Some existence results on nonlinear fractional differential equations

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In this paper, by using fixed-point methods, we study the existence and uniqueness of a solution for the nonlinear fractional differential equation boundary-value problem

$$D^{\alpha} u(t) = f(t, u(t))$$

with a Riemann–Liouville fractional derivative via the different boundary-value problems $u(0) = u(T)$, and the three-point boundary condition $u(0) = \beta_1 u(\eta)$ and $u(T) = \beta_2 u(\eta)$, where $T > 0$, $t \in I = [0, T]$, $0 < \alpha < 1$, $0 < \eta < T$, $0 < \beta_1 < \beta_2 < 1$.

1. Introduction

The field of fractional differential equations has been subjected to an intensive development of theory and applications ([1–9] and references therein). For a new history of fractional calculus, see Machado et al. [10]. It should be noted that most papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, there have been some papers dealing with the existence of solutions of nonlinear initial-value problems of fractional differential equations by using techniques of nonlinear analysis such as fixed-point results, the Leray–Schauder theorem and stability [11–19]. In fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena and aerodynamics ([6,20–22] and
2. Basic tools

First, let us recall some basic definitions of fractional calculus [4,5]. For a continuous function \( f : [0, \infty) \rightarrow \mathbb{R}, \) the Caputo derivative of fractional order \( \alpha \) is defined as

\[
D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s) \, ds \quad (n - 1 < \alpha < n, \ n = [\alpha] + 1),
\]

where \([\alpha]\) denotes the integer part of the real number \( \alpha \). In addition, the Riemann–Liouville fractional derivative of order \( \alpha \) for a continuous function \( f(t) \) is defined by

\[
D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \, ds \quad (n = [\alpha] + 1),
\]

provided the right-hand side is point-wise defined on \((0, \infty)\) [3].

Recently, the notions of \( \alpha - \psi \)-contractive and \( \alpha \)-admissible mappings were introduced in [7].

**Definition 2.1.** Let \((X, d)\) be a metric space and \( T : X \rightarrow X \) be a given mapping. We say that \( T \) is an \( \alpha - \psi \)-contractive mapping whenever there exist two functions \( \psi \in \Psi \) and \( \alpha : X \times X \rightarrow [0, \infty) \) such that \( \alpha(x, y) \, d(Tx, Ty) \leq \psi(d(x, y)) \) for all \( x, y \in X \). Also, we say that \( T \) is \( \alpha \)-admissible whenever \( \alpha(x, y) \geq 1 \) implies that \( \alpha(Tx, Ty) \geq 1 \).

Here, \( \Psi \) is the family of non-decreasing functions \( \psi : [0, \infty) \rightarrow [0, \infty) \). The following results, which we need in our results, have been proved in [7].

**Theorem 2.2.** Let \((X, d)\) be a complete metric space, \( T : X \rightarrow X \) an \( \alpha - \psi \)-contractive and \( \alpha \)-admissible self-map on \( X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) for some \( x_0 \in X \). If \( x_n \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \) and \( x_n \rightarrow x \) for some \( x \in X \), then \( \alpha(x_n, x) \geq 1 \) for all \( n \). Then, \( T \) has a fixed point.

3. Main results

In the following, we are going to state and prove our main results. We provide our results in three main parts.

**Case I.** Here, we discuss the nonlinear fractional differential equation

\[
^{c}D^\alpha x(t) = f(t, x(t)) \quad (0 < t < 1, \ 1 < \alpha \leq 2)
\]

via the integral boundary condition

\[
x(0) = 0, \quad x(1) = \int_0^\eta x(s) \, ds \quad (0 < \eta < 1),
\]

where \( D^\alpha \) denotes the Caputo fractional derivative of order \( \alpha \) and \( f : [0, 1] \times X \rightarrow X \) is a continuous function. Here, \((X, \| . \| )\) is a Banach space and \( C = C([0, 1], X) \) denotes the Banach space of continuous functions from \([0, 1]\) into \( X \) endowed with uniform topology.
Theorem 3.1. Suppose that

(i) there exists a function \( \xi : \mathbb{R}^2 \to \mathbb{R} \) and \( \psi \in \Psi \) such that

\[
|f(t, a) - f(t, b)| \leq \frac{\Gamma(\alpha + 1)}{5} \psi(|a - b|)
\]

for all \( t \in I \) and \( a, b \in \mathbb{R} \) with \( \xi(a, b) \geq 0 \);

(ii) there exists \( x_0 \in C(I) \) such that \( \xi(x_0(t), Fx_0(t)) \geq 0 \) for all \( t \in I \), where the operator \( F : C(I) \to C(I) \) is defined by

\[
Fx(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f(s, x(s)) \, ds - \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}f(s, x(s)) \, ds
\]

\[
+ \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left( \int_0^s (s - m)^{\alpha-1}f(m, x(m)) \, dm \right) \, ds \quad (t \in [0, 1]);
\]

(iii) for each \( t \in I \) and \( x, y \in C(I) \), \( \xi(x(t), y(t)) \geq 0 \) implies \( \xi(Fx(t), Fy(t)) \geq 0 \); and

(iv) if \( \{x_n\} \) is a sequence in \( C(I) \) such that \( x_n \to x \) in \( C(I) \) and \( \xi(x_n, x_{n+1}) \geq 0 \) for all \( n \), then \( \xi(x_n, x) \geq 0 \) for all \( n \).

Then, the problem (3.1) has at least one solution.

Proof. It is well known that \( x \in C(I) \) is a solution of (3.1) if and only if \( x \in C(I) \) is a solution of the integral equation

\[
x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f(s, x(s)) \, ds - \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}f(s, x(s)) \, ds
\]

\[
+ \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left( \int_0^s (s - m)^{\alpha-1}f(m, x(m)) \, dm \right) \, ds \quad (t \in [0, 1]).
\]

Then, the problem (3.1) is equivalent to finding \( x^* \in C(I) \), which is a fixed point of \( F \). Now, let \( x, y \in C(I) \) such that \( \xi(x(t), y(t)) \geq 0 \) for all \( t \in I \). By using (i), we have

\[
|Fx(t) - Fy(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f(s, x(s)) \, ds
\]

\[
- \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}f(s, x(s)) \, ds
\]

\[
+ \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left( \int_0^s (s - m)^{\alpha-1}f(m, x(m)) \, dm \right) \, ds
\]

\[
- \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f(s, y(s)) \, ds
\]

\[
+ \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}f(s, y(s)) \, ds
\]

\[
- \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left( \int_0^s (s - m)^{\alpha-1}f(m, y(m)) \, dm \right) \, ds
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t |t - s|^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \, ds
\]

\[
+ \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 |1 - s|^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \, ds
\]

\[
+ \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left( \int_0^s (s - m)^{\alpha-1} |f(m, y(m)) - f(m, x(m))| \, dm \right) \, ds
\]
Finally, from (iv) and using theorem 2.2, we deduce the existence of a function 
\[ \alpha(\cdot) \in C(I) \] such that
\[ \|F_x - F_y\| \leq \psi(\|x - y\|). \]

Now, define the function \( \alpha : C(I) \times C(I) \to [0, \infty) \) by
\[ \alpha(x, y) = \begin{cases} 1 & \xi(x(t), y(t)) \geq 0 \text{ for all } t \in I, \\ 0 & \text{otherwise.} \end{cases} \]

Hence, \( \alpha(x, y) \psi(Fx, Fy) \leq \psi(\psi(x)) \) for all \( x, y \in C(I) \). This implies that \( F \) is an \( \alpha - \psi \)-contractive mapping. By using the condition (iii), we get
\[ \alpha(x, y) \geq 1 \Rightarrow \psi(x(t), y(t)) \geq 0 \Rightarrow \psi(Fx(t), Fy(t)) \geq 0 \Rightarrow \alpha(Fx, Fy) \geq 1 \]
for all \( x, y \in C(I) \). Thus, \( F \) is \( \alpha \)-admissible. From (ii), there exists \( x_0 \in C(I) \) such that \( \alpha(x_0, Fx_0) \geq 1 \). Finally, from (iv) and using theorem 2.2, we deduce the existence of \( x^* \in C(I) \) such that \( x^* = Fx^* \).

Hence, \( x^* \) is a solution of the problem. \( \square \)

**Case II.** Now, we discuss the nonlinear fractional differential equation
\[ D^\alpha x(t) + f(t, x(t)) = 0 \quad (0 \leq t \leq 1, \alpha > 1) \quad (3.2) \]
via the two-point boundary value condition \( x(0) = x(1) = 0 \), where \( f : I \times \mathbb{R} \to \mathbb{R} \) is a continuous function and \( I = [0, 1] \). Recall that the Green function associated with the problem (3.2) is given by
\[ G(t, s) = \begin{cases} \frac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq s \leq t \leq 1, \\ \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq t \leq s \leq 1. \end{cases} \]

**Theorem 3.2.** Suppose that

(i) there exists a function \( \xi : \mathbb{R}^2 \to \mathbb{R} \) and \( \psi \in \Psi \) such that \( |f(t, a) - f(t, b)| \leq \psi(|a - b|) \) for all \( t \in I \) and \( a, b \in \mathbb{R} \) with \( \xi(a, b) \geq 0 \);

(ii) there exists \( x_0 \in C(I) \) such that \( \xi(x_0(t), \int_0^1 G(t, s)f(s, x_0(s)) \, ds) \geq 0 \) for all \( t \in I \);

(iii) for each \( t \in I \) and \( x, y \in C(I) \), \( \xi(x(t), y(t)) \geq 0 \) implies
\[ \psi \left( \int_0^1 G(t, s)f(s, x(s)) \, ds \right) \geq 0; \]

(iv) if \( \{x_n\} \) is a sequence in \( C(I) \) such that \( x_n \to x \) in \( C(I) \) and \( \xi(x_n, x_{n+1}) \geq 0 \) for all \( n \), then \( \xi(x_n, x) \geq 0 \) for all \( n \).

Then, the problem (3.2) has at least one solution.
Proof. It is well known that \( x \in C(I) \) is a solution of (3.2) if and only if is a solution of the integral equation \( x(t) = \int_0^1 G(t,s)f(s,x(s)) \, ds \) for all \( t \in I \). Define the operator \( F : C(I) \rightarrow C(I) \) by \( Fx(t) = \int_0^1 G(t,s)f(s,x(s)) \, ds \) for all \( t \in I \). Thus, for finding a solution of the problem (3.2), it is sufficient that we find a fixed point of \( F \). Now, let \( x, y \in C(I) \) be such that \( \xi(x(t), y(t)) \geq 0 \) for all \( t \in I \). By using (i), we get

\[
|Fx(t) - Fy(t)| = \left| \int_0^1 G(t,s)(f(s,x(s)) - f(s,y(s))) \, ds \right|
\leq \int_0^1 G(t,s)|f(s,x(s)) - f(s,y(s))| \, ds
\leq \int_0^1 G(t,s)|x(s) - y(s)| \, ds
\leq \psi(\|x - y\|) \sup_{t \in I} \int_0^1 G(t,s) \, ds \leq \psi(\|x - y\|).
\]

This implies that for each \( x, y \in C(I) \) with \( \xi(x(t), y(t)) \geq 0 \) for all \( t \in I \), we have

\[
\|Fx - Fy\| \leq \psi(\|x - y\|).
\]

Let us define the function \( \alpha : C(I) \times C(I) \rightarrow [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases} 1 & \xi(x(t), y(t)) \geq 0 \text{ for all } t \in I, \\ 0 & \text{else}. \end{cases}
\]

Therefore, \( \alpha(x, y)\|Fx - Fy\| \leq \psi(\|x - y\|) \) for all \( x, y \in C(I) \), that is, \( F \) is an \( \alpha - \psi \)-contractive mapping. Now by using (iii), we have

\[
\alpha(x, y) \geq 1 \Rightarrow \psi(x(t), y(t)) \geq 0 \Rightarrow \psi(Fx(t), Fy(t)) \geq 0 \Rightarrow \alpha(Fx, Fy) \geq 1.
\]

Hence, \( F \) is \( \alpha \)-admissible. From (ii), there exists \( x_0 \in C(I) \) such that \( \alpha(x_0, Fx_0) \geq 1 \). Now using (iv) and theorem 2.2, there exists \( x^* \in C(I) \) such that \( x^* = Fx^* \).

Case III. We study now the nonlinear fractional differential equation

\[
D^\alpha x(t) + D^\beta x(t) = f(t, x(t)) \quad (0 \leq t \leq 1, \ 0 < \beta < \alpha < 1)
\]

via the two-point boundary-value condition \( x(0) = x(1) = 0 \), where \( f : I \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function. Recall that the Green function associated with (3.3) is given by \( G(t) = t^\alpha E_\alpha^{-\beta}(-t^\alpha) \).

**Theorem 3.3.** Suppose that

(i) there exists a function \( \xi : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( \psi \in \Psi \) such that

\[
|f(t,a) - f(t,b)| \leq \alpha \psi(|a - b|)
\]

for all \( t \in I \) and \( a, b \in \mathbb{R} \) with \( \xi(a,b) \geq 0 \);

(ii) there exists \( x_0 \in C(I) \) such that \( \xi(x_0(t), \int_0^1 G(t,s)f(s,x_0(s)) \, ds) \geq 0 \) for all \( t \in I \);

(iii) for each \( t \in I \) and \( x, y \in C(I) \), \( \xi(x(t), y(t)) \geq 0 \) implies

\[
\psi \left( \int_0^1 G(t,s)f(s,x(s)) \, ds, \int_0^1 G(t,s)f(s,y(s)) \, ds \right) \geq 0; \quad \text{and}
\]

(iv) if \( \{x_n\} \) is a sequence in \( C(I) \) such that \( x_n \rightarrow x \) in \( C(I) \) and \( \xi(x_n, x_{n+1}) \geq 0 \) for all \( n \), then \( \xi(x_n, x) \geq 0 \) for all \( n \).

Then, the problem (3.3) has at least one solution.

**Proof.** It is well known that \( x \in C(I) \) is a solution of (3.3), if and only if is a solution of the integral equation \( x(t) = \int_0^1 G(t-s)f(s,x(s)) \, ds \) for all \( t \in I \).
Define the operator \( F : C(I) \to C(I) \) by \( Fx(t) = \int_0^t G(t - s)f(s, x(s)) \, ds \) for all \( t \in I \). Thus, for finding a solution of the problem (3.3), it is sufficient we find a fixed point of \( F \). Now, let \( x, y \in C(I) \) be such that \( \xi(x(t), y(t)) \geq 0 \) for all \( t \in I \). By using (i), we get

\[
|Fx(t) - Fy(t)| = \left| \int_0^t G(t - s)(f(s, x(s)) - f(s, y(s))) \, ds \right|
\leq \int_0^t |G(t - s)||f(s, x(s)) - f(s, y(s))| \, ds
\leq \int_0^t |G(t - s)|\alpha\psi(|x(s) - y(s)|) \, ds
\leq \alpha\psi(\|x - y\|_\infty) \sup_{t \in I} \int_0^t |G(t - s)| \, ds \leq \psi(\|x - y\|_\infty).
\]

Note that \( G(t) = t^{\alpha - 1}E_{\alpha - \beta, \alpha}(-t^{\alpha - \beta}) \leq t^{\alpha - 1}1/1 + |t^{\alpha - \beta}| \leq t^{\alpha - 1} \) for all \( t \in I \). Thus, \( \sup_{t \in I} \int_0^t |G(t - s)| \, ds \leq 1/\alpha \). Now, define the function \( \alpha : C(I) \times C(I) \to [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases} 1 & \xi(x(t), y(t)) \geq 0 \text{ for all } t \in I, \\ 0 & \text{else.} \end{cases}
\]

Hence, \( \alpha(x, y)\|Fx - Fy\|_\infty \leq \psi(\|x - y\|_\infty) \) for all \( x, y \in C(I) \), that is, \( F \) is an \( \alpha - \psi \)-contractive mapping. Now by using (iii), we have

\[
\alpha(x, y) \geq 1 \Rightarrow \psi(x(t), y(t)) \geq 0 \Rightarrow \psi(Fx(t), Fy(t)) \geq 0 \Rightarrow \alpha(Fx, Fy) \geq 1.
\]

Hence, \( F \) is \( \alpha \)-admissible. From (ii), there exists \( x_0 \in C(I) \) such that \( \alpha(x_0, Fx_0) \geq 1 \). Now by using (iv) and theorem 2.2, there exists \( x^* \in C(I) \) such that \( Fx^* = x^* \).

4. Concluding remarks

Fractional nonlinear differential equations and their applications represent a topic of high interest in the area of fractional calculus and its applications in various fields of science and engineering. As a result, new methods and techniques have been applied to this rapidly growing direction. In this article, based on the recently introduced notions of \( \alpha - \psi \)-contractive and \( \alpha \)-admissible mappings, we have proved three existence theorems for three nonlinear fractional differential equations for various boundary conditions.

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References


