We study discrete- and continuous-time consensus problems on networks in the presence of distributed time delays. We focus on information transmission delays, as opposed to information processing delays, so that each node of the network compares its current state with the past states of its neighbours. We consider directed and weighted networks where the connection structure is described by a normalized Laplacian matrix and show that consensus is achieved if and only if the underlying graph contains a directed spanning tree. This statement holds independently of the transmission delays, which is in contrast to the case of processing delays. Furthermore, we calculate the consensus value explicitly, and show that it is determined by the history of the system over an interval of time, unlike the case of processing delays where the consensus value depends only on the initial state of the system at time zero. This provides the consensus algorithm with improved robustness against noise.

1. Introduction

Consensus problems model a class of dynamical systems that find a broad range of applications in distributed computing [1], management science [2], flocking/swarming theory [3], distributed control [4] and sensor networks [5], among others. In these applications, multi-agent systems need to agree on a common value for a certain quantity of interest that depends on the states of all agents or is a pre-assigned value. Hence, it is important to design algorithms so that the agents can reach consensus. The interaction rule that specifies the information exchange between the agents is called the consensus protocol/algorithm. In addition to engineering and biological systems, such problems carry important implications for decision-making in social systems or collective behaviour of animal populations.
The underlying idea of the consensus process is updating the state of each agent by suitably processing (e.g. comparing or averaging) the states of its neighbours and its own. The question is then under which circumstances a given local update rule leads to the overall convergence of agent states, that is, whether the multi-agent system can reach consensus by the proposed algorithm. In the past decade, the stability analysis of consensus algorithms has attracted much attention in control theory and mathematics [6]. The purpose of the stability analysis is not only to obtain the algebraic conditions for consensus but also to understand the consensus properties of the network topology.

The classical consensus problem on networks can be formulated in discrete time \((t \in \mathbb{Z})\) as

\[ x_i(t + 1) = x_i(t) + u_i(t), \quad (1.1) \]

or in continuous time \((t \in \mathbb{R})\) as

\[ \dot{x}_i(t) = u_i(t), \quad (1.2) \]

with the consensus protocol given by

\[ u_i(t) = \varepsilon \sum_{j=1}^{n} a_{ij} (x_j(t) - x_i(t)). \quad (1.3) \]

Here, \(x_i(t) \in \mathbb{R}\) is the state (‘opinion’) of the agent \(i\) at time \(t\), \(i = 1, \ldots, n\), which changes under the interaction with the other agents in a manner described by the ‘consensus protocol’ \(u_i\). The coupling coefficient \(\varepsilon \geq 0\) is a measure of the overall strength of the interaction in modifying the state of individual agents, whereas \(a_{ij} \geq 0\) is the strength of the influence of agent \(j\) on \(i\). The numbers \(a_{ij}\) define a directed and weighted graph \(G\) on \(n\) vertices, where there is an arc from vertex \(j\) to \(i\) if and only if \(a_{ij} \neq 0\). Special cases of an unweighted graph are obtained if \(a_{ij}\) are restricted to binary values \(\{0, 1\}\), and an undirected graph is obtained if \(a_{ij} = a_{ji} \forall i, j\). The system (1.1) or (1.2) is said to reach consensus if for any set of initial conditions there exists some \(c \in \mathbb{R}\) (depending on the initial conditions) such that \(\lim_{t \to \infty} x_i(t) = c\) for all \(i\). The number \(c\) is then called the consensus value.

In the previously published works, delayed consensus problems have often been studied with the protocol

\[ u_i(t) = \varepsilon \sum_{j=1}^{n} a_{ij} (x_j(t - \tau) - x_i(t - \tau)), \quad (1.4) \]

where \(\tau \geq 0\) denotes the time delay; see Olfati-Saber & Murray [7] for the continuous-time case in undirected networks, Michiels et al. [8] for a distributed-delay version, and Sipahi et al. [9] for the same equation in the context of traffic stability. In these models, the interaction (1.4) involves delayed information processing, where the difference of the states \(x_j - x_i\) influences the dynamics of the agents after some time delay \(\tau\). Alternatively, one can consider the consensus problem under delayed information transmission, namely with

\[ u_i(t) = \varepsilon \sum_{j=1}^{n} a_{ij} (x_j(t - \tau) - x_i(t)). \quad (1.5) \]

Here, agent \(i\) compares its current state with the information coming from its neighbour \(j\) after some time delay \(\tau\). Information transmission delays have so far not attracted much attention in consensus problems, although they arise naturally in many dynamical processes on networks. Among the rare examples of transmission delays, one can mention Moreau [10], who studied the stability of the continuous-time consensus algorithm in a network with time-varying connections but did not obtain the consensus value, as well as Seuret et al. [11], who studied a time-invariant version with constant vertex degrees and obtained complicated conditions involving matrix inequalities. In this paper, we shall consider more general delays, derive the exact consensus value and show that the actual consensus condition is much simpler.
In the following, we study the consensus problem under delayed information transmission in a general setting, allowing both fixed and distributed delays as well as arbitrary network structure. More precisely, in discrete time, we consider the system

$$x_i(t + 1) = x_i(t) + \frac{\varepsilon}{d_i} \sum_{j=1}^{n} a_{ij} \left( \sum_{s=0}^{\tau} f_s x_j(t - s) - x_i(t) \right), \quad (1.6)$$

and in continuous time the corresponding version

$$\dot{x}_i(t) = \frac{\varepsilon}{d_i} \sum_{j=1}^{n} a_{ij} \left( \int_{0}^{\tau} f(s)x_j(t - s) \, ds - x_i(t) \right). \quad (1.7)$$

In both cases, $f$ describes an appropriate distribution of delays over the interval $[0, \tau]$, namely

$$f_s \geq 0 \quad \text{for} \quad s = 0, 1, \ldots, \tau \quad \text{and} \quad \sum_{s=0}^{\tau} f_s = 1, \quad (1.8)$$

and

$$f(s) \geq 0 \quad \text{for} \quad s \in [0, \tau] \quad \text{and} \quad \int_{0}^{\tau} f(s) \, ds = 1. \quad (1.9)$$

The summation term is normalized by the in-degree $d_i := \sum_{j=1}^{n} a_{ij}$, which is assumed to be positive for all $i$, that is, each agent receives some inputs. Here, the agents act on the average difference between their own state and those of their neighbours, regardless of how many neighbours they have. This is a natural coupling model for several important network dynamics; for instance, for opinion formation in social networks or models of synchronization in neural systems. The results of this paper will further reinforce the usefulness of normalized coupling (see §§5b,c).

The paper is organized as follows. We start with a brief review of the undelayed case in §2. The delayed systems (1.6) and (1.7) are studied in §§3 and 4, respectively. We show that consensus can be achieved regardless of the presence of the delays, provided that the network has a directed spanning tree. This result is in contrast to the case of processing delays (1.4), where consensus is not possible for large values of the delay [7,8]. Furthermore, we calculate the consensus value explicitly, and show that it depends on the initial history of the agents over a time interval as well as on the details of the network structure. The implications of the main theorems are discussed in §5, where we compare signal processing and signal transmission delays, as well as discrete and distributed delays. We also mention the application to synchronization of delay-coupled phase oscillators. Preliminary results of this paper have been presented elsewhere [12].

2. Normalized Laplacian and the undelayed consensus problem

For notation, we let $A = [a_{ij}]$ denote the weighted adjacency matrix of the network and $D := \text{diag}(d_1, \ldots, d_n)$ denote the diagonal matrix of vertex in-degrees. With $x = (x_1, \ldots, x_n)^\top$, the discrete-time system (1.6) can be written in vector form as

$$x(t + 1) = (1 - \varepsilon)x(t) + \varepsilon D^{-1}A \sum_{s=0}^{\tau} f_s x(t - s), \quad (2.1)$$

and the continuous-time system (1.7) as

$$\dot{x}(t) = -\varepsilon x(t) + \varepsilon D^{-1}A \int_{0}^{\tau} f(s)x(t - s) \, ds. \quad (2.2)$$

We denote $1 = (1, \ldots, 1)^\top$. Reaching a consensus is then equivalent to the convergence $\lim_{t \to \infty} x(t) = c1$ for some scalar $c$, which ultimately is the consensus value.

The form of the matrices in (2.1) and (2.2) motivates the definition of the normalized Laplacian matrix $L = I - D^{-1}A$. This matrix has similar properties to the combinatorial Laplacian $D - A$, and its spectral properties play a significant role in the consensus problem. Note that, if $d_i = k$ for all $i$,
then the two versions of the Laplacian matrix differ simply by a factor of $k$. Let $\{\lambda_1, \ldots, \lambda_n\}$ denote the set of eigenvalues of $L$. By an application of Gershgorin’s circle theorem [13], it can be seen that

$$|1 - \lambda_i| \leq 1, \quad \forall i.$$  \hfill (2.3)

Also note that $L1 = 0$, because $L$ has zero row sums. Hence, zero is always an eigenvalue of $L$, which we assign to $\lambda_1$, with the corresponding eigenvector $v_1 = 1$. In the following, we assume that the Laplacian matrix for the network under investigation has a complete set of eigenvectors $\{v_1, \ldots, v_n\}$ corresponding to the eigenvalues $\lambda_i$: $Lv_i = \lambda_i v_i$, $i = 1, \ldots, n$. Let $\{u^1, \ldots, u^n\}$ be the dual basis corresponding to $\{v_1, \ldots, v_n\}$; that is, the $u^i$ are linearly independent vectors such that $\langle u^i, v_j \rangle = \delta^i_j$. It is easy to see that $u^i$ are the left eigenvectors of $L$, $u^i L = \lambda_i u^i$. Furthermore, if $x = \sum_{i=1}^n \alpha_i v_i \in \mathbb{R}^n$, then one has $\alpha_i = \langle u^i, x \rangle$.

For motivation and comparison with subsequent analysis, we first briefly consider the undelayed consensus problem. When the delays are absent in (2.1) (i.e. when $\tau = 0$), the system simplifies to

$$x(t + 1) = (I - \varepsilon L)x(t),$$  \hfill (2.4)

whose solution is

$$x(t) = (I - \varepsilon L)^t x(0).$$  \hfill (2.5)

Expressing the initial condition $x(0) = x_0$ in the eigenbasis of $L$ as $x_0 = \sum_{i=1}^n (u^i, x_0) v_i$, (2.5) yields

$$x(t) = \sum_{i=1}^n (u^i, x_0) (1 - \varepsilon \lambda_i)^t v_i.$$  \hfill (2.6)

Consider now the inequality

$$\max_{i \geq 2} |1 - \varepsilon \lambda_i| < 1.$$  \hfill (2.7)

If (2.7) is satisfied, then (2.6) implies

$$x(t) \to \langle u^1, x_0 \rangle v_1,$$

because $\lambda_1 = 0$. Hence, the system reaches consensus, because $v_1 = 1$. On the other hand, if $|1 - \varepsilon \lambda_i| \geq 1$ for some $i \geq 2$, then (2.6) gives

$$|\langle u^i, x(t) \rangle| \geq |\langle u^i, x_0 \rangle|$$

for all $t \geq 0$,

so that $x(t)$ always has a component along an eigenvector different from $1$, provided $x(0)$ has a non-zero component along that eigenvector. Hence, (2.7) is also a necessary condition to reach consensus from arbitrary initial conditions.

To summarize, the discrete-time system (2.4) reaches consensus from arbitrary initial conditions if and only if the condition (2.7) is satisfied. In view of (2.3), (2.7) holds for all $\varepsilon \in (0, 1)$ if and only if zero is a simple eigenvalue of $L$; moreover, for $\varepsilon = 1$, it is additionally required that 2 is not an eigenvalue of $L$. If (2.7) holds, then the consensus value is

$$c = \langle u^1, x_0 \rangle,$$  \hfill (2.8)

i.e. a weighted average of initial values, where the weights are given by the components of the left eigenvector $u^1$ of $L$ corresponding to the zero eigenvalue.

---

1 Although this assumption is not strictly necessary, it is generically satisfied for matrices in $\mathbb{R}^{n \times n}$ and makes the subsequent analysis and notation easier to follow.
Similarly, the continuous-time consensus problem (2.2) can be written in the absence of delays as
\[ \dot{x}(t) = -\varepsilon L x(t), \quad (2.9) \]
with solution
\[ x(t) = \exp(-\varepsilon L t) x(0) = \sum_{i=1}^{n} (u_i, x_0) e^{-\varepsilon \lambda_i t} v_i. \]

By the same argument as above, we obtain that \( x(t) \rightarrow (u_1, x_0) \) 1, i.e. the system (2.9) reaches consensus, if and only if
\[ \min_{i \geq 2} \text{Re}(\lambda_i) > 0. \quad (2.10) \]
The consensus value is given by (2.8) as in the discrete case.

### 3. Consensus in discrete time

We now introduce signal transmission delays into the consensus problem and study the discrete-time system (2.1). Similar to the development in §2, we express the system state \( x = (x_1, \ldots, x_n)^T \) in the eigenbasis of \( L \) as
\[ x(t) = \sum_{i=1}^{n} \alpha_i(t) v_i, \quad \text{where} \quad \alpha_i(t) = (u_i, x(t)). \quad (3.1) \]

It follows that the evolution of \( \alpha_i \) is governed by the equation
\[ \alpha_i(t+1) = (1 - \varepsilon) \alpha_i(t) + \varepsilon (1 - \lambda_i) \sum_{s=0}^{r} f_s \alpha_i(t-s). \quad (3.2) \]

For non-zero solutions, the ansatz \( \alpha_i(t) = z^t \) yields the characteristic equation in \( z \in \mathbb{C} \),
\[ \chi_i(z) := z^{r+1} - (1 - \varepsilon) z^r - \varepsilon (1 - \lambda_i) \sum_{s=0}^{r} f_s z^{r-s} = 0. \quad (3.3) \]

We have the following result regarding (3.3).

**Lemma 3.1.** (a) Suppose \( 0 \leq \varepsilon \leq 1 \). Then, all roots \( z \) of (3.3) satisfy \( |z| \leq 1 \). (b) Suppose \( 0 < \varepsilon < 1 \). If \( \lambda_i = 0 \), then 1 is a simple root of (3.3) and all other roots have moduli less than 1. If \( \lambda_i \neq 0 \), then all roots have moduli less than 1.

**Proof.** If \( \chi_i(z) = 0 \) and \( |z| = R > 1 \), then comparing magnitudes in (3.3) gives
\[ R^{r+1} \leq (1 - \varepsilon) R^r + \varepsilon |1 - \lambda_i| \sum_{s=0}^{r} f_s R^{r-s}. \]

Dividing through by \( R^{r+1} \) yields the contradiction
\[ 1 \leq (1 - \varepsilon) \frac{R^{-1}}{R} + \varepsilon |1 - \lambda_i| \sum_{s=0}^{r} f_s \frac{R^{-s-1}}{R} \leq 1, \]
where we have used (2.3) and the fact that \( R > 1 \) to arrive at the strict inequality. Thus, all roots of (3.3) are on or inside the unit disc, which proves the first statement of the lemma. To prove (b),
suppose $0 < \varepsilon < 1$. Define the polynomial
\[
\tilde{\chi}(z) := z^r (z - (1 - \varepsilon)).
\]

Note that $z$ is a root of $\chi_i$ if and only if
\[
\tilde{\chi}(z) = \varepsilon (1 - \lambda_i) \sum_{s=0}^{r-1} f_s z^{r-s}.
\]  
(3.4)

Comparing the magnitudes in (3.4) for $z$ on the unit circle, $z = e^{i \theta}$, we have
\[
|\tilde{\chi}(e^{i \theta})| = |e^{i \theta} - (1 - \varepsilon)| = \sqrt{\sin^2 \theta + (\cos \theta - 1 + \varepsilon)^2}
\]
\[
= \sqrt{2 - 2\varepsilon + \varepsilon^2 - 2(1 - \varepsilon) \cos \theta}
\]
\[
\geq \sqrt{2 - 2\varepsilon + \varepsilon^2 - 2(1 - \varepsilon)}
\]
\[
= \varepsilon \geq \varepsilon |1 - \lambda_i|
\]
\[
\geq \varepsilon |1 - \lambda_i| \left| \sum_{s=0}^{r} f_s e^{i \theta (r-s)} \right|,
\]  
(3.5)

where we have used (1.8) in the last line. Because $0 < \varepsilon < 1$, the inequality (5.3) is strict unless $\cos \theta = 1$. Hence, $z = e^{i \theta}$ cannot satisfy (3.4), and hence cannot be a root of (3.3), unless $z = 1$. Substituting $z = 1$ into (3.3) yields $\lambda_i = 0$. Therefore, if $\lambda_i \neq 0$, then there are no roots on the unit circle, in which case part (a) implies that all roots have moduli less than 1. It remains to show that $z = 1$ is a simple root of the characteristic equation for $\lambda_i = 0$. This is trivial for $\tau = 0$, and for $\tau > 0$ we have from (3.3)
\[
\chi_i'(z) = (r + 1)z^r - \tau (1 - \varepsilon)z^{r-1} - \varepsilon \sum_{s=0}^{r-1} (r - s) f_s z^{r-s} - 1;
\]

hence using (1.8),
\[
\chi_i'(1) = 1 + \varepsilon \tau - \varepsilon \sum_{s=0}^{r} (r - s) f_s
\]
\[
= 1 + \varepsilon \sum_{s=0}^{r} s f_s > 0,
\]

implying that 1 is a simple root. This completes the proof of (b).

\[\square\]

**Theorem 3.2.** Let $\tau$ be a positive integer and $\varepsilon \in (0, 1)$. The discrete-time system (2.1) reaches consensus from arbitrary initial conditions if and only if zero is a simple eigenvalue of the Laplacian $L$. Furthermore, the consensus value is given by
\[
c = \frac{1}{1 + \varepsilon \bar{\tau}} \left( u^1, x(0) + \varepsilon \sum_{k=1}^{r} \sum_{t=k}^{\tau} f_t x(-k) \right),
\]  
(3.7)

where $u^1$ is the left eigenvector of $L$ corresponding to the zero eigenvalue, and $\bar{\tau} = \sum_{s=0}^{r} s f_s$ is the mean of the delay distribution $f$. For the special case of a single discrete delay $\tau$, the consensus value is
\[
c = \frac{1}{1 + \varepsilon \tau} \left( u^1, x(0) + \varepsilon \sum_{k=1}^{r} x(-k) \right).
\]  
(3.8)
Proof. Suppose that zero is a simple eigenvalue of \( L \). Thus, \( \lambda_i \neq 0 \) for \( i \geq 2 \), and lemma 3.1 implies that
\[
\lim_{t \to \infty} a_i(t) = 0, \quad i \geq 2,
\]
in (3.2). By (3.1), we then have
\[
\lim_{t \to \infty} \|x(t) - a_1(t)1\| = 0. \tag{3.9}
\]
From (3.2) with \( \lambda_1 = 0 \),
\[
\alpha_1(t + 1) = (1 - \varepsilon)\alpha_1(t) + \varepsilon \sum_{s=0}^{\tau} f_s \alpha_1(t - s). \tag{3.10}
\]
Letting \( a_1(t) = (\alpha_1(t), \alpha_1(t - 1), \ldots, \alpha_1(t - \tau)) \), (3.10) is written in vector form as
\[
a_1(t + 1) = E a_1(t), \tag{3.11}
\]
where \( E \) is the \((\tau + 1) \times (\tau + 1)\) matrix
\[
E = \begin{pmatrix}
1 - \varepsilon + \varepsilon f_0 & \varepsilon f_1 & \cdots & \varepsilon f_{\tau - 1} & \varepsilon f_{\tau} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}.
\]
Using (1.8) and after some calculation, it can be checked that, corresponding to the unit eigenvalue, \( E \) has the right eigenvector \( e_1 = 1 \), and the left eigenvector
\[
e_1 := \frac{1}{1 + \varepsilon \tau} (1, \beta_1, \beta_2, \ldots, \beta_\tau),
\]
where
\[
\beta_\ell = \varepsilon \sum_{\ell=1}^{\tau} f_\ell.
\]
Lemma 3.1 implies that 1 is a simple eigenvalue of (3.10), and thus of \( E \), and all other eigenvalues are inside the unit circle. The analysis of (3.11) is now similar to that of (2.4) in §2, and we conclude that \( a_1(t) \to (e^1, a_1(0))1 \) as \( t \to \infty \), i.e.,
\[
\lim_{t \to \infty} \alpha_1(t) = \frac{1}{1 + \varepsilon \tau} \left[ \alpha_1(0) + \varepsilon \sum_{k=1}^{\tau} \sum_{\ell=k}^{\tau} f_{\ell} \alpha_1(-k) \right]. \tag{3.12}
\]
Using (3.12) in (3.9), and recalling that \( \alpha_1(k) = \langle u^1, x(k) \rangle \), (3.7) follows. The special case (3.8) is immediate by setting \( f_\ell = 1 \) for \( \ell = \tau \) and zero otherwise. Finally, to prove the necessity of a simple zero eigenvalue of the Laplacian, suppose \( \lambda_j = 0 \) for some \( j \geq 2 \). Then, by part (b) of lemma 3.1, \( \alpha_j(t) \) does not approach zero from generic initial conditions. Consequently, (3.1) implies that \( \lim_{t \to \infty} x(t) \) has a component along the eigenvector \( v_j \), which is not a multiple of 1; hence, consensus is not achieved from arbitrary initial conditions.

Thus, it is seen that consensus can be achieved regardless of the delays for the whole range of \( \varepsilon \in (0, 1) \), provided that zero is a simple eigenvalue of \( L \). The main effect of delays is in the consensus values, as given by (3.7), which depends on the initial history of the system over \( \tau + 1 \) successive time points.
4. Consensus in continuous time

The continuous-time consensus problem (1.7) with transmission delays has the vector form (2.2). Carrying out the decomposition \( x(t) = \sum_{i=1}^{n} \alpha_i(t) v_i \) as before, we obtain

\[
\dot{\alpha}_i(t) = -\varepsilon \alpha_i(t) + \varepsilon (1 - \lambda_i) \int_{0}^{t} f(s) \alpha_i(t - s) \, ds,
\]

whose characteristic equation is

\[
\chi_i(z) := z + \varepsilon - \varepsilon (1 - \lambda_i) F(z) = 0,
\]

where

\[
F(z) = \int_{0}^{\tau} e^{-zs} f(s) \, ds
\]

is the Laplace transform of \( f \). We have the following result on the characteristic roots.

**Lemma 4.1.** Let \( \varepsilon > 0 \). If \( \lambda_i = 0 \), then (4.2) has a simple root at zero and all other roots have negative real parts. If \( \lambda_i \neq 0 \), then all roots have negative real parts.

**Proof.** Rearranging (4.2), \( z \) is a root of \( \chi_i \) if and only if

\[
z + \varepsilon = \varepsilon (1 - \lambda_i) F(z).
\]

We first claim that if \( \chi_i \) has a root \( z \) with \( \text{Re}(z) \geq 0 \), then \( z \) must be zero. To prove the claim, let \( z = \sigma + i\omega \) with real part \( \sigma \geq 0 \). Then, \( |F(z)| \leq 1 \) by (1.9) and (4.3), so that

\[
|z + \varepsilon| = |\sigma + \varepsilon + i\omega| \geq \varepsilon
\]

\[
\geq |\varepsilon (1 - \lambda_i)| |F(z)|,
\]

where we have used (2.3) in the last line. Now the inequality (4.5) is strict whenever \( \sigma > 0 \) or \( \omega \neq 0 \), in which case (4.4) fails to hold, contradicting the assumption that \( z \) is a characteristic root. Thus, we must have \( \sigma = \omega = 0 \), which proves the claim. Therefore, all roots of \( \chi_i \) have negative real parts, except possibly for a root at zero. Because \( F(0) = 1 \) by (1.9) and (4.3), it follows from (4.2) that \( z = 0 \) is a root of \( \chi_i \) if and only if \( \lambda_i = 0 \), in which case

\[
\chi_i'(0) = 1 - \varepsilon F'(0) = 1 + \varepsilon \int_{0}^{\tau} sf(s) \, ds \neq 0,
\]

which shows that zero is a simple root. This completes the proof. \( \blacksquare \)

We can now state the consensus result for continuous time.

**Theorem 4.2.** Let \( \varepsilon > 0 \). The continuous-time system (2.2) reaches consensus if and only if zero is a simple eigenvalue of the Laplacian \( L \). Furthermore, the consensus value is given by

\[
c = \frac{1}{1 + \varepsilon \bar{\tau}} \left\{ \mathbf{u}^1, x(0) + \varepsilon \int_{-\tau}^{\tau} f(\theta) x(\xi) \, d\xi \, d\theta \right\},
\]

where \( \mathbf{u}^1 \) is the left eigenvector of \( L \) corresponding to the zero eigenvalue, and \( \bar{\tau} = \int_{0}^{\tau} sf(s) \, ds \) is the mean of the delay distribution \( f \). For the special case of a single discrete delay \( \tau \), the consensus value is

\[
c = \frac{1}{1 + \varepsilon \tau} \left\{ \mathbf{u}^1, x(0) + \varepsilon \int_{-\tau}^{\tau} x(\xi) \, d\xi \right\}.
\]
Proof. Suppose that zero is a simple eigenvalue of the Laplacian $L$. Then, using lemma 4.1, we conclude that solutions of (4.1) satisfy $\lim_{t \to \infty} \alpha_i(t) = 0$ for $i \geq 2$. Consequently, by (3.1),

$$\lim_{t \to \infty} \|x(t) - \alpha_1(t)1\| = 0. \quad (4.8)$$

It remains to study the dynamics of $\alpha_1$, which is governed by

$$\dot{\alpha}_1(t) = -\varepsilon \alpha_1(t) + \varepsilon \int_0^t f(s)\alpha_1(t-s) \, ds. \quad (4.9)$$

To this end, we let $C = C([-\tau, 0], \mathbb{R})$ denote the Banach space of real-valued continuous functions on $[-\tau, 0]$, equipped with the supremum norm. Similarly, let $C^* = C([0, \tau], \mathbb{R})$, and define the bilinear form $(\psi, \phi)$ for $\phi \in C$ and $\psi \in C^*$ by

$$(\psi, \phi) := \psi(0)\phi(0) - \varepsilon \int_{-\tau}^0 \int_0^\theta f(-\theta)\psi(\xi - \theta)\phi(\xi) \, d\xi \, d\theta$$

$$= \psi(0)\phi(0) + \varepsilon \int_0^\tau \int_0^\theta f(\theta)\psi(\xi + \theta)\phi(\xi) \, d\xi \, d\theta. \quad (4.10)$$

Let $C_0$ and $C_0^*$ be the one-dimensional subspace of constant functions in $C$ and $C^*$, respectively. We let the constant function $\Phi(\theta) \equiv 1$ be a basis for $C_0$ and the constant function $\Psi(\theta) \equiv (1 + \varepsilon \bar{\tau})^{-1}$ be a basis for $C_0^*$, so that $(\Phi, \Psi) = 1$. Now let $a_1 \in C$ be defined by

$$a_1(\theta) = \alpha_1(t + \theta), \quad \theta \in [-\tau, 0]. \quad (4.11)$$

By lemma 4.1, the characteristic equation corresponding to (4.9) has a simple root at zero, and all other roots have negative real parts. Note that $C_0$ is the eigenspace corresponding to the characteristic value zero. By the theory of functional differential equations [14], the space $C$ can be decomposed using the invariant subspace $C_0$ and its complement. Hence, we can write $a_t$ as the sum $a_t = a_t^0 + b_t$, where $a_t^0 \in C_0$ for all $t > 0$ and $\lim_{t \to \infty} b_t = 0$ because all eigenvalues other than zero have negative real parts. Further, $a_t^0$ is given by the constant function $a_t^0 = \Phi(\Psi, a_0)$. Using (4.10) and (4.11), we conclude that $a_t$ (and hence $\alpha_1(t)$) approaches the constant function

$$a_t^0 = \Phi(\Psi, a_0) = \frac{1}{1 + \varepsilon \bar{\tau}} \left( \alpha_1(0) + \varepsilon \int_0^\tau \int_0^\theta f(\theta)\alpha_1(\xi) \, d\xi \, d\theta \right).$$

By (4.8) and (3.1), consensus is achieved, with the consensus value given by (4.6). Finally, when zero is not a simple eigenvalue of $L$, we use a similar argument as in the proof of theorem 3.2 to conclude that the system cannot reach consensus from arbitrary initial conditions.

\section{5. Observations}

(a) Zero eigenvalue and spanning trees

The main results of this paper, as given by theorems 3.2 and 4.2, indicate that the stability of the consensus algorithm does not depend on the delays but on the network structure. Namely, it is required that zero is a simple eigenvalue of the Laplacian $L$. This condition is equivalent to the more geometrical condition that the network has a spanning tree, i.e. there exists a vertex from which all other vertices can be reached along directed paths. This is a known fact for the matrix $D - A$ [15], and also holds for the normalized Laplacian $I - D^{-1}A$ with a similar proof [16]. Clearly, an undirected graph has a spanning tree if and only if it is connected.

(b) Undirected networks

Results from graph theory allow more detailed information for undirected networks [17]. Indeed, if the underlying network $G$ is undirected, then the Laplacian $L$ has real eigenvalues, which can be ordered as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$, counting multiplicities. The smallest eigenvalue $\lambda_1$ is zero,
and $\lambda_2 > 0$ if and only if the graph is connected. Furthermore, the largest eigenvalue equals 2 if and only if a connected component of $G$ is bipartite and non-trivial. By the foregoing results (for $\varepsilon > 0$ for the continuous-time system and $\varepsilon \in (0, 1)$ for the discrete-time system), the network reaches consensus if and only if it is connected, regardless of the delay. Furthermore, for $\varepsilon = 1$ the undelayed discrete-time network can reach consensus if and only if the graph is connected and non-bipartite.

Furthermore, for undirected graphs, one can calculate the left eigenvector $u^1$ of the normalized Laplacian corresponding to the zero eigenvalue to be

$$u^1 = \frac{1}{\text{vol}(G)} (d_1, \ldots, d_n),$$

(5.1)

where $\text{vol}(G) = \sum_{i=1}^n d_i$ is the volume of the graph. In other words, $u^1$ is simply the degree sequence $(d_1, \ldots, d_n)$ of the graph with a normalization. Thus, when the system reaches consensus, the consensus value is given by a weighted average of the initial opinions where the weights are determined by the vertex degrees. Consequently, the final consensus value is influenced more by those units having a larger number of neighbours. For instance, in the scale-free networks, the consensus value is essentially determined by the hubs.

(c) Normalized versus non-normalized Laplacian

In this paper, we have used the normalized Laplacian $L = I - D^{-1}A$, as opposed to most works that use $D - A$, which corresponds to the protocol

$$u_i(t) = \varepsilon \sum_{j=1}^n a_{ij} \left( \int_0^t f(s)x_j(t-s)\,ds - x_i(t) \right).$$

If $\sum_{j=1}^n a_{ij} = k$ for all $i$, then the two Laplacians differ simply by a scalar factor of $k$, and the coupled system takes the form

$$\dot{x}(t) = -\varepsilon k x(t) + \varepsilon A \int_0^t f(s)x(t-s)\,ds.$$  

(5.2)

Seuret et al. [11] considered a special case of (5.2) for a fixed delay at $\tau$ and derived convoluted expressions involving matrix inequalities and limits as the consensus condition. However, (5.2) is the same as (2.2) when $\varepsilon$ is replaced by $\varepsilon k$. Hence, the results of §4 imply that the consensus condition is actually much simpler, namely the system (5.2) reaches consensus if and only if zero is a simple eigenvalue of the Laplacian (in either definition). Furthermore, the consensus value is given by (4.7), or more generally by (4.6) for distributed delays.

Although both Laplacians have similar properties, an essential difference regarding consensus problems is the difference in their left eigenvectors. One can see this more clearly in undirected graphs: because $D - A$ is a symmetric matrix, then the left eigenvector $u^1$ corresponding to the zero eigenvalue equals the transpose of the right eigenvector $v_1 = 1$ with the normalization $\langle u^1, v_1 \rangle = 1$, i.e.

$$u_1 = \frac{1}{n} (1, 1, \ldots, 1).$$

Hence, the consensus value, say (2.8), in the case of the non-normalized Laplacian is based on the average of the agents’ initial conditions, whereas the normalized Laplacian, in view of (5.1), weights the initial conditions by the vertex degrees. This property of the normalized Laplacian makes it a more appropriate choice for modelling opinion dynamics, where one naturally expects that the highly connected hubs have a more pronounced influence on the final outcome than the poorly connected members of the network.
(d) Transmission versus processing delays

When one considers processing delays, modelled by (1.4), the discrete-time system (1.1) takes the form

\[ x(t + 1) = x(t) - \varepsilon L x(t - \tau), \]

and the expansion (3.1) gives that the perturbations \( \alpha_i \) along the eigendirections are governed by

\[ \alpha_i(t + 1) = \alpha_i(t) - \varepsilon \lambda_i \alpha_i(t - \tau). \]

Similarly, for the continuous-time system (1.2), one obtains

\[ \dot{\alpha}_i(t) = -\varepsilon \lambda_i \alpha_i(t - \tau). \] (5.3)

Although both of the above are delay equations for \( \lambda_i \neq 0 \), they become ordinary equations for \( \lambda_i = 0 \). In particular, because \( \lambda_1 = 0 \) corresponds to the eigenvector 1, one obtains an ordinary difference or differential equation for the mode that eventually determines the consensus value. Hence, if the system reaches consensus, then the consensus value depends only on the initial value of the state \( x \) at a single time point. By contrast, under signal transmission delays modelled by (1.6) and (1.7), the consensus value depends on the initial history of \( x \) over some time interval, as given by (3.7) and (4.6). This property can imply additional robustness for the consensus value. For instance, if the measured states of agents are subject to small noise, then the consensus value calculations will have some error. However, if the noise terms have zero temporal mean, then their effect is reduced by averaging over some time interval through the formulae (3.7) and (4.6). Because theorems 3.2 and 4.2 imply that reaching consensus is independent of the delays, a suitably large delay can be chosen to facilitate noise reduction in the calculation of the consensus value without impeding consensus itself. A detailed study of this feature will be given in a separate article [18]; here, we illustrate with an example.

We take a regular, undirected and unweighted network of 20 vertices in a circular arrangement, where each vertex is connected to its two nearest neighbours on either side. The initial states of 10 vertices are chosen randomly from a uniform distribution over \([-1, 1]\) (and are constant over time until \( t = 0 \)), and the remaining vertices are set to negatives of those values, so that the initial states sum up to zero. The consensus value is thus expected to be zero. We subject the system to 10% additive noise; that is, at each time step, each vertex receives the state of each of its neighbours added to a uniformly distributed random number from the interval \([-0.1, 0.1]\). The states converge to a common value, which varies over time owing to noise, as shown in figure 1. From figure 1, it is seen that the consensus protocol with a fixed positive delay (circles) performs significantly better than the undelayed protocol (triangles). Moreover, distributed delays provide even further improvement (black solid line).

Finally, we note the difference between transmission and processing delays with respect to delay magnitude. As we have shown in this paper, stability of consensus is independent of the signal transmission delays for the systems (1.6) and (1.7). This is in contrast to signal processing delays, where large delays are known to prohibit consensus [7,9].

(e) Application to synchronization of phase oscillators

The equations describing consensus also arise in several other applications through linearization near specific solutions. One example is the coupled phase oscillator model of Kuramoto [19]. We consider the special case of identical oscillators (all having equal intrinsic frequency \( \omega \)) but allowing general coupling functions and distributed coupling delays,

\[
\dot{\theta}_i = \omega + \frac{\varepsilon}{d_i} \sum_{j=1}^{N} a_{ij} g \left( \int_{0}^{\tau} \theta_j(t - s) f(s) \, ds - \theta_i(t) \right), \quad i = 1, \ldots, n. \] (5.4)
As usual, \( \theta_i \in S^1 \) is the phase of the \( i \)th oscillator, and \( \omega \) is the natural frequency of the oscillators. A special solution, namely the case when \( \theta_i(t) = \Omega t, \quad \forall i, \quad (5.5) \)
is of interest in synchronization studies and corresponds to the behaviour when all oscillators fall in perfect step and oscillate at a common frequency \( \Omega \). Substituting into (5.4) and using (1.9) gives \( \Omega \) as a solution of the equation \( \Omega = \omega + \varepsilon g(-\Omega \bar{\tau}) \), where \( \bar{\tau} = \int_0^\tau s f(s) \, ds \) is the mean delay. Linearizing (5.4) about the synchronized solution (5.5) gives the equation
\[
\dot{x}_i(t) = g'(-\Omega \bar{\tau}) \frac{\varepsilon}{d_i} \sum_{j=1}^n a_{ij} \left( \int_0^\tau x_j(t-s) f(s) \, ds - x_i(t) \right), \quad (5.6)
\]
in the perturbations \( x_i \). Hence, the system (5.4) locally synchronizes precisely when all solutions \( (x_1, \ldots, x_n) \) of (5.6) asymptotically approach the subspace spanned by \( 1 \). Obviously, (5.6) is of the same form as the consensus model (1.7). Thus, theorem 4.2 directly yields the synchronization condition: the synchronized state with frequency \( \Omega \) is asymptotically stable if and only if \( g'(-\Omega \bar{\tau}) > 0 \) and the network contains a spanning tree. This delay-independent synchronization result is essentially what has previously been obtained in [20] for fixed (discrete) delays, and the distributed-delay case differs only by the use of the mean delay.

6. Conclusion

Despite its apparent simplicity, the consensus problem arises in various contexts and presents several challenges in the investigation of the cooperative behaviour of systems. In this paper, we have studied the consensus problem in networks with distributed signal transmission delays. We have obtained exact conditions for reaching consensus and calculated the consensus value. Although we have focused on static networks and deterministic systems, there is also an interest in applications that involve stochasticity or networks whose structure changes in time. The consensus problem in such a setting has recently been studied by Lu et al. [21]. Additionally, real systems often involve nonlinearities with non-trivial global dynamics, for which a linear consensus-type equation can represent only local behaviour. A case in point is the synchronization of the oscillator network of §5e, which may exhibit different types of
synchronized or phase-locked solutions [22]. Finally, we remark that the assumption of non-negative connection weights, $a_{ij} \geq 0$, is a natural one for cooperative problems such as consensus. However, some systems, such as neural networks, may have both excitatory and inhibitory interactions, i.e. connectivities of both signs, which introduce further challenges to analysis. In general, networks involving nonlinearity, stochasticity and time delays remain a rich source of interesting problems for the mathematical sciences.

References