Linear and nonlinear parity-time-symmetric oligomers: a dynamical systems analysis

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In the present work, we focus on the cases of two-site (dimer) and three-site (trimer) configurations, i.e. oligomers, respecting the parity-time (\(PT\)) symmetry, i.e. with a spatially odd gain–loss profile. We examine different types of solutions of such configurations with linear and nonlinear gain/loss profiles. Solutions beyond the linear \(PT\)-symmetry critical point as well as solutions with asymmetric linearization eigenvalues are found in both the nonlinear dimer and trimer. The latter feature is absent in linear \(PT\)-symmetric trimers, while both of them are absent in linear \(PT\)-symmetric dimers. Furthermore, nonlinear gain/loss terms enable the existence of both symmetric and asymmetric solution profiles (and of bifurcations between them), while only symmetric solutions are present in the linear \(PT\)-symmetric dimers and trimers. The linear stability analysis around the obtained solutions is discussed and their dynamical evolution is explored by means of direct numerical simulations. Finally, a brief discussion is also given of recent progress in the context of \(PT\)-symmetric quadrimers.

1. Introduction

In the late 1990s, a radical yet well physically motivated proposal emerged in the context of the study of fundamentals of quantum mechanics. This was the suggestion of Bender & Boettcher [1] and Bender et al. [1,2] that Hamiltonians that respect two principal physical symmetries of the dynamics, namely parity (P)
and time-reversal (T), could enable the identification of real eigenvalues, which is a property that is highly desirable for operators associated with measurable quantities, even if the associated Hamiltonians are not Hermitian. A caveat in that regard, however, was that, as the gain/loss parameter introduced in these non-Hermitian Hamiltonians was varied, a transition was quantified and termed the $PT$-symmetry-breaking transition, beyond which the eigenvalues of the relevant operator were no longer purely real. It is interesting to note here that a prototypical playground where such ideas can be explored is that of standard Schrödinger Hamiltonians of the form $H = -\frac{1}{2}\Delta + V(x)$, for which in the case of a complex potential $V$ it is straightforward to see that the above constraints of $PT$ symmetry amount to the potential satisfying the condition $V(x) = V^*(-x)$.

While the initial proposal of the above possibility was one of mathematical origin (rather than one inspired by a specific physical application), in the past few years, a significant vein of potential applications of such Hamiltonians has been initiated, predominantly so in the field of nonlinear optics. There, the work of Musslimani et al. [3] and Makris et al. [4] gave rise to the realization that the synthetic systems that can be engineered therein enable a potential balance of gain (through suitable amplifiers) and the abundantly present losses in order to produce experimental realizations of $PT$-symmetric systems. An additional feature present in such settings, which made both their theoretical and experimental investigation even more interesting, was the presence of nonlinearity, which, in turn, rendered worthwhile the exploration of the dynamics of nonlinear waves (such as bright or gap solitons [3,4] and more recently dark solitons and vortices [5]). The above optical settings were in fact the ones that enabled the first experimental realizations of $PT$-symmetry. This was done in the context of waveguide couplers (i.e. either two waveguides with and without loss [6]—the so-called passive $PT$—or in the more ‘standard’ case of one waveguide with gain and one with loss [7]). More recently, electrical analogues of such systems have been engineered in the work of Schindler et al. [8]. In parallel to these pioneering steps in the realm of experiments, numerous theoretical investigations have arisen both in the context of linear $PT$-symmetric potentials [9–26] and even in the case of the so-called nonlinear $PT$-symmetric potentials (whereby a $PT$-symmetric type of gain/loss pattern appears in the nonlinear term) [27–29].

In the present work, we revisit this theme of both linear and nonlinear $PT$-symmetric potentials. We do so in the special context of the recently proposed $PT$-symmetric ‘oligomers’, namely few site configurations, which are not necessarily couplers (i.e. dimers) but rather can also be trimers, quadrimers, etc. For the sake of illustration, we restrict ourselves to dimers and trimers herein but briefly also touch upon recent works in the context of quadrimers. In what follows below, we study the case of nonlinear $PT$-symmetric dimers [27] and trimers and subsequently restrict considerations to the case of their linear $PT$-symmetric analogues [15].

One of the fundamental side-effects of the fact that the nonlinearity does not commute with the $PT$ operator is the existence of nonlinear solutions that persist past the linear $PT$ phase transition threshold. Furthermore, one of the principal consequences of the presence of gain/loss terms in the nonlinearity is the existence of both symmetric and asymmetric (in their amplitude) stationary solutions, with the latter possessing a non-symmetric linearization spectrum. Interesting bifurcation phenomena (such as spontaneous symmetry breakings) are, additionally, found to arise in this case. Our presentation is structured as follows: we first examine the nonlinear $PT$-symmetric dimer in §2 (existence of solutions in §2a, linear stability set-up in §2b and numerical results in §2c), while the corresponding analysis for the trimer is done in §3. In §4, we review the corresponding case of linear $PT$-symmetric oligomers, while in §5 we summarize our findings and present our conclusions.

## 2. Analysis of stationary solutions for the nonlinear $PT$-symmetric dimer case

We first consider the so-called $PT$-symmetric coupler or dimer, in which a gain/loss pattern appears in both the linear and nonlinear terms. The dynamical equations have the form
\[
\begin{align*}
\text{iu}_t &= -\varepsilon v + (\rho_r - i\rho_{\text{im}})|u|^2 u + i\gamma u, \\
\text{iv}_t &= -\varepsilon u + (\rho_r + i\rho_{\text{im}})|v|^2 v - i\gamma v.
\end{align*}
\] (2.1)

The model contains the Kerr nonlinearity, which is relevant to optical waveguides and is effectively a generalization of the experimental framework of Rüter et al. [7], in that nonlinear (i.e. amplitude-dependent) gain and loss processes are taken into account. We have used \(u(t)\) and \(v(t)\) to denote the two complex-valued variables for the dimer and the evolution variable is \(t\) (in optics, this is actually a spatial variable standing for the propagation distance along the optical crystal). Considering the prototypical stationary solutions of the system, we let \(u(t)\) and \(v(t)\) have the forms
\[
u(t) = a e^{iE_1} \quad \text{and} \quad v(t) = b e^{iE_1},
\] (2.2)

where \(E\) is the propagation constant while the complex numbers \(a\) and \(b\) denote the amplitudes of the dimer sites. Plugging this ansatz into equation (2.1), one finds the complex nonlinear algebraic equations
\[
\begin{align*}
E_a &= \varepsilon b - (\rho_r - i\rho_{\text{im}})|a|^2 a - i\gamma a, \\
E_b &= \varepsilon a - (\rho_r + i\rho_{\text{im}})|b|^2 b + i\gamma b.
\end{align*}
\] (2.3)

We now use a polar decomposition of \(a\) and \(b\) of the form
\[
a = A e^{i\phi_a} \quad \text{and} \quad b = B e^{i\phi_b},
\] (2.4)

for real-valued \(A\), \(B\), \(\phi_a\) and \(\phi_b\). Plugging equation (2.4) into equation (2.3) and writing these equations in terms of their real and imaginary parts, we find
\[
\begin{align*}
EA &= \varepsilon B \cos(\phi_b - \phi_a) - \rho_r A^3, \\
EB &= \varepsilon A \cos(\phi_b - \phi_a) - \rho_r B^3, \\
-\varepsilon A &\sin(\phi_b - \phi_a) - \rho_{\text{im}} B^3 + \gamma B = 0, \\
\varepsilon B &\sin(\phi_b - \phi_a) + \rho_{\text{im}} A^3 - \gamma A = 0.
\end{align*}
\] (2.5)

The last two equations yield
\[
(A^2 - B^2)[\rho_{\text{im}}(A^2 + B^2) - \gamma] = 0.
\] (2.6)

We note that equation (2.6) yields a simple algebraic condition which connects the amplitude of the two dimer sites. This allows us to distinguish several subcases of interest. We look for non-trivial solutions \(A\) and \(B\) in each of the subcases presented in the following section.

(a) **Existence of localized modes for the dimer case**

Equation (2.6) identifies the different scenarios for the values of \(A\) and \(B\). We now examine the three cases that arise from this equation for our dimer dynamical system.

(i) **Case 1:** \(A^2 = B^2\) and \(A^2 + B^2 \neq \gamma / \rho_{\text{im}}\)

Recall the equations given in (2.5),
\[
\begin{align*}
EA &= \varepsilon B \cos(\phi_b - \phi_a) - \rho_r A^3; \quad EB &= \varepsilon A \cos(\phi_b - \phi_a) - \rho_r B^3
\end{align*}
\] (2.7)

and
\[
\begin{align*}
\varepsilon B &\sin(\phi_b - \phi_a) + \rho_{\text{im}} A^3 - \gamma A = 0; \quad -\varepsilon A &\sin(\phi_b - \phi_a) - \rho_{\text{im}} B^3 + \gamma B = 0.
\end{align*}
\] (2.8)

Since \(A = B\) (i.e. these are symmetric solutions) in this case, the two equations in each set are equivalent. Thus, we have
\[
\sin(\phi_b - \phi_a) = \frac{-\rho_{\text{im}} A^2 - \gamma}{\varepsilon} \quad \text{and} \quad \cos(\phi_b - \phi_a) = \frac{\rho_r A^2 + \gamma}{\varepsilon}.
\] (2.9)
We use the relation \(\sin^2(\phi_b - \phi_a) + \cos^2(\phi_b - \phi_a) = 1\) to determine the following quadratic equation for \(A^2\):

\[
(\rho_r^2 + \rho_{\text{im}}^2)A^4 + 2(E\rho_r - \gamma\rho_{\text{im}})A^2 + \gamma^2 + E^2 - \epsilon^2 = 0. 
\]

(2.10)

The solution of the resulting bi-quadratic equation reads

\[
A^2 = B^2 = \frac{-(E\rho_r - \gamma\rho_{\text{im}}) \pm \sqrt{(E\rho_r - \gamma\rho_{\text{im}})^2 - (\rho_r^2 + \rho_{\text{im}}^2)(\gamma^2 + E^2 - \epsilon^2)}}{\rho_r^2 + \rho_{\text{im}}^2},
\]

(2.11)

with the restriction that

\[
(E\rho_r - \gamma\rho_{\text{im}})^2 \geq (\rho_r^2 + \rho_{\text{im}}^2)(\gamma^2 + E^2 - \epsilon^2).
\]

(2.12)

(ii) Case II: \(A^2 + B^2 = \gamma / \rho_{\text{im}}\) and \(A^2 \neq B^2\)

Under these conditions, one can obtain

\[
A^2 = \frac{\gamma}{2\rho_{\text{im}}} \pm \frac{\sqrt{\gamma^2 - \epsilon^2}}{2\rho_{\text{im}}},
\]

(2.13)

\[
B^2 = \frac{\gamma}{2\rho_{\text{im}}} \mp \frac{\sqrt{\gamma^2 - \epsilon^2}}{2\rho_{\text{im}}},
\]

(2.14)

\[
E = -\frac{\gamma\rho_r}{\rho_{\text{im}}},
\]

(2.15)

and

\[
\cos(\phi_b - \phi_a) = -\frac{\epsilon\rho_r}{\sqrt{\rho_r^2 + \rho_{\text{im}}^2}},
\]

(2.16)

with the restriction that

\[
\frac{\gamma^2}{4\rho_{\text{im}}^2} \geq \frac{\epsilon^2}{\rho_r^2 + \rho_{\text{im}}^2}.
\]

(2.17)

A fundamental difference of this case from case I is that here \(E\) is no longer a free parameter \[27\]. The solutions with the different amplitudes will be called asymmetric in what follows.

(iii) Case III: \(A^2 + B^2 = \gamma / \rho_{\text{im}}\) and \(A^2 = B^2\)

As a final ‘mixed’ possibility, between the above symmetric and asymmetric cases, from equation (2.5), it is straightforward to obtain

\[
A = B = \sqrt{\frac{\gamma}{2\rho_{\text{im}}}},
\]

(2.18)

\[
\cos(\phi_b - \phi_a) = \frac{2\rho_{\text{im}}E + \gamma\rho_r}{2\epsilon\rho_{\text{im}}},
\]

(2.19)

and

\[
\sin(\phi_b - \phi_a) = \frac{\gamma}{2\epsilon},
\]

(2.20)

with the restriction that

\[
\left(\frac{2\rho_{\text{im}}E + \gamma\rho_r}{2\epsilon\rho_{\text{im}}}\right)^2 + \left(\frac{\gamma}{2\epsilon}\right)^2 = 1.
\]

(2.21)

Once again this implies that, once other parameters (such as \(\gamma, \rho_{\text{im}}, \rho_r\) and \(\epsilon\)) are determined, \(E\) is not a free parameter; rather, it is obtained from equation (2.21). These will be referred to as special symmetric solutions in the following.

It is particularly important to highlight that both solutions of case II (asymmetric) and those of case III (special symmetric) are present owing to competing effects of the linear and nonlinear gain/loss profiles; note the opposite signs thereof in equation (2.1) and the necessity of \(\gamma\rho_{\text{im}} > 0\) for such solutions to exist. In the case where the linear and nonlinear gain/loss cooperate (rather than compete), such solutions would obviously be absent and the system would be inherently less wealthy in its potential dynamics. This point was also discussed by Miroshnichenko et al. \[27\].
(b) Linear stability analysis for the dimer case

We now go back to our original $PT$-symmetric dimer with linear and nonlinear gain and loss in equation (2.1) and examine the linear stability of the solutions to this equation. We begin by setting

$$u(t) = e^{iEt}[a + p e^{i\lambda t} + P e^{i\lambda^* t}], \quad v(t) = e^{iEt}[b + q e^{i\lambda t} + Q e^{i\lambda^* t}], \quad (2.22)$$

where $\lambda$ is a complex-valued eigenvalue parameter revealing the growth (instability) or oscillation (stability) of all the modes of linearization of the dimer system, the asterisk denotes the complex conjugate and $p, P, q, Q$ are perturbations to the solutions of interest. Plugging equation (2.22) into equation (2.1) and taking only the linear terms in $p, P, q$ and $Q$, we find the following eigenvalue problem:

$$AX = i\lambda X, \quad (2.23)$$

where $X = (p, -P^*, q, -Q^*)^T$ and $A$ is written as

$$A = \begin{pmatrix}
    a_{11} & -a_2(\rho_r - i\rho_{im}) & -\epsilon & 0 \\
    (a^*)^2(\rho_r + i\rho_{im}) & a_{22} & 0 & \epsilon \\
    -\epsilon & 0 & a_{33} & -b^2(\rho_r + i\rho_{im}) \\
    0 & \epsilon & (b^*)^2(\rho_r - i\rho_{im}) & a_{44}
\end{pmatrix}, \quad (2.24)$$

where

$$\begin{align*}
    a_{11} &= E + 2|a|^2(\rho_r - i\rho_{im}) + i\gamma, \\
    a_{22} &= -2|a|^2(\rho_r + i\rho_{im}) - E + i\gamma, \\
    a_{33} &= E - i\gamma + 2|b|^2(\rho_r + i\rho_{im}) \\
    a_{44} &= E - i\gamma - 2|b|^2(\rho_r - i\rho_{im}).
\end{align*} \quad (2.25)$$

The use of the symmetric, asymmetric or mixed solutions of the previous subsection into these matrix elements produces a $4 \times 4$ complex matrix whose eigenvalues will determine the spectral stability of the corresponding nonlinear solution. The existence of eigenvalues with positive real part $\lambda_r > 0$ amounts to a dynamical instability of the relevant solution, while in the case where all four eigenvalues have $\lambda_r \leq 0$ the solution is linearly stable.

(c) Numerical results for the dimer case

Figure 1 shows the profile of the different branches for the dimer case and for parameters $\epsilon = 1$, $E = 1$, $\rho_r = -2$ and $\rho_{im} = 1$ (unless noted otherwise). The branches denoted by stars and diamonds correspond to case I of the symmetric solutions; these two branches collide and disappear at the critical point $\gamma = 1.61$ (when equation (2.12) becomes an equality). The branches denoted by circles and crosses correspond to case II; and the branch denoted by squares corresponds to case III. For the last two branches, when $\gamma$ is varied, $E$ is also varied too (rather than staying fixed at $E = 1$ as for case I) according to equations (2.15) and (2.21), respectively. Similar notation is used in Figure 2, which shows the linear stability eigenvalues $\lambda = \lambda_r + i\lambda_i$ of the linearization. While the branches of case I are stable, it is interesting to note that the branch of case III (squares) is stable until a pitchfork (symmetry breaking) bifurcation arises at $\gamma = 0.895$ (when equation (2.17) becomes an equality) and acquires a real pair of eigenvalues thereafter signalling its dynamical instability. On the other hand, it is at that critical point that the two branches belonging to case II collide. While the special symmetric branch of case III (denoted by squares) persists up to the critical point of $\gamma = 2\epsilon = 2$ of equation (2.20), it should be pointed out that nonlinearity enables the asymmetric branches of case II to persist for large values of $\gamma$, in fact well past the point of the linear $PT$ phase transition. This feature has been highlighted in a number of recent works [15,30]; in the case of the dimer, the linear critical $PT$ phase transition point is identified as $\gamma = \epsilon$, while for the trimer setting considered below it is $\gamma = \sqrt{2}\epsilon$. 

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Figure 1. The solution profiles of the nonlinear $\mathcal{PT}$-symmetric dimer case with $\epsilon = 1, \rho_r = -2$ and $\rho_m = 1$. (a–d) The continuation of each branch (the amplitudes in (a), the phases in (b) and the real (c) and imaginary (d) parts of the linear stability eigenvalues) starting from the conservative system at $\gamma = 0$. The five branches are denoted by curves of stars, diamonds, squares, circles and crosses. Stars, case I with ‘−’ in the amplitude; diamonds, case I with ‘+’ in the amplitude; circles, case II with ‘+’ in the amplitude (of $A$); crosses, case II with ‘−’ in the amplitude (of $A$); squares, case III. Note that the eigenvalues of circles and crosses are opposite to each other (see the relevant discussion in the text). We always set $E = 1$ in the case I branches, namely the stars and the diamonds, which terminate at the same point when $\gamma = 1.61$. The squares are subject to a destabilizing supercritical pitchfork bifurcation at $\gamma = 0.895, E = 1.789$, whereby the circles and crosses arise. The squares branch terminates at $\gamma = 2$; the circles and crosses exist for arbitrary values of the (linear) gain/loss past the linear $\mathcal{PT}$-symmetry breaking point. (Online version in colour.)

Figure 2. The eigenvalue plots illustrating the linear stability of the nonlinear $\mathcal{PT}$-symmetric dimer with $\epsilon = 1, \rho_r = -2$ and $\rho_m = 1$ for three different values of $\gamma = (a) 0.5, (b) 1$ and (c) 1.5. For the stars and diamonds branches, we use $E = 1$ here, while for the case II (circles and crosses) and case III (squares), $E$ is determined from the remaining parameters based on equations (2.15) and (2.21), respectively. (Online version in colour.)

An additional point worthy of mention here is that, in linear $\mathcal{PT}$-symmetric chains (just as is the case in typical Hamiltonian systems), if $\lambda$ is an eigenvalue to the linearization problem around a solution, so are $-\lambda, \bar{\lambda}$ and $-\bar{\lambda}$ (where the overbar denotes the complex conjugation here). However, in our nonlinear $\mathcal{PT}$-symmetric dimer $-\lambda$ and $-\bar{\lambda}$ may not appear in the linearization around a particular branch, as is observed in figure 2. Eigenvalues of the branches denoted by circles and crosses are not symmetric about the imaginary axis, but are symmetric with respect to each other. One can see from figure 3 that the branch denoted by circles is always stable, while the branch denoted by crosses is always unstable (owing to an oscillatory instability associated with a complex eigenvalue pair). This is because the existence of asymmetry in these solutions of case II creates, in turn, asymmetries in the linearization matrix, because of the nonlinear gain/loss...
Figure 3. The dynamical evolution plots of the branches for the case of the nonlinear $PT$-symmetric dimer with the same parameter settings as in figure 2 when $\gamma = 1.5$. The symmetric stars and diamonds of case I and the asymmetric circles of case II are stable, while the squares of case III (past the pitchfork point) and crosses of case II are unstable and deviate from their initial profile during the dynamics (see also the discussion in the text). (a) Stars branch, (b) diamonds branch, (c) squares branch, (d) circles branch and (e) crosses branch. (Online version in colour.)

term, which breaks the $PT$ symmetry of the linearization matrix and produces the corresponding observable asymmetry in eigenvalues.

The dynamical evolution of the different elements of the bifurcation diagram of nonlinear $PT$-symmetric dimer is shown in figure 3 at a fixed $\gamma = 1.5$. In all the cases here and below, where a stationary solution exists for the parameter value for which it is initialized, a numerically exact solution up to $10^{-8}$ is typically used as an initial condition in the system. The system is sufficiently sensitive to dynamical instabilities that even the amplification of roundoff errors is enough to observe them. The stability of the case I branches is evident in the invariance of the relevant states during the course of the simulation (stars and diamonds). On the other hand, the branch denoted by squares is attracted towards the asymmetric (yet stable, as is evident in the corresponding simulation) branch that is denoted by circles. Finally, the asymmetric branch denoted by crosses leads to indefinite growth of the site with the larger amplitude (nonlinear gain) and the decay of the site with the smaller amplitude (nonlinear loss).

3. Analysis of stationary solutions for the nonlinear $PT$-symmetric trimer case

We now consider the generalization of the above considerations to the case of a so-called $PT$-symmetric trimer. Here, the dynamical system associated with a potential application of a three-waveguide setting is of the form

\[
\begin{align*}
    iu_t &= -\epsilon v + (\rho_r - i\rho_{im}) |u|^2 u + iyu, \\
    iv_t &= -\epsilon (u + w) - |u|^2 v \\
    iw_t &= -\epsilon v + (\rho_r + i\rho_{im}) |w|^2 w - iyw.
\end{align*}
\]

Such configurations have been considered earlier in optical applications theoretically [31,32] and even experimentally [33] in the absence of gain/loss. We examine this case with both linear and nonlinear gain/loss profiles. Once again, as in the case of the dimer, we present the richer
phenomenology setting of direct competition between linear and nonlinear gain/loss. The middle site is assumed as devoid of gain and loss. The Kerr nonlinearity is also assumed to be present in all three sites. Here, we use $u(t), v(t)$ and $w(t)$ as the complex-valued components for the trimer. For the stationary solutions, we again assume

$$u(t) = a e^{iEt}, \quad v(t) = b e^{iEt} \quad \text{and} \quad w(t) = c e^{iEt}. \quad (3.2)$$

Plugging equations (3.2) into equation (3.1), we find

$$Ea = \epsilon b - (\rho_r - i\rho_{im})|a|^2 a - i \gamma a, \quad Eb = \epsilon (a + c) + |b|^2 b$$

and

$$Ec = \epsilon b - (\rho_r + i\rho_{im})|c|^2 c + i \gamma c. \quad (3.3)$$

Since $a, b$ and $c$ are complex-valued functions, we use the polar decomposition,

$$a = A e^{i\phi_a}, \quad b = B e^{i\phi_b} \quad \text{and} \quad c = C e^{i\phi_c}, \quad (3.4)$$

where $A, B, C, \phi_a, \phi_b$ and $\phi_c$ are real-valued. Plugging equation (3.4) into equation (3.3) and separating the real and imaginary parts, we derive the following set of real-valued equations for $A, B$ and $C$:

$$
\begin{align*}
EA &= \epsilon B \cos(\phi_b - \phi_a) - \rho_r A^3, \quad EB = \epsilon A \cos(\phi_b - \phi_a) + \epsilon C \cos(\phi_b - \phi_c) + B^3, \\
EC &= \epsilon B \cos(\phi_b - \phi_c) - \rho_l C^3, \quad EB = \epsilon \sin(\phi_b - \phi_a) - \epsilon C \sin(\phi_b - \phi_c) + 0
\end{align*}
\quad (3.5)$$

and

We seek non-trivial solutions to equations (3.5), i.e. $(A, B, C) \neq (0, 0, 0)$. We can reduce equations (3.5) to the form

$$
\begin{align*}
\cos(\phi_b - \phi_a) &= \frac{EA + \rho_l A^3}{\epsilon B}, \quad \cos(\phi_b - \phi_c) = \frac{EC + \rho_l C^3}{\epsilon B}, \\
\sin(\phi_b - \phi_a) &= \frac{\gamma A - \rho_{im} A^3}{\epsilon B}, \quad \sin(\phi_b - \phi_c) = \frac{-(\gamma C - \rho_{im} C^3)}{\epsilon B}, \\
A \cos(\phi_b - \phi_a) + C \cos(\phi_b - \phi_c) &= \frac{EB - B^3}{\epsilon}
\end{align*}
\quad (3.6)$$

and

We apply the first four equations of equations (3.6) into the last two equations and obtain the following relations:

$$B^4 - EB^2 + E(A^2 + C^2) + \rho_r (A^4 + C^4) = 0, \quad (A^2 - C^2)[\gamma - \rho_{im}(A^2 + C^2)] = 0. \quad (3.7)$$

We now determine $A, B$ and $C$ for several subcases (symmetric, asymmetric and mixed), as was done for the dimer case in §2.

(a) Existence of localized modes for the trimer case

For the trimer case, the special cases that can be seen to emerge for the solutions of equations (3.7) can be classified as follows.
(i) Case I: $A^2 = C^2$ and $A^2 + C^2 \neq \gamma / \rho_{\text{im}}$

In this case, the algebraic equations assume the form

$$\begin{align*}
\cos(\phi_b - \phi_a) &= \frac{EA + \rho r A^3}{\epsilon B} = \cos(\phi_b - \phi_c) \\
\sin(\phi_b - \phi_a) &= \frac{\gamma A - \rho_{\text{im}} A^3}{\epsilon B} = -\sin(\phi_b - \phi_c).
\end{align*}$$

(3.8)

and

We now use equation (3.7) and $\cos^2(\phi_b - \phi_a) + \sin^2(\phi_b - \phi_a) = 1$ to determine:

$$\begin{align*}
(\rho_r^2 + \rho_{\text{im}}^2)A^6 + 2(E\rho_r - \gamma \rho_{\text{im}})A^4 + (E^2 + \gamma^2)A^2 - \epsilon^2 B^2 &= 0 \\
B^4 - EB^2 + 2EA^2 + 2\rho_r A^4 &= 0.
\end{align*}$$

(3.9)

One can solve equations (3.9) for $A^2$ and $B^2$ to complete the calculation of the relevant symmetric branch of solutions of case I.

(ii) Case II: $A^2 + C^2 = \gamma / \rho_{\text{im}}$ and $A^2 \neq C^2$

From equation (3.6), we obtain the four algebraic equations, as follows:

$$\begin{align*}
A^2(E + \rho r A^2)^2 + A^2(\gamma - \rho_{\text{im}} A^2)^2 &= \epsilon^2 B^2, \\
C^2(E + \rho r C^2)^2 + C^2(\gamma - \rho_{\text{im}} C^2)^2 &= \epsilon^2 B^2, \\
A^2 + C^2 &= \frac{\gamma}{\rho_{\text{im}}} \\
B^4 - EB^2 + E(A^2 + C^2) + \rho_r (A^4 + C^4) &= 0.
\end{align*}$$

(3.10)

We now have four equations but with only three unknowns ($A$, $B$ and $C$). Therefore, in contrast to the previous symmetric branch of case I, one of the parameters $E, \epsilon, \rho_r, \rho_{\text{im}}, \gamma$ is determined by the other four, i.e. not all five of these parameters can be picked independently in order to give rise to a solution of the trimer. Once again, we should, nevertheless, highlight here that these asymmetric solutions only exist because of the interplay of linear gain/loss and nonlinear loss/gain profiles.

(iii) Case III: $A^2 + C^2 = \gamma / \rho_{\text{im}}$ and $A^2 = C^2$

In this mixed case, we have

$$\begin{align*}
A^2 &= C^2 = \frac{\gamma}{2\rho_{\text{im}}} \\
B^4 - EB^2 + 2EA^2 + 2\rho_r A^4 &= 0.
\end{align*}$$

(3.11)

with the restriction that

$$E^2 - \frac{4\gamma E}{\rho_{\text{im}}} - \frac{2\rho_r \gamma^2}{\rho_{\text{im}}^2} \geq 0.$$  

(3.12)

One can solve the following equations for $B$ and $E$:

$$\begin{align*}
B^4 - EB^2 + 2E\frac{\gamma}{2\rho_{\text{im}}} + 2\rho_r \frac{\gamma^2}{4\rho_{\text{im}}^2} &= 0 \\
(\rho_r^2 + \rho_{\text{im}}^2)\frac{\gamma^3}{8\rho_{\text{im}}^3} + (2\rho_r E - 2\rho_{\text{im}} \gamma)\frac{\gamma^2}{4\rho_{\text{im}}^3} + (E^2 + \gamma^2)\frac{\gamma}{2\rho_{\text{im}}} &= \epsilon^2 B^2.
\end{align*}$$

(3.13)

These equations imply that one of the parameters (e.g. $E$) will be determined once the parameters $\gamma, \epsilon, \rho_r$ and $\rho_{\text{im}}$ are chosen.
We again consider the nonlinear $\mathcal{PT}$-symmetric trimer model with linear and nonlinear gain/loss and examine the linear stability of its solutions given in equation (3.1) for the solutions given in the previous section. We begin by positing the linearization ansatz,

$$u(t) = e^{i\lambda t}[a + p e^{i\xi} + P e^{i\xi^*}], \quad v(t) = e^{i\lambda t}[b + q e^{i\xi} + Q e^{i\xi^*}]$$

and

$$w(t) = e^{i\lambda t}[c + r e^{i\xi} + R e^{i\xi^*}],$$

where $p, P, q, Q, r$ and $R$ are perturbations to the solutions of interest. Plugging equation (3.14) into equation (3.1) and truncating at the linear order in $p, P, q, Q, r$ and $R$, we derive the following eigenvalue problem,

$$AY = i\lambda Y,$$  \hspace{1cm} (3.15)

where $Y = (p, -P^*, q, -Q^*, r, -R^*)^T$ and $A$ is the $(6 \times 6)$ matrix,

$$A = 
\begin{bmatrix}
    a_{11} & -a^2(\rho_r - i\rho_{im}) & -\epsilon & 0 & 0 & 0 \\
    (a^*)^2(\rho_r + i\rho_{im}) & a_{22} & 0 & 0 & 0 & 0 \\
    -\epsilon & 0 & E - 2|b|^2 & b^2 & -\epsilon & 0 \\
    0 & \epsilon & -(b^*)^2 & -E + 2|b|^2 & 0 & \epsilon \\
    0 & 0 & -\epsilon & -\epsilon & a_{55} & -c^2(\rho_r + i\rho_{im}) \\
    0 & 0 & 0 & \epsilon & (c^*)^2(\rho_r - i\rho_{im}) & a_{66}
\end{bmatrix},$$  \hspace{1cm} (3.16)

where

$$a_{11} = E + i\gamma + 2|a|^2(\rho_r - i\rho_{im}),$$
$$a_{22} = -E + i\gamma - 2|a|^2(\rho_r + i\rho_{im}),$$
$$a_{55} = E - i\gamma + 2|c|^2(\rho_r + i\rho_{im}),$$
$$a_{66} = -E - i\gamma - 2|c|^2(\rho_r - i\rho_{im}).$$  \hspace{1cm} (3.17)

The solution of this $6 \times 6$ eigenvalue problem (and whether the corresponding eigenvalues $\lambda$ possess a positive real part) will determine the spectral stability properties of the solutions of the nonlinear $\mathcal{PT}$-symmetric trimer.

(c) Numerical results for the trimer case

(i) Trimer case I

The numerical results for the symmetric solutions of the nonlinear $\mathcal{PT}$-symmetric trimer (case I) are shown in figures 4–6, with similar notation as in the dimer case. Solutions are found by numerically solving equation (3.9). A typical example of the branches that may arise in case I of the trimer is shown for the parameters $\epsilon = 1$, $E = 1$, $\rho_r = -1$ and $\rho_{im} = 1$. In this case, we find three branches in the considered interval of parameter values. There are two branches which exist up to the point $\gamma = 2.59$, where they collide in a saddle-node bifurcation. One of these, the diamonds branch, is mostly unstable except for $\gamma \in [1.26, 1.33] \cup [2, 2.11]$. For $\gamma < 1.26$, this branch has one real and one imaginary pair, both of which become imaginary for $\gamma > 1.26$ until they collide for $\gamma = 1.33$ and yield a complex quartet, which subsequently splits into two imaginary pairs for $\gamma = 2$ and finally into one real and one imaginary pair for $\gamma > 2.11$. The other one, the squares branch, is always unstable owing to one real and one imaginary pair. When these two modes collide, a collision arises between both their real and their imaginary (respectively) eigenvalue pairs.

Aside from the other two branches, the branch denoted by the stars emerges from $\gamma = 1$ and persists beyond the above critical point (and for all values of $\gamma$ that we have monitored). In our case, this branch is only stable for $\gamma < 1.25$, at which two pairs of imaginary eigenvalues collide and lead to a complex quartet, which renders the branch unstable thereafter. This branch behaves
Figure 4. The symmetric solution profiles of case I in the nonlinear $\mathcal{PT}$-symmetric trimer with $\epsilon = 1, E = 1, \rho_r = -1$ and $\rho_m = 1$. The three branches are denoted by stars, diamonds and squares and their (a) amplitudes, (b) phases, (c) real and (d) imaginary parts of the corresponding eigenvalues are shown. See also the relevant discussion in the text. (Online version in colour.)

Figure 5. The spectral plane of the linear stability analysis for the symmetric solutions of case I with $\epsilon = 1, E = 1, \rho_r = -1$ and $\rho_m = 1$, for three different values of $\gamma = (a) 0.5, (b) 1.5$ and (c) 2.5. Each branch is associated with three eigenvalue pairs, one of which is at 0 owing to symmetry. (Online version in colour.)

Figure 6. The time evolution plots of the trimer case I with $\epsilon = 1, E = 1, \rho_r = -1$ and $\rho_m = 1$ when $\gamma = 1.5$. For each branch, the blue line denotes the nonlinear loss/linear gain site, the red line denotes the nonlinear gain/linear loss site, while the green represents the inert site between the two. Note the oscillatory evolution of the blue solid branch, while the red diamonds and black squares lead to ultimate unbounded increase of at least one site within the trimer. (a) Stars branch, (b) diamonds branch and (c) squares branch. (Online version in colour.)
Figure 7. The time evolution plots of the trimer case I with $\epsilon = 1$, $E = 1$, $\rho_r = -1$ and $\rho_{im} = 1$ when $\gamma = 1.5$. For each branch, the solid line denotes the nonlinear loss/linear gain site, the dot-dashed line denotes the nonlinear gain/linear loss site, while the dashed line represents the inert site between the two. Notice the oscillatory evolution of the stars branch, while the diamonds and squares branches lead to ultimate unbounded increase of at least one site within the trimer. (a) Stars branch, (b) diamonds branch and (c) squares branch. (Online version in colour.)

very similarly to the one in the linear $\mathcal{PT}$ trimer case reported by Li & Kevrekidis [15]. Both of them bifurcate from zero amplitude after a certain value of $\gamma$, persist beyond the linear $\mathcal{PT}$ critical point and have similar stability properties.

To monitor the dynamical evolution of the different branches, we used the direct numerical simulations illustrated in figure 6 for the case of $\gamma = 1.5$. Two of the branches, the stars branch in figure 6a and the diamonds branch in figure 6b, are oscillatory unstable for this value of $\gamma$, while the squares branch (figure 6c) is always unstable owing to a real eigenvalue pair. The last has been found to generically cause the unbounded gain of at least one node within the trimer. The oscillatory instability, on the other hand, in the case of the branch denoted by stars and for $\gamma = 1.5$ can be observed to lead to a long-lived periodic exchange of ‘power’ between the three sites. On the other hand, for the diamonds branch in figure 6b, while there is an intermediate stage of power oscillations between the three nodes, the ultimate fate of the configuration favours the unbounded growth of at least one node (in fact, two nodes in the example shown) of the trimer.

(ii) Trimer case II

According to equation (3.10), one of the parameters should be determined by the others. We hereby set $\epsilon = 1$, $\rho_r = -1$ and $\rho_{im} = 1$, then $E$ is obtained self-consistently for a given choice of $\gamma$. The solution profiles, which are obtained by solving equation (3.10) numerically, are plotted in figure 7. There are three pairs of branches, i.e. six branches of solutions in total are found in this case. Only one out of each pair is shown in figures 7–9 (to avoid cluttering of the relevant figures), namely the stars, diamonds and squares. The other three branches are mirror symmetric to these branches, respectively. For example, the existence profile of the mirror symmetric branch of the stars would be identical to the stars shown in figure 7, while its stability plot would be mirror symmetric about the imaginary axis to the stars shown in figure 8. Among the three branches shown in figure 7, two of them emerge at $\gamma = 2$ and persist throughout the range of $\gamma$ values considered. It should be noticed that the amplitudes are different within each branch. The stars (dynamically stable) branch has a large $A$ and small $B$ and $C$, while the diamonds and squares branches have a fairly small $A$ and large values of $B$ and $C$. In fact, precisely at the critical point of the branches’ emergence, the stars and the squares are exact mirror images of each other.
Figure 8. The spectral stability plots of the trimer case II with $\epsilon = 1$, $\rho_r = -1$ and $\rho_{im} = 1$, illustrating the stability of the stars branch and the instability of the other two. (Online version in colour.)

Figure 9. The time evolution plots of the trimer case II with $\epsilon = 1$, $\rho_r = -1$ and $\rho_{im} = 1$ when $\gamma = 3$. The two unstable branches (diamonds and squares) tend to a dynamically stable configuration which is a mirror image of the diamonds branch. (a) Stars branch, (b) diamonds branch and (c) squares branch. (Online version in colour.)

(i.e. they have the same amplitude for $B$ and the one’s $A$ is the other’s $C$ and vice versa). This mirror symmetry is in fact directly reflected in the eigenvalues of the linearization around the two configurations, one set of which (for the stars) possesses negative real parts, while the other (squares) has mirror symmetric positive ones. As can be perhaps intuitively anticipated, the more stable configuration is the one having large amplitude at the nonlinear loss/linear gain site. The third branch (diamonds) is also highly unstable and emerges out of a bifurcation at $\gamma = 2.05$ (to which we will return when discussing case III).

Figure 8 shows the eigenvalues of the three branches clearly illustrating the fact that they are not symmetric about the imaginary axis. This can once again be justified by the asymmetry of the configurations of case II which, in turn, break the $PT$ symmetry of the linearization matrix and hence lead to asymmetric spectra. The stars branch is always stable, as indicated above, and the other two branches are always unstable as $\gamma$ increases. For instance, for the diamond branch, there exists (in addition to a zero eigenvalue) an imaginary pair, a complex conjugate pair (with a positive real part) and a real eigenvalue. In the case of the squares branch, there are (in addition to the zero eigenvalue) three real pairs (two positive and one negative) and a complex conjugate pair (with positive real part).

Figure 9 shows the dynamical plots of the three distinct branches of solutions. The stars branch clearly preserves its configuration owing to its dynamical stability, while for the two unstable branches, their evolution gives rise to asymmetric dynamics favouring the loss of the power in a single site (the nonlinear gain/linear loss one), and quickly absorbed by a stable state. The latter state appears to be the mirror symmetric of the diamonds asymmetric branch in both cases. This is indeed also a stable dynamical state of the original stationary system of equations. Furthermore, this state is expected to exist based on the symmetry-breaking bifurcation that we will discuss
Figure 10. The solution profile (amplitude (a), phase (b)) and the stability (real (c) and imaginary (d) part of the eigenvalues) for the nonlinear $PT$-symmetric trimer of case III (special symmetric solutions) with $\epsilon = 1, \rho_r = -1$ and $\rho_{im} = 1$. The norms are not squared in (a) to improve the visibility of the branches (given the disparity of the relevant amplitudes). The four branches are denoted by stars, diamonds, squares and circles. The stars, diamonds and squares branches always have two pairs of purely imaginary eigenvalues, while the circles branch always has a complex quartet. The stars branch terminates with the circles at $\gamma = 2.1$, while the diamonds and squares terminate together at $\gamma = 0.65$. (Online version in colour.)

below as giving rise to the diamonds branch. As remarked above, the stability properties of the mirror symmetric branches are mirror symmetric to those shown in figure 8. This implies that only the mirror symmetric branch of the diamonds is stable, which is, in turn, consonant with our observation that it is a potential attractor for the dynamics for $\gamma = 3$ shown in figure 9.

(iii) Trimer case III

Finally, we turn to a consideration, using the same parametric setting as in case II, of the numerical results by solving equation (3.13) for case III in figures 10–12. Four distinct branches of solutions are observed in this case. The branches denoted by diamonds and squares exist only for small values of the linear gain/loss parameter $\gamma$, are stable and terminate at $\gamma = 0.65$. The other two branches, namely the stars and the circles, collide and terminate at $\gamma = 2.1$. The circles branch is unstable in this case, owing to a complex quartet of eigenvalues (observed in figure 11). On the other hand, the stars branch is stable up to $\gamma = 2.05$, a critical point at which branches of case II (the diamonds branch referred to in case II as having a pitchfork bifurcation at the same value and its mirror symmetric image) emerge. Note that this detail is not discernible in the eigenvalue plots of figure 10.

The dynamics of the different configurations are shown in figure 12. The stars, diamonds and squares, special symmetric branches of solutions, are stable and thus preserve their shape. On the other hand, the circles for $\gamma = 0.5$ are subject to the oscillatory instability predicted by the linear stability analysis. This, in turn, results in long-lived oscillatory dynamics of the system, as indicated in figure 12d.

4. The special case of linear $PT$-symmetric oligomers

As a special case with $\rho_r = -1$ and $\rho_{im} = 0$ of what we did above, the linear $PT$-symmetric oligomer case example has been addressed in the earlier work of Li & Kevrekidis [15].

Figure 13 shows the profile of the two branches which are analogues of the nonlinear $PT$-symmetric dimer displayed in figure 1. The branch denoted by dashed line corresponds to
the stars branch of case I within the nonlinear \( \mathcal{PT} \)-symmetric dimer with the \((-\)) sign in equation (2.11) and is stable when \( \gamma^2 \leq k^2 - E^2/4 \), whereas the solid line branch corresponding to the diamonds branch (of case I in the nonlinear \( \mathcal{PT} \)-symmetric dimer) is always stable. The linearization around these branches can be performed explicitly in this simpler linear \( \mathcal{PT} \)-symmetric case yielding the non-zero eigenvalue pairs \( \pm 2i\sqrt{2(\epsilon^2 - \gamma^2) - E\sqrt{\epsilon^2 - \gamma^2}} \) for the first and \( \pm 2i\sqrt{2(\epsilon^2 - \gamma^2) + E\sqrt{\epsilon^2 - \gamma^2}} \) for the second (note that the latter can never become real).

It is relevant to note here that the two branches ‘die’ in a saddle–centre bifurcation at \( \gamma = \epsilon \), as shown in the figure. This point coincides with the linear \( \mathcal{PT} \)-symmetric dimer phase transition. As indicated before, in the nonlinear dimer, the two branches die when the restriction (2.12) is
Figure 13. The two branches of solutions for the dimer problem are shown for parameter values $\epsilon = E = 1$. (a) The amplitude of the sites, (b) their relative phase and (c) the (non-trivial) squared eigenvalue of the two branches. The solid line corresponds to the always stable branch while the dashed line corresponds to the branch which acquires a real eigenvalue pair above a certain $\gamma = \sqrt{\epsilon^2 - E^2}/4$. Adapted from Li & Kevrekidis [15]. (Online version in colour.)

no longer satisfied. Nevertheless, the nonlinear solutions of the latter case can generally exist past the linear phase transition (and even arbitrarily past that, as in the case II solutions) and moreover asymmetric solutions can exist owing to the interplay of linear and nonlinear gain/loss, a feature absent in the simpler linear $\mathcal{PT}$-symmetric dimer.

As an analogue of the case I solutions of the nonlinear $\mathcal{PT}$-symmetric trimer, we present a prototypical example of the branches that may arise in the case of the linear $\mathcal{PT}$-symmetric trimer in figure 14 for $E = \epsilon = 1$. There are three distinct branches. Two of them collide in a saddle–centre bifurcation (for $\gamma = 1.043$) and disappear thereafter. The other one goes to 0 amplitude at $\gamma = \sqrt{2\epsilon^2 - E^2}$ but grows in amplitude beyond that point and persists beyond the critical point of the linear $\mathcal{PT}$ phase transition $\gamma_{\mathcal{PT}} = \sqrt{2\epsilon}$ (although it is unstable in that regime). Hence, the feature of solutions persisting past the linear $\mathcal{PT}$ phase transition exists even in the linear $\mathcal{PT}$-symmetric trimer, yet other more complex features do not appear in this setting. A canonical example thereof is the existence of asymmetric solutions (which, in turn, possess asymmetric spectra). The latter trait is amply present in the nonlinear $\mathcal{PT}$-symmetric trimer of the previous section.

It should be added that, in addition to the oligomers of the dimer and the trimer variety, recently there has also been considerable interest towards the study of quadrimer settings. Such settings were again initiated in the examination by Li & Kevrekidis [15], where the particular (and simpler) case of $(-i\gamma, -i\gamma, +i\gamma, +i\gamma)$, i.e. a linear $\mathcal{PT}$-symmetric quadrimer with two lossy sites on the one side and two gain nodes on the other side, was considered. The considerations of Li & Kevrekidis [15] were generalized in the very recent study of Zezyulin et al. [29], which examined the configuration $(-i\gamma_1, -i\gamma_2, +i\gamma_2, +i\gamma_1)$, i.e. the most general bi-parametric gain/loss family of quadrimers. In both cases, the possibility of solutions that emerge purely in the nonlinear regime and of those that continue past the critical point of the underlying linear $\mathcal{PT}$ phase transition was identified. Another reason why such quadrimers are of interest is that they can be thought of as the prototypical building blocks (if placed on a square ‘plaquette’) of a two-dimensional $\mathcal{PT}$-symmetric lattice [34]. To the best of our knowledge, nonlinear $\mathcal{PT}$-symmetric quadrimers, along the lines considered herein, have not been examined to date.
Figure 14. Existence and stability of solutions for the case of the trimer analogous to Figure 13 with parameters $E = 1$ and $\phi_b$ normalized to 0 (without loss of generality). There are three branches: $u^{(1)}$ (dashed-dotted line), $u^{(2)}$ (dashed line), and $u^{(3)}$ (solid line). For each branch, two curves in (a) stand for $A, B, C$ (since $C = A$), and two curves in (b) stand for $\phi_a$ and $\phi_c$. 

(c, d) The illustration of both real and imaginary parts for the eigenvalues. $u^{(1)}$ is mostly unstable except for a small interval of $\gamma \in [1, 1.035]$, $u^{(2)}$ is mostly stable except for $\gamma \in [1.035, 1.043]$. In (c), the eigenvalues of $u^{(1)}$ and $u^{(2)}$ in $\gamma \in [1.035, 1.043]$ are very close to each other but not identical. These two branches collide and disappear for $\gamma = 1.043$. The third branch $u^{(3)}$ decreases to zero in amplitude at $\gamma = 1$, but persists and grows thereafter (for all values of $\gamma$ considered), although it becomes unstable for $\gamma > 1.134$. Adapted from Li & Kevrekidis [15]. (Online version in colour.)

5. Conclusions and future challenges

In the above study, we illustrated some interesting characteristics which emanate from the interplay of nonlinearity with $\mathcal{PT}$-symmetric linear Hamiltonians in the case of oligomer configurations. The basic underlying premise which has been explained in the recent works of Zezyulin [30] and Li et al. [34] and has been explored by others [15, 27, 30] is that the nonlinear and the $\mathcal{PT}$-symmetric-linear part of the system at hand no longer commute and hence give rise to novel phenomenology that is not expected for linear $\mathcal{PT}$ symmetric systems. A principal element of this phenomenology that arises even for $\mathcal{PT}$-symmetric oligomers (trimers, quadrimers, etc.) with merely linear gain/loss is the fact that nonlinear solutions may exist that do not have a corresponding linear limit and which, in fact, defy the threshold for the linear $\mathcal{PT}$ phase transition in that they exist for arbitrary gain/loss parameter values past that critical point. The introduction of a nonlinear gain/loss pattern considered herein for dimers and trimers (and earlier in a more cursory way in [27] for dimers) presents additional possibilities stemming from the interplay of linear and nonlinear gain/loss profiles. These include among others the emergence of asymmetric solutions which not only are involved in symmetry-breaking (pitchfork) bifurcations but also produce asymmetric linearization matrices with spectral properties that reflect this asymmetry.

We believe that this direction of studies is particularly intriguing for further progress, especially as the complexity of the problem increases within the confines of a full one-dimensional chain, but also even for fundamental two-dimensional entities, such as the quadramer-based plaquettes. These themes constitute pristine territory for further exploration at the theoretical level. Naturally, on the other hand, a potential generalization of the earlier experiments of Rüter et al. [7] towards the inclusion of nonlinear amplification and amplitude-dependent loss in balance with each other would be most worthwhile to consider in order to take advantage of the considerable additional wealth of phenomenology of the latter system.

P.G.K. gratefully acknowledges support from the US National Science Foundation (grant DMS-0806762), the Alexander von Humboldt Foundation and the Alexander S. Onassis Public Benefit Foundation.
References


