect to the decomposition \( C = h_s \oplus (h_s \otimes \mathbb{R}) \oplus h_f \oplus (h_f \otimes \mathbb{R}) \), we have
\[
G^{(k)} = [1 \ 1 \ 1][G_0 + G'(k)] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \] (4.2)
where
\[
G_0 = \begin{bmatrix} B_{ss} & G_{ss} & A_{sf} & G_{sf} \\ -N_{sa}G_{sa}^* & N_{ss} - I & -N_{sa}F_{fa}^* & N_{sf} \\ A_{fs} & F_{fs} & Y_{ff} & F_{ff} \\ -N_{fa}G_{fa}^* & N_{fs} & -N_{fa}F_{fa}^* & N_{ff} - I \end{bmatrix},
\]
and \( \lim_{k \to \infty} G'(k) \phi = 0 \) for all \( \phi \in D \). We then observe that
\[
G_0 / Y_{ff} = \begin{bmatrix} \hat{K}_{ss} & \hat{L}_s & \hat{L}_f \\ \hat{M}_s & \hat{N}_{ss} - I & \hat{N}_{sf} \\ \hat{M}_f & \hat{N}_{fs} & \hat{N}_{ff} - I \end{bmatrix},
\]
where
\[
\hat{K}_{ss} = B_{ss} - A_{sf}Y_{ff}^{-1}A_{fs}, \quad \hat{L}_a = G_{sa} - A_{sf}Y_{ff}^{-1}F_{fa},
\]
\[
\hat{M}_a = -N_{ab}G_{ab}^* + N_{ab}F_{tb}Y_{ff}^{-1}A_{fs} \quad \text{and} \quad \hat{N}_{ab} = N_{ab} + N_{ac}F_{fc}Y_{ff}^{-1}F_{fb}.
\]
We also assume that
\[
\hat{L}_f = \hat{N}_{sf} = \hat{N}_{fs} = 0, \] (4.3)
and this will ensure that the limit dynamics excludes the possibility of transitions that terminate in any of the fast states. In this case, \( \hat{N}_{ss} \) and \( \hat{N}_{ff} \) are unitary. In particular,
\[
\hat{G} = \begin{bmatrix} \hat{K}_{ss} & \hat{L}_s \\ \hat{M}_s & \hat{N}_{ss} - I \end{bmatrix} = \begin{bmatrix} \hat{K} & \hat{L} \\ \hat{M} & \hat{N} \end{bmatrix} \] (4.4)
is an Itô generator matrix \( \hat{M}_s = -\hat{N}_{ss}\hat{L}_s^* \) on the coefficient space \( C_s = h_s \otimes (\mathbb{C} \oplus \mathbb{R}) \). The final assumption is a technical condition. For any \( \alpha, \beta \in \mathbb{C}^n \) (represented as column vectors), \( P_sD \) is a core for the operator \( \mathcal{L}^{(\alpha \beta)} \) defined by
\[
\mathcal{L}^{(\alpha \beta)} = \alpha^* \hat{N} \beta + \alpha^* \hat{M} + \hat{L} \beta + \hat{K} - \frac{|\alpha|^2 + |\beta|^2}{2}, \] (4.5)
with \( \hat{K}, \hat{L}, \hat{M}, \hat{N} \) as defined in (4.4).

Phil. Trans. R. Soc. A (2012)
Theorem 4.1. [17] Suppose that all the assumptions above hold. If the right QSDEs with coefficients $G(k)$ possess a unique solution that extends to a contraction co-cycle $U^{(k)}(t)$ on $\mathfrak{h} \otimes \Gamma(L^2_\mathbb{R}[0, \infty))$ for all $k > 0$, and the right QSDE with coefficients $\hat{G}$ has a unique solution that extends to a unitary co-cycle $\hat{U}(t)$ on $\mathfrak{h} \otimes \Gamma(L^2_\mathbb{R}[0, \infty))$, then $U^{(k)}(t)$ converges to the solution $\hat{U}(t)$ uniformly in a strong sense,

$$\lim_{k \to \infty} \sup_{0 \leq t \leq T} \| U^{(k)}(t)^* \phi - \hat{U}(t)^* \phi \| = 0, \quad \forall \phi \in \mathfrak{h} \otimes \Gamma(L^2_\mathbb{R}[0, \infty)),$$

for each fixed $T \geq 0$.

Theorem 4.1 is theorem 3 of Bouten et al. [17].

5. Adiabatic elimination of quantum stochastic differential equations: Schur complements

In this section, we will show how the singular perturbation limit of the QSDE can be related to the Schur complementation of a certain matrix with operator entries. To this end, define the extended Itô generator matrix $G_E$ as

$$G_E = \begin{bmatrix} B & A_{sf} & G \\ A_\xi & Y_{\xi \xi} & F_\xi \\ -NG^* & NF^*_\xi & N - I \end{bmatrix},$$

where $A_\xi = P_\xi A$, $F_\xi = P_\xi F$.

Lemma 5.1. The limit QSDE $\hat{U}(t)$ has the Itô generator matrix $\hat{G}$ given by $\hat{G} = P_\mathfrak{h}(G_E/Y_{\xi \xi})P_\mathfrak{h}|_{\mathfrak{h}}$, where $G_E/Y_{\xi \xi}$ is the Schur complement of $G_E$ with respect to the sub-block with entry $Y_{\xi \xi}$.

Proof. Direct calculation shows that

$$G_E/Y_{\xi \xi} = \begin{bmatrix} B & A_{sf} & G \\ -NG^* & N - I \end{bmatrix} - \begin{bmatrix} A_{sf} \\ -NF^*_\xi \end{bmatrix} Y_{\xi \xi}^{-1} \begin{bmatrix} A_\xi & F_\xi \end{bmatrix},$$

$$= \begin{bmatrix} B - A_{sf} Y_{\xi \xi}^{-1} A_\xi & G - A_{sf} Y_{\xi \xi}^{-1} F_\xi \\ -NG^* + NF^*_\xi Y_{\xi \xi}^{-1} A_\xi & N + NF^*_\xi Y_{\xi \xi}^{-1} F_\xi - I \end{bmatrix}.$$ (5.1)

Thus,

$$P_\mathfrak{h}(G_E/Y_{\xi \xi})P_\mathfrak{h} = \begin{bmatrix} P_\mathfrak{h}(B - A_{sf} Y_{\xi \xi}^{-1} A_\xi)P_\mathfrak{h} & P_\mathfrak{h}(G - A_{sf} Y_{\xi \xi}^{-1} F_\xi)P_\mathfrak{h} \\ P_\mathfrak{h}(-NG^* + NF^*_\xi Y_{\xi \xi}^{-1} A_\xi)P_\mathfrak{h} & P_\mathfrak{h}(N + NF^*_\xi Y_{\xi \xi}^{-1} F_\xi)P_\mathfrak{h} - P_\mathfrak{h} \end{bmatrix}.$$ 

Therefore, because $P_\mathfrak{h}(G_E/Y_{\xi \xi})P_\mathfrak{h}|_{\mathfrak{h}}$ equals

$$\begin{bmatrix} P_\mathfrak{h}(B - A_{sf} Y_{\xi \xi}^{-1} A_\xi)P_\mathfrak{h} & P_\mathfrak{h}(G - A_{sf} Y_{\xi \xi}^{-1} F_\xi)P_\mathfrak{h} \\ P_\mathfrak{h}(-NG^* + NF^*_\xi Y_{\xi \xi}^{-1} A_\xi)P_\mathfrak{h} & P_\mathfrak{h}(N + NF^*_\xi Y_{\xi \xi}^{-1} F_\xi)P_\mathfrak{h} - P_\mathfrak{h} \end{bmatrix},$$ (5.2)

it follows from (4.4) that $\hat{G} = P_\mathfrak{h}(G_E/Y_{\xi \xi})P_\mathfrak{h}|_{\mathfrak{h}}$.  \[\blacksquare\]
We then denote by $\mathcal{A}$ the map that takes $G^{(k)}$ to the Itô generator matrix $\hat{G}$ in the lemma by $\mathcal{A}: G^{(k)} \mapsto \hat{G}$.

We conclude by remarking that the instantaneous feedback limit operation $\mathcal{F}$ and the adiabatic elimination operation $\mathcal{A}$ can be cast as structure-preserving transformations, that is, transformations that preserve the structure of Itô generator matrices or convert Itô generator matrices to Itô generator matrices (possibly of lower initial space and multiplicity space dimensions).

6. Sequential application of the instantaneous feedback and adiabatic elimination operations

(a) The adiabatic elimination operation followed by the instantaneous feedback operation

When the adiabatic elimination operation is first applied followed by the instantaneous feedback operation, we have the following.

**Lemma 6.1.** Under the standing assumptions in §4 and taking $N_{\tilde{i}} + N_{\tilde{e}} F_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} - I$ to be invertible, we have

$$P_s((G_E / Y_{\tilde{e}})/(N_{\tilde{i}} + N_{\tilde{e}} F_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} - I)) P_s = \mathcal{F} \mathcal{A} G^{(k)},$$

where $F_{\tilde{e}} = P_{\tilde{e}} F_{\tilde{e}}$.

**Proof.** Partition the extended Itô generator with respect to $\mathfrak{h}_e \oplus \mathfrak{h}_i$ to get

$$G_E / Y_{\tilde{e}} = \begin{bmatrix} B & A_{\tilde{e}} & G_{\tilde{i}} & G_{\tilde{e}} \\ A_{\tilde{e}} & Y_{\tilde{e}} & F_{\tilde{i}} & F_{\tilde{e}} \\ -N_i G^* & -N_i F_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} - I & N_{\tilde{e}} \\ -N_{\tilde{e}} G^* & -N_{\tilde{e}} F_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} - I & N_{\tilde{ee}} - I \end{bmatrix} / Y_{\tilde{e}}$$

$$= \begin{bmatrix} B - A_{\tilde{e}} Y_{\tilde{e}}^{-1} A_{\tilde{e}} & G_{\tilde{e}} - A_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} & G_{\tilde{e}} - A_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} \\ -N_i G^* + N_i F_{\tilde{e}} Y_{\tilde{e}}^{-1} A_{\tilde{e}} & N_{\tilde{i}} + N_i F_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} - I & N_{\tilde{ie}} + N_i F_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} \\ -N_{\tilde{e}} G^* + N_{\tilde{e}} F_{\tilde{e}} Y_{\tilde{e}}^{-1} A_{\tilde{e}} & N_{\tilde{ee}} + N_{\tilde{e}} F_{\tilde{e}} Y_{\tilde{e}}^{-1} F_{\tilde{e}} - I & N_{\tilde{ee}} - I \end{bmatrix}$$

$$= \begin{bmatrix} \hat{B} & \hat{G}_{\tilde{i}} & \hat{G}_{\tilde{e}} \\ \hat{M}_{\tilde{i}} & \hat{N}_{\tilde{i}} - I & \hat{N}_{\tilde{ie}} \\ \hat{M}_{\tilde{e}} & \hat{N}_{\tilde{ee}} - I \\ \hat{M}_{\tilde{e}} & \hat{N}_{\tilde{ee}} - I \\ \hat{M}_{\tilde{e}} & \hat{N}_{\tilde{ee}} - I \end{bmatrix},$$

where $N_a = [N_{ae} N_{al}]$, $F_{fa} = P_{\tilde{e}} F_{\tilde{e}}$ for $a = i, e$, and $[F_{\tilde{e}} F_{\tilde{e}}] = F$ and $[G_{\tilde{i}} G_{\tilde{e}}] = G$, and we used (5.1). We now apply the operation $\mathcal{F}$ to get $(G_E / Y_{\tilde{e}}) / (\hat{N}_{\tilde{i}} - I)$ equal to

$$\begin{bmatrix} \hat{B} - \hat{G}_{\tilde{i}}(\hat{N}_{\tilde{i}} - I)^{-1} \hat{M}_{\tilde{i}} & \hat{G}_{\tilde{e}} - \hat{G}_{\tilde{i}}(\hat{N}_{\tilde{i}} - I)^{-1} \hat{N}_{\tilde{ie}} \\ \hat{M}_{\tilde{e}} - \hat{N}_{\tilde{ei}}(\hat{N}_{\tilde{i}} - I)^{-1} \hat{M}_{\tilde{i}} & \hat{N}_{\tilde{ee}} - \hat{N}_{\tilde{ei}}(\hat{N}_{\tilde{i}} - I)^{-1} \hat{N}_{\tilde{ie}} - I \end{bmatrix}.$$
Next, note that $N_i + N_i F_j^* Y^{-1} F_i$ has the representation
\[
N_i + N_i F_j^* Y^{-1} F_i = \begin{bmatrix}
P_f (N_i + N_i F_j^* Y^{-1} F_i) P_f & 0 \\
0 & P_s (N_i + N_i F_j^* Y^{-1} F_i) P_s
\end{bmatrix},
\]
with respect to the decomposition $D = P_f D \oplus P_s D$. Moreover, we also note the representation
\[
G_a - A s \hat{Y}_f Y^{-1} F_a = \begin{bmatrix}
P_f G_a P_f & P_f G_a P_s \\
0 & P_s (G_a - A s \hat{Y}_f Y^{-1} F_a) P_s
\end{bmatrix}, \quad a = i, e.
\]
By using these representations, we can verify the following sequence of identities:
\[
\begin{align*}
P_s (G_E / Y) /\left(\hat{N}_i - I\right) P_s &= P_s (G_E / Y) P_s / P_s (N_i + N_i F_j^* Y^{-1} F_i) P_s, \\
&= \left(P_s (G_E / Y) P_s \big|_{\hat{h}_s}\right) / \left(P_s (N_i + N_i F_j^* Y^{-1} F_i) P_s - I\right),
\end{align*}
\]
where the last equality follows from the fact that $P_s (N_i + N_i F_j^* Y^{-1} F_i - I) P_s \big|_{\hat{h}_s} = P_s (N_i + N_i F_j^* Y^{-1} F_i) P_s - I$. Finally, because
\[
\mathcal{F}AG^{(k)} = \left(P_s (G_E / Y) P_s \big|_{\hat{h}_s}\right) / \left(P_s (N_i + N_i F_j^* Y^{-1} F_i) P_s - I\right),
\]
by definition, we thus obtain the desired result.

\[\Box\]

(b) The instantaneous feedback operation followed by the adiabatic elimination operation

We now turn to consider the alternative sequence of first applying the instantaneous feedback operation followed by the adiabatic elimination operation. The main result in this section as follows.

**Lemma 6.2.** Suppose that the assumptions of §4 are satisfied, $N_i - I$ is invertible, $\ker(Y + F_i (N_i - I)^{-1} F_i) = \hat{h}_s$, and there exists an operator $\hat{Y}_s$ such that $\hat{Y}_s, \hat{Y}_s$ have $D$ as a common invariant domain and $\hat{Y} \hat{Y}_s = \hat{Y}_s \hat{Y} = P_f$, where $\hat{Y} = Y + F_i (N_i - I)^{-1} F_i$. Then,
\[
\mathcal{A} \mathcal{F} G^{(k)} = P_s (\left(G_E / (N_i - I)\right) / (Y) / F_i (N_i - I)^{-1} N_i F_j^*) P_s \big|_{\hat{h}_s}.
\]

**Proof.** We first compute the extended Itô generator matrix corresponding to $\mathcal{F}G^{(k)}$. With $\tilde{N}_{ee} = N_{ee} - N_{ei} (N_i - I)^{-1} N_{ie}$, this is
\[
(\mathcal{F}G^{(k)})_E = \begin{bmatrix}
B + G_i (N_i - I)^{-1} N_i G^* \\
A_f + F_i (N_i - I)^{-1} N_i G^* + P_f G_i (N_i - I)^{-1} N_i F^* \\
-\tilde{N}_{ee} (G_e - G_i (N_i - I)^{-1} N_i G^*) \\
A_s \hat{Y}_s + P_s G_i (N_i - I)^{-1} N_i F_j^* \\
\hat{F}_s = F_i (N_i - I)^{-1} N_i F_j^* \\
\hat{N}_{ee} (F_e - F_i (N_i - I)^{-1} N_i G^*) \\
-\tilde{N}_{ee} (F_e - F_i (N_i - I)^{-1} N_i G^*) \\
\end{bmatrix}.
\]

Let $\hat{Y} = Y + F_i (N_i - I)^{-1} N_i F_j^*$. Then, under the structural assumptions of §4 and the hypothesis that $\ker(Y + F_i (N_i - I)^{-1} N_i F_j^*) = \ker(Y)$, we have that $\hat{Y}$

\[\text{Phil. Trans. R. Soc. A (2012)}\]
has a representation, with respect to the decomposition $\mathcal{D} = P_\xi \mathcal{D} \oplus P_s \mathcal{D}$, with the special structure

$$
\hat{Y} = \begin{bmatrix} P_\xi Y P_\xi + P_\xi F_1 (N_{ii} - I)^{-1} N_{i} F_s^* P_\xi & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Y_{\xi \xi} + F_{\xi i} (N_{ii} - I)^{-1} N_{i} F_{s}^* & 0 \\ 0 & 0 \end{bmatrix}.
$$

Moreover, because there exists an operator $\hat{Y}^{-}$ that satisfies the hypothesis of the theorem, we have that $\hat{Y}^{-} = (Y + F_{\xi i} (N_{ii} - I)^{-1} N_{i} F_s^*)^{-1}$ with respect to the same decomposition has the diagonal structure

$$
\hat{Y}^{-} = \begin{bmatrix} P_\xi \hat{Y}^{-} P_\xi & 0 \\ 0 & P_s \hat{Y}^{-} P_s \end{bmatrix},
$$

with $\hat{Y}_{\xi \xi} = P_\xi \hat{Y}^{-} P_\xi$ invertible. In fact, we have that

$$
\hat{Y}_{\xi \xi} = (Y_{\xi \xi} + F_{\xi i} (N_{ii} - I)^{-1} N_{i} F_s^*)^{-1}.
$$

Introduce the additional notations

$$
\hat{A}_{s\xi} = A_{s\xi} + P_s G_{i} (N_{ii} - I)^{-1} N_{i} F_s^*,
$$

$$
\hat{A}_{\xi} = A_{\xi} + F_{\xi i} (N_{ii} - I)^{-1} N_{i} G_s^*
$$

and

$$
\hat{F}_{\xi} = F_{\xi e} - F_{\xi i} (N_{ii} - I)^{-1} N_{ie}.
$$

From the partitioning of $(G^{(k)}/(N_{ii} - I))_{E}$, we can compute $\mathcal{A} \mathcal{F} G^{(k)}$ by lemma 5.1 as

$$
\mathcal{A} \mathcal{F} G^{(k)} = P_s((G^{(k)}/(N_{ii} - I))_{E}/\hat{Y}_{\xi \xi}) P_s |_{h_s} \equiv \begin{bmatrix} \tilde{K} & \tilde{L} \\ \tilde{M} & \tilde{N} - I \end{bmatrix},
$$

where

$$
\tilde{K} = P_s (B + G_{i}(N_{ii} - I)^{-1} N_{i} G_s^*) P_s - P_s \hat{A}_{s\xi} \hat{Y}_{\xi \xi}^{-1} \hat{A}_{\xi} P_s,
$$

$$
\tilde{L} = P_s (G_{e} - G_{i}(N_{ii} - I)^{-1} N_{ie}) P_s - P_s \hat{A}_{s\xi} \hat{Y}_{\xi \xi}^{-1} \hat{F}_{\xi} P_s,
$$

$$
\tilde{M} = -P_s \tilde{N}_{ee} (G_{e} - G_{i}(N_{ii} - I)^{-1} N_{ie})^* P_s + P_s \tilde{N}_{ee} \hat{F}_{\xi} \hat{Y}_{\xi \xi}^{-1} \hat{A}_{\xi} P_s,
$$

and

$$
\tilde{N} = P_s \tilde{N}_{ee} P_s + P_s (\tilde{N}_{ee} \hat{F}_{\xi} \hat{Y}_{\xi \xi}^{-1} \hat{F}_{\xi} P_s).
$$

We also compute $(G_{E}/(N_{ii} - I))/(Y_{\xi \xi} + F_{\xi i} (N_{ii} - I)^{-1} N_{i} F_s^*)$. To begin with,

$$
(G_{E}/(N_{ii} - I))_{E} \text{ is given by}
$$

$$
\begin{bmatrix}
B + G_{i}(N_{ii} - I)^{-1} N_{i} G_s^* & A_{s\xi} + G_{i}(N_{ii} - I)^{-1} N_{i} F_s^* & G_{e} - G_{i}(N_{ii} - I)^{-1} N_{ie} \\
A_{\xi} + F_{\xi i} (N_{ii} - I)^{-1} N_{i} G_s^* & Y_{\xi \xi} + F_{\xi i} (N_{ii} - I)^{-1} N_{i} F_s^* & F_{\xi e} - F_{\xi i} (N_{ii} - I)^{-1} N_{ie} \\
-N_{ee} (G_{e} - G_{i}(N_{ii} - I)^{-1} N_{ie})^* & -N_{ee} (F_{\xi e} - F_{\xi i} (N_{ii} - I)^{-1} N_{ie})^* & \tilde{N}_{ee} - I
\end{bmatrix}.
$$

Continuing the calculation, we then find that

$$
(G_{E}/(N_{ii} - I))/(Y_{\xi \xi} + F_{\xi i} (N_{ii} - I)^{-1} N_{i} F_s^*)
$$

$$
= \begin{bmatrix}
B + G_{i}(N_{ii} - I)^{-1} N_{i} G_s^* & A_{s\xi} \hat{Y}_{\xi \xi}^{-1} \hat{A}_{\xi} & G_{e} - G_{i}(N_{ii} - I)^{-1} N_{ie} - \hat{A}_{s\xi} \hat{Y}_{\xi \xi}^{-1} \hat{F}_{\xi} \\
-N_{ee} (G_{e} - G_{i}(N_{ii} - I)^{-1} N_{ie})^* + \hat{N}_{ee} \hat{F}_{\xi} \hat{Y}_{\xi \xi}^{-1} \hat{A}_{\xi} & \hat{N} + \hat{N}_{ee} \hat{Y}_{\xi \xi} \hat{F}_{\xi} & I
\end{bmatrix}.
$$

Phil. Trans. R. Soc. A (2012)
By direct comparison of the entries of \( \mathcal{A}\mathcal{F}\mathcal{G}^{(k)} \) as given above with the corresponding entries of \( P_s((G_E/(N_i - I))/(Y_{\ast \xi} + F_{\xi}(N_i - I)^{-1}N_iF_{\xi}^*)) P_s |_{\mathcal{H}_a} \), we conclude that

\[
\mathcal{A}\mathcal{F}\mathcal{G}^{(k)} = P_s((G_E/(N_i - I))/(Y_{\ast \xi} + F_{\xi}(N_i - I)^{-1}N_iF_{\xi}^*)) P_s |_{\mathcal{H}_a} .
\]

\( \blacksquare \)

(c) Commutativity of the adiabatic elimination and instantaneous feedback operations

We are now in a position to investigate the commutativity of the adiabatic elimination and instantaneous feedback limit operations for a dynamical quantum network with Markovian components. First, note that if \( (G_E/Y_{\ast \xi})/(N_i + N_iF_{\xi}Y_{\ast \xi}^{-1}F_{\xi} - I) = (G_E/(N_i - I))/(Y_{\ast \xi} + F_{\xi}(N_i - I)^{-1}N_iF_{\xi}^*) \), then \( \mathcal{A}\mathcal{F}\mathcal{G}^{(k)} = \mathcal{F}\mathcal{A}\mathcal{G}^{(k)} \). Next, let us introduce the following notation. Let \( \mathcal{I} = \{1, 2, \ldots, n\} \) and let \( X \) be an \( n \times n \) matrix with operator entries. For any set of distinct indices \( I_1 = \{j_1, j_2, \ldots, j_m\}, I_2 = \{l_1, l_2, \ldots, l_m\} \subset \mathcal{I} \) (with \( m < n \)), define the matrix \( X_{I_1,I_2}^{\ast} \) as \( [X_{jl}] \) with \( j \in I_1 \) and \( l \in I_2 \). Denoting set complements as \( I_1^c = \mathcal{I} \setminus I_1 \) and \( I_2^c = \mathcal{I} \setminus I_2 \), we define the Schur complement of \( X \) with respect to a sub-matrix \( X_{I_1,I_2} \) (if it exists), denoted by \( X/X_{I_1,I_2} \), as

\[
X/X_{I_1,I_2} = X_{I_1^c,I_2^c} - X_{I_1^c,I_2}X_{I_1,I_2}^{-1}X_{I_1,I_2}^c .
\]

We are now ready to establish commutativity of successive Schur complementations, via the following lemma.

**Lemma 6.3.** Let \( X \) be a matrix of operators whose entries have \( \mathcal{D} \) as a common invariant domain, and let \( I_1, I_2, I_3 \) be a disjoint partitioning of the index set \( \mathcal{I} \) of \( X \) (i.e. \( \cap_{j=1}^3 I_j = \emptyset \) and \( \cup_{j=1}^3 I_j = \mathcal{I} \)). If the Schur complements

\[
X/X_{I_1 \cup I_2,I_1 \cup I_2}, (X/X_{I_2,I_2})/(X/X_{I_2,I_2})_{I_1,I_1}, (X/X_{I_1,I_1})/(X/X_{I_1,I_1})_{I_2,I_2}
\]

exist, then the successive Schur complementation rule holds,

\[
X/X_{I_1 \cup I_2,I_1 \cup I_2} = (X/X_{I_2,I_2})/(X/X_{I_2,I_2})_{I_1,I_1} = (X/X_{I_1,I_1})/(X/X_{I_1,I_1})_{I_2,I_2} .
\]

**Proof.** The proof of this lemma follows *mutatis mutandis* from the proof of Gough et al. [22, lemma 9] and here is somewhat simpler because the lemma concerns ordinary Schur complements rather than generalized Schur complements as in Gough et al. [22, lemma 9]. Therefore, the image and kernel inclusion conditions for the uniqueness of the generalized Schur complement (where the inverse is replaced by a generalized inverse) are not required. \( \blacksquare \)

**Theorem 6.4.** Under the conditions of lemmas 6.1 and 6.2, we have \( \mathcal{A}\mathcal{F}\mathcal{G}^{(k)} = \mathcal{F}\mathcal{A}\mathcal{G}^{(k)} \). Furthermore, if in addition

- \( \mathcal{D} \) is a core for the operator \( \mathcal{L}^{(a,b)} \) given in (4.5),
- \( \mathcal{F}\mathcal{G}^{(k)} \) corresponds to a QSDE that has a unique solution which extends to a contraction co-cycle on \( \mathfrak{h} \otimes \Gamma(L^2_\mathcal{K}[0, \infty)) \).

*Phil. Trans. R. Soc. A* (2012)
— $\mathcal{D}$ is a core for the operator $\mathcal{L}^{(\alpha \beta)}$ given in (4.5) with $\hat{K}, \hat{L}, \hat{M}, \hat{N}$ being replaced therein by the corresponding coefficients of $\mathcal{F}A\mathcal{G}^{(k)}$,

then the instantaneous feedback and adiabatic elimination operations can be commuted. That is, applying adiabatic elimination followed by instantaneous feedback or, conversely, applying instantaneous feedback followed by adiabatic elimination yields the same QSDE, and this QSDE has a unique solution that extends to a unitary co-cycle on $\mathfrak{h}_s \otimes \Gamma(L^2_{\mathbb{R}}[0, \infty))$.

Proof. If

$$\begin{bmatrix}
Y_{\xi \xi} & F_{\xi i} \\
-N_{\xi} F_{\xi}^* & N_{ii} - I
\end{bmatrix}$$

is invertible, the Schur complement

$$G_E \begin{bmatrix}
Y_{\xi \xi} & F_{\xi i} \\
-N_{\xi} F_{\xi}^* & N_{ii} - I
\end{bmatrix}$$

is well defined. However, because $Y_{\xi \xi}$ is invertible and $N_{ii} + N_{\xi} F_{\xi}^* Y_{\xi \xi}^{-1} F_{\xi i} - I$ is also invertible by the conditions of lemmas 6.1 and 6.2, the matrix

$$\begin{bmatrix}
Y_{\xi \xi} & F_{\xi i} \\
-N_{\xi} F_{\xi}^* & N_{ii} - I
\end{bmatrix}$$

is indeed invertible by the Banachiewicz matrix inversion formula [22, §III-A]. The first result follows from this and lemma 6.3.

Because now $\mathcal{A}\mathcal{F}\mathcal{G}^{(k)} = \mathcal{F}\mathcal{A}\mathcal{G}^{(k)}$, if the QSDEs corresponding to $\mathcal{A}\mathcal{F}\mathcal{G}^{(k)}$ and $\mathcal{F}\mathcal{A}\mathcal{G}^{(k)}$ have unique solutions that extend to a unitary co-cycle on $\mathfrak{h}_s \otimes \Gamma(L^2_{\mathbb{R}}[0, \infty))$ then they will coincide. Moreover, from this, it follows by inspection that the remaining three conditions of the theorem guarantee that all the requirements of theorem 4.1 are met so that:

— $U^{(k)}(t)$ converges to $\hat{U}(t)$ in the sense of theorem 4.1 and
— the solution of the QSDE corresponding to $\mathcal{F}\mathcal{G}^{(k)}$ converges to the solution of the QSDE corresponding to $\mathcal{A}\mathcal{F}\mathcal{G}^{(k)}$ in the sense of theorem 4.1.

Thus, we conclude that under the sufficient conditions for each of the sequence of operations $\mathcal{A}\mathcal{F}$ and $\mathcal{F}\mathcal{A}$, the two sequences of operations are equivalent and yield the same reduced-complexity QSDE model. This generalizes the results of Gough et al. [22] for quantum feedback networks with fast oscillatory components to be eliminated. Remarkably, the structural constraints imposed in Bouten et al. [17] to establish rigorous adiabatic elimination results for open Markov quantum systems, originally introduced for considerations unrelated to the goals of this paper, play a crucial role in the algebra required for us to establish our results. By exploiting these constraints, we proved that both the instantaneous feedback limit and adiabatic elimination operations correspond to Schur complementation of a common extended Itô generator matrix but with respect to different sub-blocks of this matrix. From this, we then showed that the instantaneous feedback and adiabatic elimination operations are consistent and can be commuted once each sequence of operations is well defined.
References


