In this note, we correct an error in the first two paragraphs of §2b of the paper by Ashwin et al. [1]. This section attempts to generalize sufficient conditions for R-tipping in the linear model [1, equation (2.1)] with steady drift in §2a to the case of time-varying rates \( r(t) \). Starting with [1, equation (2.3)] and noting that \( |e^{Mu}| \leq \|e^{Mu}\| |v| \), we have the upper bound

\[
|x(t) - \tilde{x}(t)| \leq r_{\text{max}}(t) \int_0^\infty \|e^{Mu}\| \, du.
\]

If \( M \) is stable, then

\[
\|e^{Mu}\| \leq c e^{-\beta u},
\]

for some real \( c, \beta > 0 \) [2], and so

\[
|x(t) - \tilde{x}(t)| \leq r_{\text{max}}(t) \frac{c}{\beta}.
\]

Hence, one can guarantee that [1, equation (2.1)] avoids R-tipping by time \( t \) if

\[
\frac{c}{\beta} r_{\text{max}}(t) < R.
\]

If \( M \) is scalar, then we can choose \( c = 1 \), \( \beta = -M \), and (1.2) reduces to [1, equation (2.9)]. On the other hand, if \( M \) is a matrix, then we need a good choice of \( c \) and \( \beta \) in (1.1) to make the tipping condition (1.2) optimal, but this depends on the matrix structure and not simply the norm; see, for example, the text by Hinrichsen & Pritchard [2] and the elegant estimates of Godunov [3, equation (13)]. Incidentally, we remark that within the unnumbered equation between [1, equation (2.1)] and [1, equation (2.2)] there should be a minus sign before the integral, though this is corrected in the rest of the paper.
The converse condition [1, equation (2.10)] is not correct, but can be corrected as follows. From the formula between [1, equation (2.3)] and [1, equation (2.4)] recall

\[ x(t) - \tilde{x}(t) = M^{-1} \frac{d\tilde{x}}{dt}(t) - M^{-1} \int_0^\infty e^{Mu} \frac{d^2\tilde{x}}{dt^2}(t - u) \, du \]

where we define

\[ \tilde{r}(t) = \frac{d\tilde{x}}{dt}(t) - \int_0^\infty e^{Mu} \frac{d^2\tilde{x}}{dt^2}(t - u) \, du. \]

Note that \( \tilde{r}(t) = r(t) \) in the case of constant drift, while in the more general case, the expression for \( \tilde{r}(t) \) includes an additional term depending on \( M \) and the history of the rate of change of drift. Because \( |x(t) - \tilde{x}(t)| \geq \|M\|^{-1} |\tilde{r}(t)| \), one can guarantee that an R-tipping occurs before time \( t \) if

\[ \|M\|^{-1} |\tilde{r}(t)| > R. \]

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References

