Chaos in networks with time-delayed couplings

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Networks of nonlinear units coupled by time-delayed signals can show chaos. In the limit of long delay times, chaos appears in two ways: strong and weak, depending on how the maximal Lyapunov exponent scales with the delay time. Only for weak chaos, a network can synchronize completely, without time shift. The conditions for strong and weak chaos and synchronization in networks with multiple delay times are investigated.

1. Introduction

The cooperative behaviour of networks of nonlinear units is an active field of research, not only from a fundamental point of view but also with applications in different scientific disciplines, from neurons to lasers [1]. The nonlinear units interact by transmitting a signal to their neighbours. Often, the transmission time is longer than the internal time scales of these units; the coupling has a long delay time. When the evolution of a deterministic system depends not only on its present state but also on its past state, the mathematical space of its solution becomes infinite dimensional, and the system is flexible enough to develop instabilities which lead to high-dimensional chaos [2,3]. For example, if the beam of a semiconductor laser is reflected by an external mirror back to the laser cavity, then a chaotic intensity on the time scale of picoseconds is observed [4].

The same is true for a network of nonlinear units with time-delayed couplings. The network can develop high dimensional space–time chaos. However, under some conditions, the network can synchronize to a common chaotic trajectory [5,6]. Although the transmission time may be very long, the units move without time shift in lockstep, but chaotically. For two coupled semiconductor lasers, complete chaos synchronization has been demonstrated experimentally in [7].
Chaos is extremely sensitive to initial conditions, and the Lyapunov exponents (LEs) describe how fast two nearby trajectories separate from each other. Recently, two types of chaos have been reported for time-delayed systems: strong and weak, depending on how the LE scales with the delay time [8]. It was shown that systems with weak chaos can only synchronize completely. Chaos synchronization is related to the eigenvalue gap of the coupling matrix and the LE of a single unit with feedback [9]. For networks without eigenvalue gap and/or with multiple delay times, an argument related to mixing of information determines the conditions and patterns of chaos synchronization [10].

In this contribution, I summarize our recent results on chaotic networks. In §2, strong and weak chaos is investigated for a single unit with self-feedback. Networks are considered in §3, and, in §4, units with multiple delay times are studied.

2. Strong and weak chaos of a single unit with feedback

Here, we consider the chaotic attractor of a single unit with time-delayed self-feedback. This unit may represent the dynamics of the synchronization manifold of a chaotic network, but it may also be a model for a single laser where the laser beam is reflected back into the laser cavity with an external mirror. In general, the dynamics is assumed to be given by the equation

\[ \dot{s}(t) = F[s(t)] + \sigma H[s(t - \tau)], \]  

where \( s(t) \) is a set of \( k \) local variables, and the instantaneous dynamics is described by the function \( F \). The feedback is added with some delay time \( \tau \), with a function \( H \) and a strength \( \sigma \). In the following, we will discuss the properties of this unit as a function of the feedback strength \( \sigma \). For example, semiconductor lasers have a stable fixed point for \( \sigma = 0 \). Already, a small feedback generates instabilities and chaos. Mathematically, the \( k \)-dimensional system becomes infinite dimensional when the feedback is added, hence there is enough phase space available for a chaotic attractor. The LE of this unit is described by linearizing equation (2.1). One obtains

\[ \delta \dot{s}(t) = DF[s(t)]\delta s(t) + \sigma DH[s(t - \tau)]\delta s(t - \tau). \]  

Here, \( DF \) and \( DH \) are the corresponding Jacobi matrices which depend on the trajectory \( s(t) \). Although this is a linear equation for a small deviation \( \delta s \) between two trajectories, the delay term leads to a complex behaviour which usually can be investigated only numerically. In the limit of large delay times \( \tau \), however, some general properties can be derived [8]. Roughly speaking, if the first term of equation (2.2) explodes, then the second term cannot avoid this. If the first term does not explode, then the maximal LE \( \lambda_m \) decreases to zero with increasing delay time, but the product \( \lambda_m \tau \) goes to a constant value. Thus, we find two different types of chaos in the limit of large delay times, which we call strong and weak chaos. For strong chaos, we have

\[ \lim_{\tau \to \infty} \lambda_m = \mu, \]  

with a constant \( \mu \), whereas, for weak chaos, we have

\[ \lim_{\tau \to \infty} \lambda_m \tau = \text{const.} \]  

In fact, strong and weak instabilities of fixed points of time-delayed systems have been found in Wolfrum et al. [11]. For iterated maps with time-delayed feedback, different scaling of the LE has already been found by Lepri et al. [12]. Here, we investigate this behaviour for general chaotic networks.

It turns out that an artificial LE determines whether the chaos is strong or weak. It is given by the equation

\[ \dot{\xi}(t) = DF[s(t)]\xi(t). \]  

The maximal LE of this equation is denoted by \( \lambda_0 \). We call it sub-LE; previously, it was called anomalous [12] or instantaneous LE [8]. Note that \( \lambda_0 \) still depends on the feedback, because the
matrix $DF$ depends on the trajectory of the complete dynamics, equation (2.1). Hence, $\lambda_0$ is also a function of $\sigma$.

The sign of the sub-LE $\lambda_0$ determines the type of chaos. If $\lambda_0$ is positive, then chaos is strong and $\lambda_0$ agrees with the value of $\mu$ in equation (2.3). If the sub-LE is negative, then chaos is weak, and the limit of $\lambda_m\tau$ has to be calculated from equation (2.2).

Precise scaling relations have been derived in Heiligenthal et al. [8]. However, already an intuitive argument shows the properties of equation (2.2): small deviations separate from each other with $\delta s(t) \propto \exp(\lambda_m t)$. If, in the limit of large delay times $\tau$, the largest LE $\lambda_m$ goes to a constant, then the second term of equation (2.2) is weakened to zero with the factor $\exp(-\lambda_m \tau)$. Consequently, $\lambda_m$ agrees with the sub-LE of equation (2.5). This cannot happen when the sub-LE is negative, because we consider only chaotic systems. In this case, the second term of equation (2.2) is important, and this can happen only if $\lambda_m \tau$ goes to a constant in the limit of infinite delay times.

The sign of the sub-LE may change with the coupling strength $\sigma$ or with other parameters of the model. In this case, we observe a transition from weak to strong chaos. The properties of this transition can be derived from the behaviour of corresponding linear equations with multiplicative noise (T. Jüngling 2012, unpublished data). It turns out that close to the transition the noise term (modelling the time dependence of the chaotic coefficients of equation (2.2)) becomes important, and the delay-normalized exponent $\lambda_m\tau$ increases with $\sqrt{\tau}$ just at the transition $\lambda_0 = 0$.

The predictions of the scaling relations equations (2.3) and (2.4) have been tested with numerical simulations of the rate equation of semiconductor lasers, the Lang–Kobayashi (LK) equations:

\[
\begin{align*}
\dot{E}(t) &= \frac{1}{2} - G_N n(t) E(t) \\
\dot{n}(t) &= (p - 1)I_{th} - \gamma n(t) - (\Gamma + G_N n(t)) |E(t)|^2.
\end{align*}
\]

(2.6)

Realistic parameters of this equation are taken from Heiligenthal et al. [8]. Figure 1 shows the maximal LE and the sub-LE as a function of the feedback strength $\sigma$. Without feedback, the laser relaxes to a constant intensity. However, for already very weak feedback, the laser is unstable and its intensity becomes chaotic. With increasing parameter $\sigma$, one observes a sequence of transitions: weak to strong to weak chaos.

Figure 2 shows how the maximal LE scales with the delay time. For strong chaos, it goes to the sub-LE exponentially fast with $\tau$, and, for weak chaos, the product $\lambda_m\tau$ goes to a constant with increasing delay $\tau$. This product diverges at the transition to strong chaos. Its scaling close to the transition can be understood from an analytic calculation of linear time-delayed systems with multiplicative noise (T. Jüngling 2012, unpublished data).
Figure 2. (a) Difference between the maximal LE and the sub-LE as a function of the delay time. The difference decreases exponentially with increasing delay in the case of strong chaos. (b) Scaled maximal LE in the case of weak chaos. With increasing delay time, the scaled LE saturates to a constant. Results are obtained from the laser equations (S. Heiligenthal 2012, unpublished data).

Figure 3. (a) Cross correlation between lasers A and B as a function of the coupling strength $\sigma$. (b) When A and B are synchronized, $C = 1$, the laser A is in the state of weak chaos. Adapted from S. Heiligenthal 2012, unpublished data.

It is interesting that the sub-LE, which has been introduced as a mathematical tool, is accessible to an experiment. In figure 3, the unit B, which is identical to the original unit A, is driven by A with the same strength as the feedback. Hence, the synchronized trajectory of A and B is a solution of corresponding dynamical equations. It turns out that the stability of this solution is determined by the sub-LE. A and B synchronize if and only if the sub-LE is negative.

3. Networks

Strong and weak chaos has also been observed in networks of nonlinear units which are coupled by signals which are transmitted with a delay time $\tau$. Consider a network of $N$ units $x_i(t)$ with the equations of motion

$$\dot{x}_i(t) = F[x_i(t)] + \sigma \sum_k G_{i,k} H[x_k(t - \tau)].$$

In this case, each node may have its own sub-LE. For example, if three lasers are coupled on a chain (figure 4), the inner laser has a different sub-LE from the two outer ones. In the case of strong chaos, the maximum of all sub-LEs is identical to the maximal LE of the complete network. This is shown in figure 4 from the simulation of the LK equations. For small coupling strength $\sigma$, the outer sub-LE is larger than the inner one. But with increasing $\sigma$, the inner one passes the outer one until finally both cross zero. The LE of the network follows the larger one of the two sub-LEs. When both become negative, the two outer units synchronize to a common chaotic trajectory. Synchronization is mediated by the inner laser, but the inner one does not synchronize [13].
The row sum of the coupling matrix \( G \) is unity; hence, the synchronized trajectory \( x_i(t) = s(t) \) is a solution of the dynamics, equation (2.1). Within the synchronization manifold, the dynamics is identical to that of a single unit with feedback, as discussed earlier. However, the stability of the synchronization manifold is determined by the master stability function, which is the largest LE of the following equation:

\[
\delta \dot{s}_k(t) = DF[s(t)]\delta s_k(t) + \sigma \gamma_k DH[s(t - \tau)]\delta s_k(t - \tau).
\]

Here, the coupling \( \sigma \) is multiplied by the eigenvalues \( \gamma_k \) of the matrix \( G \). \( \gamma_1 = 1 \) describes perturbations inside the synchronization manifold. Because we consider chaotic systems, the largest LE is always positive for \( \gamma_1 = 1 \). When the LEs for all other eigenvalues are negative, the synchronization manifold is stable, and complete chaos synchronization occurs. When only some of the eigenmodes are stable, it depends on the structure of the corresponding eigenvectors whether clusters of the network can still synchronize.

In the limit of long delay times, a remarkable result has been derived \[8,9\]. The condition for complete chaos synchronization reads as

\[
|\gamma_2| < \exp(-\lambda_m \tau),
\]

where \( \gamma_2 \) is the eigenvalue of \( G \), whose absolute value is the second largest, and \( \lambda_m \) is the largest LE of a single unit with feedback. Thus, the dynamics of a single unit and the eigenvalue gap of the coupling matrix determine the stability of chaos synchronization for any network of identical units with time-delayed coupling.

As a consequence of equation (3.3), we immediately derive the following results. Chaos synchronization is ruled out for a pair of units without self-feedback, for any bi-partite network, for any directed ring. For these cases, the eigenvalue gap is zero, \( |\gamma_2| = 1 \). A triangle or an all-to-all network with bidirectional couplings, however, can synchronize, because one finds \( \gamma_2 = -\frac{1}{3} \) or \(-1/(N - 1)\), respectively. Adding self-feedback opens an eigenvalue gap; in this case, even a pair of semiconductor lasers has been synchronized to a common chaotic intensity \[7\].

The existence of a non-zero eigenvalue gap depends on the topological properties of the loop structure of the network graph. Consider the lengths of all loops of a network. It turns out that the greatest common divisor (GCD) of the loop lengths determines the eigenvalue gap. If the GCD = 1, a non-zero eigenvalue gap exists, and the system can synchronize if the condition equation (3.3) is fulfilled. If GCD = \( m \), then complete synchronization is ruled out, but the network can still synchronize to \( m \) many clusters \[14–16\]. In the latter case, the condition equation (3.3) is more complex and depends on the dynamics of the individual clusters \[16\].

As an example, consider the network shown in figure 5. A directed ring of three or four units cannot synchronize. However, when we combine them, we find GCD(3, 4) = 1, and the system can synchronize according to equation (3.3). This example shows that the properties of network motifs do not determine the collective behaviour of the network. Global properties of...

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**Figure 4.** (a,b) The upper line shows the maximal LE of a chain of three lasers. The dashed lines show the two different sub-LEs for the outer and inner lasers. For strong chaos, the maximal LE follows the largest of the sub-LEs. Adapted from S. Heiligenthal 2012, unpublished data. When both sub-LEs are negative, the lasers A and C synchronize. (Online version in colour.)
Figure 5. Directed rings of chaotic units cannot synchronize. However, when two rings are connected, and the GCD of the two loops is 1 (here GCD(3, 4) = 1), then the network can synchronize completely.

Figure 6. When the network mixes all colours, the network can synchronize completely. (Online version in colour.)

the graph determine chaos synchronization. This phenomenon has also been observed for cluster synchronization of non-chaotic laser networks and for neural networks [17,18].

Remarkably, a simple intuitive argument shows the condition for complete cluster synchronization [10]. Information on the trajectories of the individual units has to mix in order to synchronize them. For example, paint the units of the network with different colours, as in figure 6. Then, mix the colours in discrete steps according to the graph of the network. If finally all colours are mixed, then complete synchronization is possible. If the final coloured configuration is periodic with period $m$, then cluster synchronization with $m$ clusters is possible. For example, in the case of three units on a chain (figure 4), one obtains $m = 2$, and therefore only the two outer units can synchronize. Thus, if the information about trajectories is not mixed, then synchronization is ruled out. On the other hand, if all nodes are mixed, complete synchronization is possible if chaos is sufficiently weak. However, even if the second largest mode is unstable, the system may still synchronize to several clusters, depending on the stability of the corresponding eigenvectors. The validity of these arguments has been tested experimentally on a large networks of lasers [17].

4. Multiple delay times

The argument for mixing information is also valid for multiple delay times [19–21]. For example, a pair of units cannot synchronize because its eigenvalue gap is zero. But, when we add an additional coupling channel with delay $\tau_2 = 2\tau_1$, one finds GCD(3, 2) = 1. Therefore, the two units can synchronize.

For semiconductor lasers, the second delay has been realized by splitting the laser beam and constructing a detour with mirrors. Complete chaos synchronization was measured when $\tau_2$ was
Figure 7. For multiple delays, the loop structure is defined in units of the smallest delay time. In (a), there are closed loops of length 2 and 3, and (b) for the triangle, there exist loops of length 3 and 4. For both (a,b), the GCD is 1; therefore, both networks can synchronize. (Online version in colour.)

Figure 8. Two lasers are mutually coupled by their laser beams with two different delay times, as in figure 7a. The stability of the synchronization manifold is described by the LE of the \((-1, -1)\)-mode, which is shown as a function of the ratio of the two delay times. If this ratio is a ratio of odd/odd relative primes, then the LE is positive and synchronization is not possible. For odd/even or even/odd ratios, however, the LE may cross zero, and the two lasers synchronize. (Online version in colour.)

carefully adjusted to \(2\tau_1\) [21]. In fact, the mixing argument rules out synchronization when the ratio of \(\tau_2/\tau_1 = p/q\) is a ratio of two odd relative prime numbers \(p\) and \(q\), in the limit of large delay times. For other ratios, synchronization is not ruled out, although the parameter space available for synchronization shrinks with increasing \(p\) and \(q\).

The corresponding master stability function for two lasers is shown in figure 8 as a function of the ratio \(\tau_2 = 2\tau_1\). For odd/odd ratios, the LE describing the stability of the synchronization manifold is identical to that describing chaos, thus synchronization is unstable.

Thus, in the limit of infinitely large delay times, keeping their ratios constant, we find resonances of stability of the synchronization manifold. However, synchronization occurs for weak chaos, i.e. only when all sub-LEs are negative.

For multiple delay times, one can distinguish two limiting cases: first, all delay times diverge but their ratios are kept constant. This case has been discussed already and may lead to resonances. Second, all delay times are large but separated from each other, \(\tau_1 \ll \tau_2 \ll \cdots \ll \tau_M\). For example, consider the single unit with multiple feedbacks, which is identical to the dynamics of the synchronization manifold of a network with multiple couplings. The LE is given by the linear equation

\[
\delta \dot{s}(t) = DF[s(t)]\delta s(t) + \sum_k \sigma_k DH[s(t - \tau_k)]\delta s(t - \tau_k). \tag{4.1}
\]

Now, we obtain a hierarchy of phases with strong and weak chaos. If the sub-LE \(\lambda_0\) without delay, equation (2.5), is positive, we have strong chaos, as before, and the maximal LE converges
to $\lambda_0$ in the limit of infinite delays. If, however, $\lambda_0$ is negative, we have to consider the sub-LE $\lambda_1$, which is defined by including the smallest delay $\tau_1$,

$$
\dot{\xi}(t) = DF[s(t)]\xi(t) + \sigma_1 DH[s(t - \tau_1)]\xi(t - \tau_1). \tag{4.2}
$$

If $\lambda_1$ is positive, the maximal LE converges to $\lambda_1$ when all larger delay times go to infinity. If $\lambda_1$ is negative, then we have to continue with the sub-LE $\lambda_2$ defined by the corresponding linear equation containing $\tau_1$ and $\tau_2$. In fact, this different scaling behaviour can already be seen in the complete spectrum of LEs. There are some LEs which do not scale with any delay. However, there is a part of the spectrum which scales with $1/\tau_1$, another part scales with $1/\tau_2$, etc. Because the time scales have to be well separated, say at least a factor 10 for iterated maps, it is possible to observe this sequential scaling behaviour only for up to three delay times. But, for this case, the hierarchy of strong and consecutive weak chaos has been found in simulations of the iterated functions and semiconductor rate equations, and in analytical calculations for Bernoulli units [22].

References

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