Quantum ergodicity of random orthonormal bases of spaces of high dimension

Steve Zelditch

Department of Mathematics, Northwestern University, Evanston, IL 60208-2370, USA

We consider a sequence $\mathcal{H}_N$ of finite-dimensional Hilbert spaces of dimensions $d_N \to \infty$. Motivating examples are eigenspaces, or spaces of quasi-modes, for a Laplace or Schrödinger operator on a compact Riemannian manifold. The set of Hermitian orthonormal bases of $\mathcal{H}_N$ may be identified with $U(d_N)$, and a random orthonormal basis of $\bigoplus N \mathcal{H}_N$ is a choice of a random sequence $U_N \in U(d_N)$ from the product of normalized Haar measures. We prove that if $d_N \to \infty$ and if $(1/d_N) \text{Tr} A|_{\mathcal{H}_N}$ tends to a unique limit state $\omega(A)$, then almost surely an orthonormal basis is quantum ergodic with limit state $\omega(A)$. This generalizes an earlier result of the author in the case where $\mathcal{H}_N$ is the space of spherical harmonics on $S^2$. In particular, it holds on the flat torus $\mathbb{R}^d/\mathbb{Z}^d$ if $d \geq 5$ and shows that a highly localized orthonormal basis can be synthesized from quantum ergodic ones and vice versa in relatively small dimensions.

1. Introduction

The purpose of this article is to prove a general result on the quantum ergodicity of random orthonormal bases $\{\psi_{N,j}\}_{j=1}^{d_N}$ of finite-dimensional Hilbert spaces $\mathcal{H}_N \subset L^2(M)$ of dimensions $d_N \to \infty$ of a compact Riemannian manifold $(M,g)$. The proof is based on a ‘moment polytope’ interpretation of quantum ergodicity from [1]: the quantum variances of a Hermitian observable $A \in \Psi^0(M)$ (where $\Psi^0(M)$ is the class of pseudo-differential operators of degree zero) are identified with moments of inertia of the convex polytopes $P_\lambda$ defined as the convex hull of the vectors $\lambda = (\lambda_1, \ldots, \lambda_{d_N})$ of eigenvalues (in all possible orders) of $\Pi_N A \Pi_N$ where $\Pi_N : L^2(M) \to \mathcal{H}_N$ is the orthogonal projection. Equivalently, $P_\lambda$ is the image of the coadjoint orbit $O_\lambda$ of the diagonal matrix $D(\lambda)$ under the moment map for the Hamiltonian action of the
maximal torus $T_{d_N} \subset U(d_N)$ of diagonal matrices acting by conjugation on $O_{d_N}$. In particular, the main estimates of quantum ergodicity can be formulated in terms of estimates of the first four moments of inertia of $T_{d_N}$. The main result, theorem 1.3, states that random orthonormal bases are almost surely quantum ergodic as long as $d_N \to \infty$ and $(1/d_N) \text{Tr} \Pi_N A \Pi_N \to \omega(A)$ for all $A \in \Psi^0(M)$, where $\omega(A)$ is the Liouville state. More generally, if these traces have any unique limit state, then almost surely it is the quantum limit of a random orthonormal basis. The proof is essentially implicit in [1], but we bring it out explicitly here and also give detailed calculations of the moments of inertia, which seem of independent interest.

Quantum ergodicity of random orthonormal bases is a rigorous result on the ‘random wave model’ in quantum chaos, according to which eigenfunctions of quantum chaotic systems should behave like random waves. It also has implications for the approximation of modes by quasi-modes. As eigenfunctions of the Laplacian $\Delta$ of a compact Riemannian manifold $(M, g)$ form an orthonormal basis, it is natural to compare the orthonormal basis of eigenfunctions to a ‘random orthonormal basis’. In [1], the result of this article was proved for the special case, where $\mathcal{H}_N$ is the space of degree $N$ spherical harmonics on the standard $S^2$. In [2], the quantum ergodic property was generalized to any compact Riemannian manifold, with $\mathcal{H}_N$ the span of the eigenfunctions in a spectral interval $[N, N + 1]$ for $\sqrt{\Delta}$; in [3], essentially the same result was proved for holomorphic sections of line bundles over Kähler manifolds. Related results for eigenfunctions have recently been proved in [4,5]. The dimension of such $\mathcal{H}_N$ grows at the rate $N^{m-1}$, where $m = \dim M$, and thus a random element of $\mathcal{H}_N$ is a superposition of $N^{m-1}$ states. The results of this article show that the same quantum ergodicity property holds for sequences of eigenspaces (or linear combinations) whose dimensions $d_N$ tend to infinity at any rate. For instance, the results show that random orthonormal bases of eigenfunctions on a flat torus of dimension $\geq 5$ are quantum ergodic (for the precise statement, see §4a, and for further discussion, see §1c.)

To explain the moment map interpretation and the variance formula, recall that quantum ergodicity is concerned with quantum variances, i.e. with the dispersion from the mean of the diagonal part of a Hermitian matrix on a large-dimensional vector space $\mathcal{H}_N$. The matrix is the restriction

$$T^A_N := \Pi_N A \Pi_N$$

(1.1)

to a pseudo-differential operator $A \in \Psi^0(M)$; here $\Pi_N$ is the orthogonal projection to $\mathcal{H}_N$ and $\Psi^0(M)$ is the space of pseudo-differential operators of order zero. The same methods and results apply to other contexts such as semiclassical pseudo-differential operators or to Toeplitz operators on holomorphic sections of powers of a positive line bundle [3]. Given an orthonormal basis (ONB) $\{\psi_{nj}\}_{j=1}^{d_N}$ of $\mathcal{H}_N$ we define the (normalized) quantum variances of the ONB (indexed by $A \in \Psi^0(M)$) by

$$V^A_N(\{\psi_{nj}\}) := \frac{1}{d_N} \sum_{j=1}^{d_N} |\langle A \psi_{nj}, \psi_{nj} \rangle - \omega(A) |^2.$$ (1.2)

Here, $\omega(A) = \int_{S^2} \sigma_A d\mu_L$, where $d\mu_L$ is normalized Liouville measure (of mass one), and $\sigma_A$ is the principal symbol of $A$.

**Definition 1.1.** A sequence $\{\psi_{nj}\}_N$ of ONBs of $\mathcal{H}_N$ is a quantum ergodic ONB of $L^2(M)$ if

$$(\mathcal{E}P) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} V^A_N(\{\psi_{nj}\}_{j=1}^{d_N}) = 0, \quad \forall A \in \Psi^0(M).$$ (1.3)

We say sequence $\{\psi_{nj}\}_N$ is fine quantum ergodic if

$$(\mathcal{FE}P) \quad \lim_{N \to \infty} V^A(\{\psi_{nj}\}_{j=1}^{d_N}) = 0, \quad \forall A \in \Psi^0(M).$$ (1.4)

By a standard diagonal argument, this implies a subsequence of density of one of the individual elements $\langle A \psi_{nj}, \psi_{nj} \rangle$ tends to $\omega(A)$. The fine notion of density is of course stronger, because there
could exist a sparse subsequence of \( N \) for which the individual normalized variances do not tend to zero. As this aspect of quantum ergodicity is the same as in [1,3] (e.g.), we do not discuss it here.

To define random orthonormal bases, we introduce the probability space \((ONB,d\nu)\), where \( ONB \) is the infinite product of the sets \( ONB_N \) of orthonormal bases of the spaces \( \mathcal{H}_N \), and \( \nu = \prod_{N=1}^\infty v_N \), where \( v_N \) is the Haar probability measure on \( ONB_N \). A point of \( ONB \) is thus a sequence \( \Psi = (\{\psi_{N,1}, \ldots, \psi_{N,d_N}\})_{N \geq 1} \) of orthonormal basis. Given one orthonormal basis \( \{e_{N,j}\} \) of \( \mathcal{H}_N \), any other is related to it by a unique unitary matrix. So the probability space is equivalent to the product

\[
(ONB,d\nu) \simeq \prod_{N=1}^\infty (\mathcal{U}(d_N), dU),
\]

where \( dU \) is the unit mass Haar measure on \( \mathcal{U}(d_N) \). We denote by \( \mathbb{E} \) the expectation with respect to \( d\nu \). Here, we are working with Hermitian orthonormal bases and Hermitian pseudo-differential operators. We could also work with real self-adjoint operators and real orthonormal bases, which are then related by the orthogonal group. The results in that setting are essentially the same but the proofs are somewhat more complicated; for expository simplicity, we stick to the unitary Hermitian framework.

Let \( A \in \Psi^0 \) and denote the eigenvalues of \( T_N^A \) by \( \lambda_1, \ldots, \lambda_{d_N} \). The empirical measure of eigenvalues of \( T_N^A \) is defined by

\[
v_{\lambda_N} := \frac{1}{d_N} \sum_{j=1}^{d_N} \delta_{\lambda_j}.
\]

Its moments are given by

\[
p_k(\lambda_1, \ldots, \lambda_{d_N}) = \sum_{j=1}^{d_N} \lambda_j^k = \text{Tr}(T_N^A)^k.
\]

To obtain quantum ergodicity, we put the following constraint on the sequence \( \{\mathcal{H}_N\} \):

**Definition 1.2.** We say that \( \mathcal{H}_N \) has local Weyl asymptotics if, for all \( A \in \Psi^0(M) \),

\[
\frac{1}{d_N} \text{Tr} T_N^A = \omega(A) + o(1).
\]

In fact, the results generalize to the case where \( \omega(A) \) is replaced by any other limit state, i.e.

\[
\int_{S^*M} \sigma_A \, d\mu,
\]

where \( d\mu \) is another invariant probability measure for the geodesic flow.

Our main result is:

**Theorem 1.3.** Let \( \mathcal{H}_N \) be a sequence of subspaces of \( L^2(M) \) of dimensions \( d_N = \dim \mathcal{H}_N \to \infty \). Assume that \( (1/d_N)\text{Tr} \Pi_N A \Pi_N = \omega(A) + o(1) \) for all \( A \in \Psi^0(M) \). Then with probability one in \((ONB,d\nu)\), a random orthonormal basis of \( \bigoplus_N \mathcal{H}_N \) is quantum ergodic.

A natural question is whether a random orthonormal basis is quantum uniquely ergodic (QUE), i.e. whether

\[
\max \{|\langle A\psi_{N,j}, \psi_{N,j} \rangle - \omega(A)|^2, j = 1, \ldots, d_N \} \to 0 \quad (a.s.) \, d\nu?
\]

As a tail event, the probability of a random orthonormal basis being QUE is either 0 or 1. Recently, this has been proved in [4,5] when the spectral intervals satisfy certain growth assumptions.

**(a) Outline of the proof**

The first step is to reformulate quantum ergodicity properties (1.3) and (1.4) of a random ONB in terms of moment maps and polytopes. As mentioned above, quantum ergodicity of a random orthonormal basis concerns the dispersion from the mean of the diagonal part of \( T_N^A \), which depends on the choice of an orthonormal basis of \( \mathcal{H}_N \). Once an orthonormal basis is fixed, \( iT_N^A \) can be identified with an element \( H_N \) of the Lie algebra \( u(d_N) \) of \( \mathcal{U}(d_N) \), and a unitary change of the
orthonormal basis results in the conjugation $H_N \rightarrow U_N^* H_N U_N$ of $H_N$. If the vector of eigenvalues of $H_N$ is denoted $\lambda_N$, then the conjugates sweep out the orbit $O_{\lambda_N}$. Let $t(d_N)$ denote the Cartan subalgebra of diagonal elements in $u(d_N)$, and let $\| \cdot \|^2$ denote the Euclidean inner product on $t(d_N)$. Also let

$$J_{d_N} : iu(d_N) \rightarrow i t(d_N)$$

denote the orthogonal projection (extracting the diagonal). Extracting the diagonal from each element of the orbit is precisely the moment map

$$J_{d_N} : O_{\lambda_N} \rightarrow {\mathcal P}_{\lambda_N} \subset i t(d_N), \quad J_{d_N}(U d(\lambda) U^*) = \left( \ldots, \sum_{j=1}^{d_N} \lambda_j |U_{ij}|^2, \ldots \right)$$

(1.9)

of the conjugation action of the Cartan subgroup $T_{d_N}$ of diagonal matrices (see [1] for background and references). Finally, let

$$\bar{J}_{d_N}(H) = \left( \frac{1}{d_N} \text{Tr } H \right) \text{Id}_{d_N} \quad \text{and} \quad D_0(\lambda_N) = D(\lambda_N) - \left( \frac{1}{d_N} \text{Tr } H \right) \text{Id}_{d_N},$$

for Hermitian matrices $H \in iu(d_N)$. We also introduce notation for the diagonal of $D_0(\lambda_N)$

$$D_0(\lambda) = D(\lambda), \quad \text{with } \Lambda_j := \lambda_j - \frac{1}{d_N} \sum_{j=1}^{d_N} \lambda_j. \quad (1.10)$$

Thus,

$$H = H^0 + \bar{J}_{d}(H), \quad \text{resp., } \quad D(\lambda_N) = D(\lambda_N) + \left( \frac{1}{d_N} \text{Tr } H \right) \text{Id}_{d_N},$$

with $H^0$ traceless, corresponds to the decomposition $u(d_N) = su(d_N) \oplus \mathbb{R}$.

We now rewrite quantum variances (1.2) in terms of the polytopes $P_{\lambda_N}$. Define

$$X^A_N(\Psi) := \frac{1}{d_N} \| J_{d_N}(U_N^* D(\lambda_N) U_N) - \omega(A) I \|^2. \quad (1.11)$$

The following is immediate from the definitions:

**Lemma 1.4.** The ergodic property of an ONB $\Psi$ (cP) is equivalent to

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_N} X^A_N(\Psi) = 0, \quad \forall A \in \Psi^0(M). \quad (1.12)$$

Similarly, (1.4) is equivalent to $(1/d_N)X^A_N(\Psi) \rightarrow 0$ almost surely.

As lemma 1.4 indicates, quantum ergodicity of random orthonormal bases is essentially a result about the asymptotic geometry of the polytopes $P_{\lambda_N}$ corresponding to a sequence $T_N^A$ of Toeplitz operators satisfying trace condition (1.8). Under assumption (1.8), the point $\omega(A)I$ is almost the centre of mass of $P_{\lambda_N}$. We now modify the random variables $X^A_N$ slightly to replace $\omega(A)I$ by the exact centre of mass:

**Definition 1.5.**

$$Y^A_N : ONB_N \rightarrow [0, + \infty), \quad \Psi = (U_{d_1}, U_{d_2}, \ldots)$$

$$Y^A_N(\Psi) := \| J_{d_N}(U_N^* D(\lambda) U_N) - \bar{D}(\lambda) \|^2 = \| J_{d_N}(U_N^* D_0(\lambda) U_N) \|^2. \quad (1.13)$$

In lemma 2.1, we will show that the ergodic properties of an orthonormal basis are equivalent to the statements in lemma 1.4 with $X^A_N$ replaced by $Y^A_N$.

The main step in the proof of theorem 1.3 is to determine the asymptotic mean $EY^A_N$ and variance

$$\text{Var}(Y^A_N) := E((Y^A_N)^2) - (E(Y^A_N))^2$$

of these random variables with respect to $d\nu$. For the proof of theorem 1.3, it suffices to prove that the mean tends to zero and the variance is bounded.
In lemma 3.1, we have given an exact formula for $E\|J_{d_n}(U^*D_0(\lambda)U)\|^2$ for all $A \in \Psi^0(M)$. From the formula, it is obvious that the mean is bounded, so that

$$E\left(\frac{1}{d_n}\|J_{d_n}(U^*D_0(\lambda)U)\|^2\right) \rightarrow 0 \quad \text{as long as } d_n \rightarrow \infty,$$

i.e. the mean of the quantum variances (1.2) tends to zero. In corollary 1.8 and lemma 1.7, we have given an exact formula for $\text{Var}(Y^A_N)$ which shows that it too is bounded. As in [1,3], we then apply the Kolmogorov SLLN (strong law of large numbers) to the sequence $\{Y^A_N\}$ of independent random variables. The SLLN implies that the partial sums

$$S_N := \sum_{n \leq N} \frac{1}{d_n}(Y^A_n - EY^A_n)$$

have the property

$$\frac{1}{N}S_N \rightarrow 0, \quad \text{hence } \frac{1}{N}\sum_{n \leq N} \frac{1}{d_n} Y^A_n \rightarrow 0 \quad \text{almost surely.} \quad (1.14)$$

Moreover, if $d_n$ grows at a sufficiently fast rate we obtain stronger results such as (1.4) from the Borel–Cantelli lemma: e.g. if $\sum_{n=1}^{\infty}(1/d_n) < \infty$, one obtains almost sure convergence $(1/d_n)Y^A_n \rightarrow 0$ (a.s.). By lemma 2.1, the same results then hold for $X^A_N$. By lemma 1.4, this will conclude the proof of theorem 1.3.

Thus, the main step is to compute the mean and variance of $Y^A_N$. To do so, we rewrite them in terms of moments of inertia of $P_{\lambda_N}$. The pushforward of the $U(d_N)$-invariant normalized measure on $O_{\lambda}$ to $P_{\lambda}$ is the so-called Duistermaat–Heckman measure $\nu_{\lambda_N}^{\text{DH}}$, a piecewise polynomial measure on $P_{\lambda}$. We denote the centre of mass of the polytope $P_{\lambda} \subset \mathbb{R}^N$ by

$$\mathcal{D}(\lambda) = \frac{1}{\text{Vol}(P_{\lambda})} \int_{P_{\lambda}} x \, d\nu_{\lambda_N}^{\text{DH}}.$$

By volume we mean the integral of the function 1 with respect to Duistermaat–Heckman measure, which is simply the volume of the orbit $O_{\lambda}$. It is normalized to equal to 1, but we include the volume normalization to emphasize that the measure on the polytope is a probability measure. We further denote the moments of the polytope by

$$m_{2k}(P_{\lambda}) = \frac{1}{\text{Vol}(P_{\lambda})} \int_{P_{\lambda}} |x - \mathcal{D}(\lambda)|^{2k} \, d\nu_{\lambda_N}^{\text{DH}}. \quad (1.15)$$

Unravelling the definitions gives

$$m_2(P_{\lambda_N}) := E\|J_{d_n}(U^*D_0(\lambda)U)\|^2 = EY^A_N$$

and

$$m_4(P_{\lambda_N}) := E\|J_{d_n}(U^*D_0(\lambda)U)\|^4 = \text{Var} Y^A_N.$$  \quad (1.16)

Thus, it suffices to show that for all sequences $T^A_N$ obtained from $A \in \Psi^0(M)$ with spectra $\{\lambda_N\}$, the second and fourth moments of inertia of $P_{\lambda}$ with respect to $d\nu_{\lambda_N}^{\text{DH}}$ are bounded. Note that the concentration of mass of these convex bodies in high-dimensional spaces is evidently quite different from that of isotropic convex bodies [6].

We asymptotically evaluate the moments of inertia of $P_{\lambda_N}$ using the Fourier transform

$$\hat{\mu}_\lambda(X) := \int_{O_{\lambda}} e^{i\langle X, \text{diag}(Y) \rangle} \, d\mu_\lambda(Y)$$

of the $\delta$-function on $O_{\lambda}$. Here, we assume $X \in \mathbb{R}^{d_N}$. We may identify $X$ with a diagonal matrix, so that $\langle X, \text{diag}(Y) \rangle = \text{Tr} XY$, so that $\hat{\mu}_\lambda(X)$ is the standard Fourier transform. We translate $\lambda$ by its centre of mass to make the centre of mass of $P_{\lambda}$ equal to 0, i.e. $\sum \lambda_j = 0$, i.e. we replace $\lambda$ by $A$ (1.10).

Then, differentiating under the integral sign gives
Lemma 1.6. Let $\Delta$ be the Euclidean Laplacian of $\mathbb{R}^{dN}$ acting in the $X$ variable. Then,

\[
\begin{align*}
m_2(P_A) &= - \Delta \hat{\mu}_A(X)|_{X=0} \\
m_4(P_A) &= \Delta^2 \hat{\mu}_A(X)|_{X=0}.
\end{align*}
\]

These formulae are useful because we can calculate $\hat{\mu}_A(X)$ in terms of Schur polynomials. The definition of the Schur polynomials is recalled in §1 and the calculation of $\hat{\mu}_A(X)$ in terms of Schur polynomials is recalled in lemma 2.3. It leads to the following:

Lemma 1.7. Let $p_k$ be power functions (1.7). Then for any $\lambda$ such that $p_1(\lambda) = 0$,

\[
\begin{align*}
\Delta \hat{\mu}_A(0) &= \frac{p_2(\lambda)}{d_N + 1}, \\
\Delta^2 \hat{\mu}_A(0) &= \beta_4(d_N) \cdot p_2^2(\lambda),
\end{align*}
\]

with $\beta_4(d_N) = \left( \frac{4d_N(d_N - 1)}{(d_N + 1)d_N^2(d_N - 1)} - \frac{4d_N(d_N - 1)}{(d_N + 2)(d_N + 1)d_N(d_N - 2)} + \frac{(12d_N^2 + 4d_N(d_N - 1))}{(d_N + 3)(d_N + 2)(d_N + 1)d_N} \right)$.

Combining (1.16) and lemmas 1.6 and 1.7, we find that the variances of $Y^A_N$ are bounded, and indeed have the asymptotics given in

Corollary 1.8. We have

\[
\text{Var}(Y^A_N) = \left( \beta_4(d_N) - \frac{1}{(d_N + 1)^2} \right) p_2^2(A_N) \simeq \frac{3}{d_N^2} p_2^2(A_N).
\]

Hence, $\text{Var}(Y^A_N) \leq 3\|A\|^2$.

Boundedness holds because $A \in \Psi^0(M)$ is a bounded operator, hence the numerator is bounded by $3\|A\|^2$ times the number $d_N$ of terms.

(b) More on limit shapes of $P_{\lambda_N}$

As mentioned above, the calculations of lemma 1.7 go beyond what is necessary for almost sure quantum ergodicity. They have their own intrinsic interest in the asymptotic geometry of the polytopes $P_{\lambda_N}$. This sequence of polytopes has nice asymptotic properties as long as empirical measures (1.6) tend to a weak limit $\nu$. It seems of interest to explore the limit shapes of any sequence of polytopes with this property. Independently of any connection to quantum ergodicity, the results of this article show:

Proposition 1.9. Let $\lambda_N \in \mathbb{R}^{dN}$ be a sequence of vectors with the property that empirical measures (1.6) tend to a weak limit $\nu$. Let $P_{\lambda_N}$ be the associated polytopes. Then,

\[
\begin{align*}
m_2(P_{\lambda_N}) &\to \int_{\mathbb{R}} (t - \tilde{\nu})^2 \, d\nu \\
m_4(P_{\lambda_N}) &\to 4 \left( \int_{\mathbb{R}} (t - \tilde{\nu})^2 \, d\nu \right)^2.
\end{align*}
\]

This proposition is closely related to the ‘Weingarten theorem’ that the matrix elements $\sqrt{d_N}U_{ij}$ are asymptotically complex normal random variables, where $U_{ij}$ are the matrix elements of $U \in \mathcal{U}(d_N)$ [7]. Perhaps this explains why the fourth moment is a constant multiple of the square of the second moment. It would be interesting to see whether the pattern continues to higher moments.
In the context of quantum ergodicity and pseudo-differential operators, one may ask when the sequence of polytopes associated with \(A \in \Psi^0(M)\) and the Hilbert spaces \(\mathcal{H}_N\) has the property in proposition 1.9. It is a much stronger condition than the one in definition 1.8.

**Definition 1.10.** We say that the sequence \(\{\mathcal{H}_N\}\) has Szegö asymptotics if, for all \(A \in \Psi^0(M)\), there exists a unique weak* limit, \(v_{\mathcal{H}_N} \to v_A \in \mathcal{M}(\mathbb{R})\) as \(N \to \infty\). Here, \(\mathcal{M}(\mathbb{R})\) is the set of probability measures on \(\mathbb{R}\).

It seems of some interest in spectral asymptotics to determine when the condition in this definition holds. For instance, we do not know if it holds for the sequence of eigenspaces of flat tori in dimensions \(\geq 5\).

(c) **Discussion**

The motivation for proving quantum ergodicity of random orthonormal bases for \(\mathcal{H}_N\) of any dimensions tending to infinity was prompted by the general question: how many diffuse states (modes or quasi-modes) does it take to synthesize localized modes or quasi-modes? Vice versa, how many localized states does it take to synthesize diffuse states? We would like to synthesize entire orthonormal bases rather than individual states and measure the dimensions of the space of states in terms of the Planck constant \(\hbar\). Let us consider some examples.

In the case of the standard \(S^2\), the eigenspaces \(\mathcal{H}_N\) of \(\Delta\) are the spaces of spherical harmonics of degree \(N\). They have the well-known highly localized basis \(Y^N_m\) of joint eigenfunctions of \(\Delta\) and of rotations around the \(x_3\)-axis. By localized we mean that a sequence \(\{Y^N_m\}\) with \(m/N \to \alpha\) microlocally concentrates on the invariant tori in \(S^*S^2\) where \(p_\theta = \alpha\). Here, \(p_\theta(x, \xi) = \xi(\partial/\partial \theta)\) where \(\partial/\partial \theta\) generates the \(x_3\)-axis rotations. On the other hand, it is proved in [1] that independent ‘random’ orthonormal bases of \(\mathcal{H}_N\) are quantum ergodic, i.e. are highly diffusive in \(S^*S^2\). As \(\dim \mathcal{H}_N = 2N + 1\), it is perhaps not surprising that the same eigenspace can have both highly localized and highly diffuse orthonormal bases when its dimension is so large. The question is, how large must it be for such incoherently related bases to exist?

A setting where the eigenvalues have high multiplicity but of a lower order of magnitude than on \(S^2\) is that of flat rational tori \(\mathbb{R}^n/L\), such as \(\mathbb{R}^n/\mathbb{Z}^n\). Of course it has an orthonormal basis of localized eigenfunctions, \(\alpha^{(k,x)}\). The key feature of such rational tori is the high multiplicity of eigenvalues of the Laplacian \(\Delta\) of the flat metric. It is well known and easy to see that the multiplicity is the number of lattice points of the dual lattice \(L^*\) lying on the surface of a Euclidean sphere. We denote the distinct multiple \(\Delta\)-eigenvalues by \(\mu_N\), the corresponding eigenspace by \(\mathcal{H}_N\) and the multiplicity of \(\mu^2_N\) by \(d_N = \dim \mathcal{H}_N\). In dimensions \(n \geq 5\), \(d_N \sim \mu^{-2}_N\), one degree lower than the maximum possible multiplicity of a \(\Delta\)-eigenvalue on any compact Riemannian manifold, achieved on the standard \(S^n\). Furthermore, \((1/d_N)\Tr \Pi_N A \Pi_N \to \omega(A)\). Hence, the results of this article show that despite the relatively slow growth of \(d_N\) on a flat rational torus, orthonormal bases of \(\mathcal{H}_N\) in dimensions \(\geq 5\) are almost surely quantum ergodic. The statement for dimensions 2, 3, 4 is more complicated (see §4d).

An interesting setting where the behaviour of eigenfunctions is largely unknown is that of Kolmogorov–Arnold–Moser (KAM) systems. For these, one may construct a ‘nearly’ complete and orthonormal basis for \(L^2(M)\) by highly localized quasi-modes associated with the Cantor set of invariant tori. It seems unlikely that the actual eigenfunctions are quantum ergodic; but the results of this article show that if they resemble random combinations of the quasi-modes, then it is possible that they are. Further discussion is in §4b.

2. **Background**

In this section, we review the definition of random orthonormal basis and relate it to properties of the moment map for the diagonal action of the maximal torus \(T_{d_N}\) on co-adjoint orbits of \(U(d_N)\).
(a) Random orthonormal bases of eigenspaces

Suppose that we have a sequence of Hilbert spaces \( \mathcal{H}_N \) of dimensions \( d_N = \dim \mathcal{H}_N \to \infty \). We define the large Hilbert space

\[
\mathcal{H} = \bigoplus_{N=1}^{\infty} \mathcal{H}_N
\]

and orthogonal projections

\[
\Pi_N : \mathcal{H} \to \mathcal{H}_N.
\]

We then consider orthonormal bases (1.5) of \( \mathcal{H} \) which arise from sequences of orthonormal bases of \( \mathcal{H}_N \).

(b) The basic random variables

Let \( A \in \Psi^0(M) \) be a zeroth-order pseudo-differential operator. By a Toeplitz operator, we mean the compression \( T^A_N \) of \( A \) to \( \mathcal{H}_N \).

Given one ONB \( \{ e^N_j \}_{j=1}^{d_N} \) of \( \mathcal{H}_N \), \( T^A_N \) can be identified with a Hermitian \( d_N \times d_N \) matrix.

Moreover, any other ONB \( \{ \psi^N_j \}_{j=1}^{d_N} \) is related to \( \{ e^N_j \}_{j=1}^{d_N} \) by a unitary matrix \( U_N \). We thus introduce the random variables on (1.5)

\[
A^N_j(\Psi) = \frac{1}{d_N} \sum_{j=1}^{d_N} \langle A \psi^N_j, \psi^N_j \rangle - \omega(A)
\]

where \( \Psi = \{ U_N \}, U_N \in U(d_N) \equiv OBN_N \). We also define

\[
\hat{A}^N_j(\Psi) = \frac{1}{d_N} \left( U^*_N T^A_N U_N e^N_j, e^N_j \right) - \frac{1}{d_N} \text{Tr} T^A_N
\]

Evidently,

\[
\frac{1}{d_N} Y^A_N(\Psi) = \frac{1}{d_N} \sum_{j=1}^{d_N} \hat{A}^N_j(\Psi) = \frac{1}{d_N} \sum_{j=1}^{d_N} A^N_j(\Psi) + o(1)
\]

(where the \( o(1) \) term is independent of \( \Psi \)). It follows that

\[
\sup_{\mathcal{O}_N B_N} \left| X^A_N - \frac{1}{d_N} Y^A_N(\Psi) \right| = o(1).
\]

Combining with lemma 1.4,

**Lemma 2.1 ([1,3]).** The ergodic property of an ONB \( \Psi (\mathcal{E}P) \) is equivalent to

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_n} Y^A_n(\Psi) = 0, \quad \forall A \in \Psi^0(M).
\]

Similarly, (1.4) is equivalent to \( \frac{1}{d_n} Y^A_n(\Psi) \to 0 \) almost surely.

As mentioned in the Introduction, it follows by a standard diagonal argument that almost all the individual elements \( \langle A \psi^N_j, \psi^N_j \rangle \) tend to \( \omega(A) \) for all \( A \). We do not discuss this step because it is nothing new.

(c) Moment map interpretation

In the case where the components of \( \lambda_N \) are distinct, the convex polytope \( \mathcal{P}_{\lambda_N} \) is the permutahedron determined by \( \lambda \), that is, the simple convex polytope defined as the convex hull
of the points $\{\sigma(\lambda_N)\}$, where $\sigma \in S_N$ runs over the symmetric group on $d_N$ letters (i.e. the Weyl group of $U(d_N)$). The centre of mass is the unique point $X \in \mathcal{P}_{\lambda_N}$ so that

$$\sum_{\sigma \in S_{d_N}} X\sigma(\lambda_N) = 0 \iff X = \frac{1}{(d_N)!} \sum_{\sigma \in S_{d_N}} \sigma(\lambda_N),$$

where $XY = X - Y$ is the vector from $X$ to $Y$. The centre of mass is evidently invariant under $S_{d_N}$, hence has the form $(a, a, \ldots, a)$ for some $a$ and clearly $a = (1/d_N) \sum_{j=1}^{d_N} \lambda_j$. In effect, we want to asymptotically calculate the moments of inertia of the sequence of permutahedra (figure 1) associated with a Toeplitz operator.

(d) Symmetric polynomials and Schur polynomials

The elementary symmetric polynomial of degree $k$ in $d$ variables is defined by

$$e_k(X_1, \ldots, X_d) = \sum_{i_1 < i_2 < \cdots < i_k \leq d} X_{i_1} \cdots X_{i_k}.$$

If one replaces $<$ by $\leq$ one obtains the complete symmetric polynomials $h_k$.

Remark 2.2. As we use the notation $d_N = \dim \mathcal{H}_N$, we use $d$ to denote the number of variables in general.

Schur polynomials $S_{\mu_1, \ldots, \mu_d}(X_1, \ldots, X_d)$ of degree $k$ in $d$ variables are symmetric polynomials parametrized by partitions $\mu = (\mu_1, \ldots, \mu_d)$ of the degree $k = \mu_1 + \mu_2 + \cdots + \mu_d$ into $d$ parts. They are defined by

$$S_{\mu} = \det(h_{\mu_i+j-i}) = \det(e_{\mu_i' + j-i}),$$

where $\mu'$ is a dual partition to $\mu$. For instance, $S_{(2,1,1)}(X_1, X_2, X_3) = X_1X_2X_3(X_1 + X_2 + X_3)$. We refer to [8] (or Wikipedia) for background.

(e) Fourier transform of the orbit

We now compute the moments of inertia using Fourier transform (1.17) of the orbital measure on the orbit of $D(\lambda)$.

An explicit formulae for $\hat{\mu}_2(X)$ is given in the first line of the proof of theorem 5.1 in Olshanski & Vershik:
Lemma 2.3. In dimension $d$,

$$
\hat{\mu}_p(X) = (d-1)! \cdots 0! \sum_{\mu: \ell(\mu) \leq d} \frac{S_{\mu}(X)S_{\mu}(i\lambda)}{(\mu_1 + d - 1)!((\mu_2 + d - 2)! \cdots \mu_d)!}.
$$

(2.7)

Here, $\ell(\mu)$ is the number of rows of the partition $\mu$.

As we would like to shift the centre of mass of $\mathcal{P}_\lambda$ to the origin, we mainly consider $\hat{\mu}_p(X)$ the Fourier transform of the traceless orbit (see (1.10)).

3. Proof of lemma 1.7: asymptotics of $m_2(\mathcal{P}_{\lambda_N})$ and $m_4(\mathcal{P}_{\lambda_N})$

(a) Asymptotics of $m_2(\mathcal{P}_{\lambda_N})$

We now prove

**Lemma 3.1 ([1–3]).** Let $\lambda = (\lambda_1, \ldots, \lambda_{d_N}) \in \mathbb{R}^{d_N}$, and let $D_0(\lambda_N)$ denote the trace zero diagonal matrix with entries (1.10). Thus, $p_1(\Lambda) = 0$ (1.10). Then,

$$
\mathbb{E} Y_N^2 = m_2(\mathcal{P}_{\Lambda_N}) = \int_{U(d)} \|J_{d_N}(U^*D_0(\lambda)U)\|^2 dU = \frac{p_2(\Lambda)}{d_N + 1},
$$

(3.1)

whereas above, $dU$ is the normalized Haar probability measure on $U(d_N)$.

This lemma was proved in [1–3] using the so-called Itzykson–Zuber–Harish–Chandra formula for the Fourier transform of the orbit, and again using Gaussian integrals. The proof we give here generalizes better to higher moments. We also sketch a proof using the Weingarten formulae.

**Proof.** We use lemma 2.3 to obtain

$$
\mathbb{E}\|J_{d_N}(U^*D(\lambda)U)\|^2 = (d_N - 1)! \cdots 0! \sum_{\mu: \mu|\mu| \leq 2, \ell(\mu) \leq d_N} \Delta S_{\mu}(0)S_{\mu}(i\lambda)(\mu_1 + d_N - 1)!((\mu_2 + d_N - 2)! \cdots \mu_{d_N})!.
$$

We sum over the Young diagrams with exactly two boxes and $\leq d_N$ rows. There are just two of them: one row of two boxes or two rows of one box each corresponding, respectively, to the Schur functions $S_{(2,0)}$, $S_{(1,1)}$. Note that $S_{1k} = e_k$ is the $k$th elementary symmetric function and $S_{(k)} = h_k$ is the complete $k$th degree symmetric function.

We then translate $\lambda$ to $\Lambda$ so that $\sum_j \ell_j = 1\ell(\Lambda) = 0$, i.e. we replace $D(\lambda_N)$ by $D_0(\lambda_N)$.

As the degree $|\mu| = 2$, the only partitions are $\mu = (2)$, (11) and the associated Schur polynomials are

$$
S_{(1,1)} = e_2 = \sum_{i<j} x_i x_j \quad \text{and} \quad S_{(2,0)} = e_1^2 - e_2 = \sum_j x_j^2 + \sum_{i<j} x_i x_j.
$$

But

$$
\Delta e_2 \equiv 0 \quad \text{and} \quad \Delta (e_1^2 - e_2) = 2\|\nabla e_1\|^2 = 2d_N.
$$

For each monomial $x_i x_j$, we have $\Delta x_i x_j = 2\delta_{ij}$. Thus, $\Delta S_{(1,1)} = 0$ and $\Delta S_{(2,0)} = 2d_N$. As the Schur polynomials are homogeneous of degree 2, we can remove the $i$ under the Schur polynomials to get an overall factor of $-1$, which is cancelled by the $-\$ sign from $\Delta$. Thus,

$$
\mathbb{E}\|J_{d_N}(U^*D_0(\lambda)U)\|^2 = (2d_N)(d_N - 1)! \frac{S_{(2,0)}(i\Lambda)}{(d_N + 1)!} = \frac{(2d_N)(d_N - 1)!}{(d_N + 1)!} \frac{S_{(2,0)}(i\Lambda)}{d_N + 1}.
$$

Thus,

$$
\mathbb{E}\|J_{d_N}(U^*D_0(\lambda)U)\|^2 = \frac{2}{d_N + 1} S_{(2,0)}(\Lambda_N) = \frac{2}{d_N + 1} (e_1^2 - e_2)(\Lambda).
$$

As $e_1(\Lambda_N) = 0$, we find that

$$
\mathbb{E}\|J_{d_N}(U^*D_0(\lambda)U)\|^2 = -\frac{2}{d_N + 1} e_2(\Lambda) = \frac{1}{d_N + 1} p_2(\Lambda_N).
$$
Here, we use that  
\[ e_1 = p_1 \quad \text{and} \quad 2e_2 = e_1p_1 - p_2. \]
The formula agrees with the one stated in lemma 3.1.

(b) Weingarten formulae for the expectation

As the second proof, we use the Weingarten formula for integrals of polynomials over \( U(N) \) [7,10]. As before, we denote the eigenvalues of \( D_0(\lambda) \) by \( \Lambda \). Then,

\[
\|\text{diag}(U^* D_0(\lambda) U)\|^2 = \sum_{j_1,j_2} A_{j_1} A_{j_2} \sum_i |U_{ij_1}|^2 |U_{ij_2}|^2.
\]

The Weingarten formulae for these special polynomials state that asymptotically \( \sqrt{d_N} |U_{ij}|^2 \) is a complex Gaussian random variable of mean zero and variance one. Thus, to leading order,

\[
\int_{U(d_N)} |U_{ij}|^2 |U_{ij}|^2 \, dU \simeq d_N^{-2}(1 + \delta_{ij})
\]

and

\[
\sum_{j_1,j_2} A_{j_1} A_{j_2} \int_{U(d_N)} |U_{ij_1}|^2 |U_{ij_2}|^2 \, dU \simeq d_N^{-1}\left(2 \sum_j A_j^2 + \sum_{j \neq j_1} A_j A_{j_1}\right). \tag{3.2}
\]

As

\[
0 = \left(\sum A_j\right)^2 = \sum A_j^2 + \sum_{j \neq k} A_j A_k,
\]

we get

\[
(3.2) \simeq d_N^{-1} \sum_j A_j^2.
\]

(c) Proof of lemma 1.7 and corollary 1.8: variance and fourth moment asymptotics

To prove the formula for the fourth moment \( m_4(\mathcal{P}_{\lambda_0}) \) in lemma 1.7, we use the expression in lemma 1.6 in terms of \( \hat{\mu}_k \), and then use the formula of lemma 2.3.

When the degree \( k = 4 \) and the number of variables \( d \geq 4 \) there are five partitions of the degree into \( d \) parts (most zero), i.e.

\[
4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,
\]

and the corresponding Schur polynomials are

\[
S_{1,1,1,1}(X_1, \ldots, X_d) = e_4 = \sum_{1 \leq i < j < k < \ell} X_i X_j X_k X_\ell;
\]

\[
S_{2,1,1}(X_1, \ldots, X_d) = e_1 e_3,
\]

\[
S_{2,2,0} = e_2^2 - e_1 e_3,
\]

\[
S_{4,0,0} = e_4^2 - 3e_2^2 e_2^2 + 2e_1 e_3 + e_2^2
\]

and

\[
S_{3,1,0} = e_3^2 e_2^2 - e_2^2 - e_1 e_3.
\]

We note that \( \Delta e_k(X) = 0 \) for all \( k \), so \( \Delta e_k u_n = 2 \nabla e_k \cdot \nabla u_n \). Also, \( \nabla e_1 \) is a constant vector. So, \( \Delta^2 e_1 e_3 = \nabla e_1 \cdot \nabla e_3 e_3 = 0 \) and

\[
\Delta^2 e_1^2 e_2 = 4 \Delta (e_1 \nabla e_1 \cdot \nabla e_2) = 8 \nabla e_1 \cdot \nabla (\nabla e_1 \cdot \nabla e_2) = 8 \text{Tr Hess}(e_2) = 0.
\]

Here, Hess denotes the Hessian. We also use that \( \Delta(\nabla f \cdot \nabla g) = 2 \text{Hess}(f) \cdot \text{Hess}(g) \) when \( \Delta f = \Delta g = 0 \). Also,

\[
\nabla e_1 \cdot \nabla (\nabla e_1 \cdot \nabla e_2) = (1, 1, \ldots, 1) \cdot \sum_{j,k} \frac{\partial^2 e_2}{\partial x_j \partial x_k} \frac{\partial}{\partial x_k} = \text{Tr Hess}(e_2) = 0.
\]
Then,
\[ \Delta^2 e_2^1 = 2\Delta(\nabla e_1 \cdot \nabla e_2) = 4\|\text{Hess}(e_2)\|^2 = 4d_N(d_N - 1). \]

Furthermore, \( \Delta^2 e_1^1 = 2\nabla e_1 \cdot \nabla e_1 = 2d_N \), so that
\[ \Delta^2 e_1^1 = \Delta(2(\Delta t_1^1) e_1^1 + 2\nabla e_1^1 \cdot \nabla e_1^1) = \Delta(4d_N e_1^1 + 2e_1^2 d_N) = 12d_N. \]

We recall Newton’s identities
\[
\begin{align*}
&\begin{align*}
e_1 &= p_1 \\
2e_2 &= e_1 p_1 - p_2 \\
3e_3 &= e_2 p_1 - e_1 p_2 + p_3 \\
4e_4 &= e_3 p_1 - e_2 p_2 + e_1 p_3 - p_4.
\end{align*}
\end{align*}
\]

We note that \( \Delta e_k \equiv 0 \) for all \( k \), so at \( X = 0 \),
\[
\begin{align*}
\Delta^2 S_{1,1,1,1} &= \Delta e_4 \equiv 0; \\
\Delta S_{2,1,1} &= \Delta^2 e_1 e_3 = 2\Delta(\nabla e_1 \cdot \nabla e_3) = \nabla e_1 \cdot \nabla \Delta e_3 = 0; \\
\Delta^2 S_{2,2,0} &= \Delta^2 e_2^2 = 2\Delta(\nabla e_2 \cdot \nabla e_2) = 4\|\text{Hess}(e_2)\|^2 = 4d_N(d_N - 1) \\
\Delta S_{4,0,0} &= \Delta^2 e_4^1 - (3)8\nabla e_1 \cdot (\nabla e_1 \cdot \nabla e_2) + 4\|\text{Hess}(e_2)\|^2 \\
&= 12d_N^2 + 4d_N(d_N - 1)
\end{align*}
\]
and
\[
\begin{align*}
\Delta S_{3,1,0} &= 8\nabla e_1 \cdot (\nabla e_1 \cdot \nabla e_2) - \|\text{Hess}(e_2)\|^2 = -4\|\text{Hess}(e_2)\|^2 \\
&= -4d_N(d_N - 1).
\end{align*}
\]

By routine calculations and lemma 2.3, we have
\[
\begin{align*}
\Delta^2 \bar{\mu}_A(0) &= (d_N - 1)! \cdots 0! \sum_{\mu : |\mu| = 4} \frac{\Delta^2 S_\mu (0) S_\mu(i\mathbf{A})}{(\mu_1 + d_N - 1)! (\mu_2 + d_N - 2)! \cdots \mu_d!} \\
&= (d_N - 1)!(d_N - 2)! \frac{\Delta^2 S_{2,2,0}(0) S_{2,2,0}(i\mathbf{A})}{(d_N + 1)!(d_N)!} \\
&\quad + (d_N - 1)! \frac{\Delta^2 S_{4,0,0}(0) S_{4,0,0}(i\mathbf{A})}{(d_N + 3)!} \\
&\quad + (d_N - 1)!(d_N - 2)! \frac{\Delta^2 S_{3,1,0}(0) S_{3,1,0}(i\mathbf{A})}{(d_N + 2)!(d_N - 1)!} \\
&= \frac{\Delta^2 S_{2,2,0}(0) S_{2,2,0}(i\mathbf{A})}{(d_N + 1)d_N^2(d_N - 1)} \\
&\quad + \frac{\Delta^2 S_{4,0,0}(0) S_{4,0,0}(i\mathbf{A})}{(d_N + 3)(d_N + 2)(d_N + 1)d_N} + \frac{\Delta^2 S_{3,1,0}(0) S_{3,1,0}(i\mathbf{A})}{(d_N + 2)(d_N + 1)d_N(d_N - 2)}. \\
\end{align*}
\]

By (3.4), we then have
\[
\begin{align*}
\Delta^2 \bar{\mu}_A(0) &= \frac{4d_N(d_N - 1) S_{2,2,0}(i\mathbf{A})}{(d_N + 1)d_N^2(d_N - 1)} + \frac{(12d_N^2 + 2d_N(d_N - 1)) S_{4,0,0}(i\mathbf{A})}{(d_N + 3)(d_N + 2)(d_N + 1)d_N} \\
&\quad + \frac{-4d_N(d_N - 1) S_{3,1,0}(i\mathbf{A})}{(d_N + 2)(d_N + 1)d_N(d_N - 2)}. \\
\end{align*}
\]
Recalling (3.3) and that \( e_1(A) = 0 \), we get
\[
\Delta^2 \hat{\mu}_A(0) = \frac{4d_N(d_N - 1)p_2^2(iA)}{(d_N + 1)d_N^2(d_N - 1)} + \frac{(12d_N^2 + 4d_N(d_N - 1))p_2^2(iA)}{(d_N + 3)(d_N + 2)(d_N + 1)d_N} \\
+ \frac{-4d_N(d_N - 1)p_2^2(iA)}{(d_N + 2)(d_N + 1)d_N(d_N - 2)}. 
\] (3.7)

Further recalling that \( 2e_2 = e_1p_1 - p_2 \), we finally get
\[
\Delta^2 \hat{\mu}_A(0) = \frac{d_N(d_N - 1)p_2^2(iA)}{(d_N + 1)d_N^2(d_N - 1)} + \frac{(3d_N^2 + d_N(d_N - 1))p_2^2(iA)}{(d_N + 3)(d_N + 2)(d_N + 1)d_N} \\
+ \frac{-d_N(d_N - 1)p_2^2(iA)}{(d_N + 2)(d_N + 1)d_N(d_N - 2)}. 
\] (3.8)

As the polynomials are homogeneous of degree 4, the factor of \( i \) inside the polynomials may be removed and we get
\[
\mathbb{E}\|J_{d_N}(U^* D_0(\lambda) U)\|^4 = \frac{d_N(d_N - 1)p_2^2(A)}{(d_N + 1)d_N^2(d_N - 1)} + \frac{(3d_N^2 + d_N(d_N - 1))p_2^2(A)}{(d_N + 3)(d_N + 2)(d_N + 1)d_N} \\
+ \frac{-d_N(d_N - 1)p_2^2(A)}{(d_N + 2)(d_N + 1)d_N(d_N - 2)}. 
\] (3.9)

As \( N \to \infty \) the leading asymptotics of the outer terms cancel and the middle term is asymptotic to \( (4/d_N^2)p_2^2(A_N) \). We note that \( p_2(A_N)/d_N \) is bounded. If the empirical measure of eigenvalues tends to a limit measure, then \( p_2(A_N)/d_N \) tends to its second moment.

Together with lemma 3.1, this completes the proof of lemma 1.7. Corollary 1.8 follows by subtracting the square of the expectation.

Following the outline in the Introduction and combining with lemma (2.1), this completes the proof of theorem 1.3.

4. Applications

(a) Flat tori

Theorem 1.3 applies to eigenspaces of the Laplacian on the flat torus \( \mathbb{R}^d/\mathbb{Z}^d \) (or other rational lattices) of dimension \( \geq 5 \) and for many eigenspaces in dimensions \( d = 2, 3, 4 \).

Proposition 4.1. Random orthonormal bases of \( \Delta \)-eigenspaces of the flat torus \( \mathbb{R}^d/\mathbb{Z}^d \) are quantum ergodic for \( d \geq 5 \). Also for \( d = 2, 3, 4 \) for special eigenspaces (specified below).

The only condition on the eigenspaces for theorem 1.3 is that (1.8) holds, and we now recall the known results on this problem. Given \( A \in \Psi^0 \), we denote the eigenspaces on a flat torus, enumerated of the order of the eigenvalue by \( \mathcal{H}_N \) and by \( \Pi_N \) the orthogonal projection to \( \mathcal{H}_N \).

Lemma 4.2. Condition (1.8) is valid in dimensions \( \geq 5 \) on \( \mathbb{R}^d/\mathbb{Z}^d \). That is,
\[
\frac{1}{d_N} \text{Tr} A \Pi_N \sim \int_{S^T \mathbb{R}^d} a(x, \omega) \, dx \, d\omega.
\]

It follows that \( (1/d_N) Y_{N}^{2} \to 0 \) almost surely.

In dimensions 2, resp. 3, resp. 4 there are restrictions on the sequence of eigenvalues given in [11], resp. [12], resp. [13]. For eigenvalues in the allowed sequences, (1.8) is valid.

\(^1\)Thanks to Z. Rudnick for explanations and references.
Proof. We use the basis \( e_{N,k} = e^{i(k,x)} \) with \( |k| = \mu_N \). Then,
\[
\langle Ae_{N,k}, e_{N,k} \rangle = \int_{\mathbb{R}^n / \mathbb{Z}^n} \sigma_A(x,k) \, dx.
\]
Hence,
\[
\frac{1}{d_N} \text{Tr} \Pi_N A = \sum_{k:|k| = \mu_N} \int_{\mathbb{R}^n / \mathbb{Z}^n} \sigma_A(x,k) \, dx.
\]
In dimensions \( n \geq 5 \), \( d_N \sim \mu_N^{n-2} \). It is proved that lattice points of fixed norm on a sphere of radius \( \sqrt{n} \) become uniformly distributed as \( n \to \infty \) [13]. It follows from the spectral theorem that for any \( A(1/d_N)X_N^4 \) is summable when \( n \geq 5 \).

The Liouville limit formula is true in dimension 4 when the number of lattice points grows linearly in \( n \). The condition on \( n \) is given in [13]. In dimension 3, the equidistribution result is proved in [12] with similar conditions on the sequence of integers \( n \).

Dimension 2 is more complicated. In dimension 2, the eigenvalues of integers \( n \) for which there exist lattice points \( (a,b) \) on the circle \( a^2 + b^2 = n \). It is necessary that all prime factors of \( n \) are congruent to 1 modulo 4. In [11], it is shown that for almost all such \( n \), the lattice points on the circle become uniformly distributed as \( n \to \infty \).

Remark 4.3. In the case of a generic lattice \( L \subset \mathbb{R}^d \), the multiplicity of eigenvalues of \( \Delta \) on \( \mathbb{R}^d / L \) is two. The analogue of the eigenspaces above are spectral subspaces for \( \sqrt{\Delta} \) of shrinking width \( \omega \). Thus, one considers the exponentials \( e^{i(\ell,x)} \) for \( \ell \in L \) with \( |\ell| \leq \sqrt{\lambda} - C\omega, \lambda + \omega \). It follows from the lattice point results of [14] that in dimensions \( d \geq 5 \), the number of eigenvalues of an irrational flat torus in \( [\lambda, \lambda + O(\lambda^{-1})] \) is of order \( \lambda^{d-2} \). The question whether trace asymptotics (1.8) hold for the span of the corresponding eigenfunctions does not appear to have been studied.

(b) Quasi-modes

Theorem 1.3 is not restricted to eigenspaces of the Laplacian and is equally valid for spaces of quasi-modes. We refer to [15,16] for background on quasi-modes. Following [16], we define a \( C^\infty \) quasi-mode of infinite order for \( h^2 \Delta \) with index set \( M_h \) to be a family
\[
Q = \{ (\psi_m(\cdot,h), \mu_m(h)) : m \in M_h \}
\]
of approximate eigenfunctions satisfying
\[
\begin{align*}
(\text{i}) \quad & \| (h^2 \Delta - \mu_m(h)) \psi_m(\cdot,h) \|_{H^r} = O_M(h^M), \quad (\forall M \in \mathbb{Z}^+) \\
(\text{ii}) \quad & |\langle \psi_m, \psi_n \rangle - \delta_{mn}| = O_M(h^M), \quad (\forall M \in \mathbb{Z}^+).
\end{align*}
\]
(4.1)

It follows by the spectral theorem that for any \( M \in \mathbb{Z}^+ \), there exists at least one eigenvalue of \( h^2 \Delta \) in the interval
\[
\begin{align*}
\begin{cases}
(1) \quad & \mu_m(h) \in [\mu_m(h) - \delta_{mn}, \mu_m(h) + \delta_{mn}] \\
(2) \quad & \| E_{\mu_m} \psi_k - \psi_k \|_{H^r} = O_M(h^M).
\end{cases}
\end{align*}
\]
(4.2)

Here, \( E_I \) denotes the spectral projection for \( h^2 \Delta \) corresponding to the interval \( I \). We denote the quasi-classical eigenvalue spectrum of \( h \sqrt{\Delta} \) by
\[
QSp_h = \{ \mu_m(h) : m \in M_h \}.
\]
As quasi-eigenvalues \( \mu_m(h) \) are only defined up to errors of order \( h^\infty \), there is a notion of ‘multiple quasi-eigenvalue’ defined as follows: we say \( \mu_m(h) \sim \mu_n(h) \) if \( \mu_m - \mu_n = O(h^\infty) \) and define the multiplicity of \( \mu_m(h) \) by

\[
\text{mult}(\mu_m(h)) = \# \{ n : \mu_m(h) \sim \mu_n(h) \} = \dim \text{Span}\{ \psi_n(\cdot, h) : (h^2 \Delta - \mu_m(h))\psi_n = O(h^\infty) \}.
\]

We then introduce slightly larger intervals \( I_{m,h} \) (if need be) so that

\[
QSp(h) \subset \bigcup_{m \in M'} I_{m,h} \quad \text{and} \quad I_{m,h} \cap I_{n,h} = \emptyset \quad (m \neq n).
\]

Here, \( M' \) consists of equivalence classes of indices (corresponding to equivalence classes of quasi-modes). We denote by \( H^h_m \) the span of the quasi-modes \( \{ \psi_m(\cdot, h) : \mu_m(h) \in I_{m,h} \} \). Then,

\[
\| E_{I_{m,h}} v - v \| = O(h^\infty), \quad \text{if} \ v \in H^h_m.
\]

Theorem 1.3 applies to quasi-mode spaces \( H^h_m \) as long as their dimensions tend to infinity and as long as there exists a unique limit state for \( (1/\dim H^h_m) \text{Tr} A|_{H^h_m} \). One might expect true modes (eigenfunctions) with eigenvalues in the intervals \( I_{m,M} \) to be close to linear combinations of the quasi-modes with quasi-eigenvalues in that interval. The question raised by theorem 1.3 is whether they behave like random linear combinations or not. If they do, theorem 1.3 gives their quantum limits.

In particular, this bears on the question whether \( \Delta \)-eigenfunctions of compact Riemannian manifolds \((M, g)\) with KAM geodesic flow might be quantum ergodic. It seems unlikely that they are, but we are not aware of a proof that they are not. For such KAM \((M, g)\), a large family of quasi-modes is constructed in [15,16] which localizes on the invariant tori of the KAM Cantor set of tori. Without reviewing the results in detail, the ‘large’ family has positive spectral density, i.e. the number of quasi-eigenvalues \( \leq \mu \) grows like a positive constant times \( \mu^n \) where \( n = \dim M \).

To our knowledge, the multiplicities and trace asymptotics for KAM quasi-modes have not been studied at this time. As in the discussion of flat tori, one would need to determine the equidistribution law of the tori in the invariant Cantor set corresponding to eigenvalues (or pseudo-eigenvalues) of \( \sqrt{\Delta} \) in very short intervals \( I_\lambda = [\lambda - w, \lambda + w] \). The orthonormal basis of eigenfunctions is not simple to relate to the near orthonormal basis of quasi-modes in this case, but we might expect that a positive density of the eigenfunctions are mainly given as linear combinations of KAM quasi-modes with quasi-eigenvalues very close to the true eigenvalues. Whether or not they are quantum ergodic would reflect the extent to which they are sufficiently random combinations of quasi-modes and the extent to which the collection of quasi-modes in \( H^h_{M,m} \) is Liouville distributed.

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\textbf{References}


