Remarks on nodal volume statistics for regular and chaotic wave functions in various dimensions

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We discuss the statistical properties of the volume of the nodal set of wave functions for two paradigmatic model systems which we consider in arbitrary dimension \( s \geq 2 \): the cuboid as a paradigm for a regular shape with separable wave functions and planar random waves as an established model for chaotic wave functions in irregular shapes. We give explicit results for the mean and variance of the nodal volume in the arbitrary dimension, and for their limiting distribution. For the mean nodal volume, we calculate the effect of the boundary of the cuboid where Dirichlet boundary conditions reduce the nodal volume compared with the bulk. Boundary effects for chaotic wave functions are calculated using random waves which satisfy a Dirichlet boundary condition on a hyperplane. We put forward several conjectures on what properties of cuboids generalize to general regular shapes with separable wave functions and what properties of random waves can be expected for general irregular shapes. These universal features clearly distinguish between the two cases.

1. Introduction

We consider real square-integrable eigenfunctions \( \Phi(q) \) of the free stationary Schrödinger (or Helmholtz) equation

\[
- \Delta_M \Phi(q) = E \Phi(q)
\]

on an \( s \)-dimensional smooth connected compact Riemannian manifold \( M \) with local coordinates \( q \equiv (q^1, \ldots, q^s) \). Here, \( \Delta_M \) is the Laplace–Beltrami operator on \( M \) and \( E \) is an energy eigenvalue. We have set the value of the physical constant \( \hbar^2/2m \) of
Planck’s constant squared over twice the mass of the particle equal to 1 by appropriate choice of units.

If $\mathcal{M}$ has a boundary we will impose Dirichlet boundary conditions.

Compactness ensures a discrete and non-negative energy spectrum that we arrange in an ascending order as $0 \leq E_1 < E_2 \leq \cdots \leq E_N \leq E_{N+1} \leq \cdots$. The eigenfunction with eigenvalue $E_N$ will be denoted by $\Phi_N(q)$ (for degenerate eigenvalues this requires a choice of eigenbasis and order).

For a given eigenfunction $\Phi_N(q)$, the nodal set

$$\mathcal{N}[\Phi_N(q)] = \Phi_N^{-1}(0) \setminus \partial \mathcal{M} \subset \mathcal{M}$$

(1.2)

consists of all interior points on the manifold where the eigenfunction vanishes. For the real wave functions that are under consideration here the nodal set is a collection of hypersurfaces.

More than 200 years ago, Chladni [1] visualized the vibration modes of plates with sand that accumulates along nodal lines. He analysed in detail the geometric patterns formed by the nodal lines. About 30 years later Sturm’s oscillation theorem [2,3] that the $n$th eigenfunction of a Sturm–Liouville differential operator has $n-1$ nodal points may have been the first rigorous mathematical result concerning the nodal set of wave functions. Since this time the nodal set for wave functions of various types has attracted the attention of many mathematicians and physicists—and many seminal results followed.

In the present work we will focus on the size of the nodal set which we measure by its hypersurface volume. We denote the hypersurface volume of the nodal set of the $N$th eigenfunction by $\mathcal{H}_N$—we will refer to it just as the nodal volume.

With increasing energy, the wavelength becomes smaller and so does the typical distance between two nearby nodal surfaces. One thus expects the typical $\mathcal{H}_N$ increase with the energy $E_N$. Comparison of nodal volumes between eigenfunctions at very different energies or of eigenfunctions on different manifolds is possible through the dimensionless rescaled nodal volume

$$\sigma_N = \frac{\mathcal{H}_N}{\sqrt{V} \sqrt{E_N}}$$

(1.3)

where $V$ is the volume of the manifold $\mathcal{M}$. Note that the rescaled nodal volume is only defined for positive energies $E_N > 0$. In manifolds without boundary this excludes the ground state with energy $E_1 = 0$ which has a vanishing nodal set and thus $\mathcal{H}_1 = 0$.

In the mathematical literature, the nodal volume has been a central object and the main results are best summarized by Yau’s conjecture [4], which states that the rescaled nodal volumes are bounded from below and above for all smooth compact manifolds. That is,

$$c_1 \leq \sigma_N \leq c_2$$

(1.4)

for all $N \geq 2$, where the constants $c_1$ and $c_2$ depend only on the manifold and the metric. For real analytic manifolds, this is a classic theorem by Donnelly & Fefferman [5]. For the smooth case lower bounds have been established [6–8]. The general proof of Yau’s conjecture remains a central problem in spectral theory—we refer to the recent survey by Zelditch for a more complete overview and additional results [9].

In this work, we will consider how the rescaled nodal volume for a given manifold is distributed statistically. Such a statistical approach is well established for the number of nodal domains [10,11], which revealed that shapes with a chaotic ray dynamics have a universal distribution that can clearly be distinguished from distributions for shapes with a separable Laplacian and thus integrable ray dynamics. For separable shapes, these distributions can often be calculated explicitly [10,12] and share some universal features. Here, we will consider nodal volumes for two models:

— The cuboid. This is a paradigm of a regular shape with separable Laplacian and integrable ray dynamics.
The boundary-adapted planar random-wave model [13]. In two dimensions, statistical properties of nodal volumes and nodal densities have been discussed in detail for various types of random superposition of eigenstates [14–20]. The boundary-adapted random-wave model is an extension of the standard Gaussian random-wave model introduced by Berry [21], who conjectured that eigenfunction statistics for chaotic systems follow the predictions of the Gaussian random-wave model in a semiclassical limit. The boundary-adapted random-wave model is able to predict systematic corrections near the boundary—for the two-dimensional case the effect of a boundary on nodal densities has been discussed in detail [13–15]. We will add some results for higher dimensions and discuss implications for manifolds with chaotic ray dynamics.

In §2, we will give a full derivation of limiting distributions for the nodal volume based on Poisson summation. Moreover, we will consider boundary corrections to the mean of the nodal volume. In §3, we will summarize Berry’s results on the nodal volumes in boundary-adapted random waves and derive some extensions. The main result of this section is formulated as a conjecture on the finite energy correction to the mean nodal volume in an irregular shape for arbitrary dimension. Implications of our findings for more general regular and irregular shapes will be discussed further in §4.

2. The regular case: nodal volume statistics for an $s$-dimensional cuboid

Let $a_\ell$ ($\ell = 1, \ldots, s$) be the side lengths of an $s$-dimensional cuboid with volume $V = \prod_{\ell=1}^{s} a_\ell$. Separation of variables leads to a unique basis of (normalized) eigenfunctions

$$\phi_n(q) = \frac{(2\pi)^{s/2}}{V^{1/2}} \prod_{\ell=1}^{s} \sin \left( \frac{\pi n_\ell q_\ell}{a_\ell} \right),$$

(2.1)

which are labelled by $s$ positive integers $n_\ell$ ($\ell = 1, \ldots, s$). The corresponding energies are

$$E_n = \pi^2 \sum_{\ell=1}^{s} \frac{n_\ell^2}{a_\ell^2},$$

(2.2)

and the rescaled nodal volumes are given by

$$\sigma_n = \frac{1}{\sqrt{E_n}} \sum_{\ell=1}^{s} \frac{n_\ell - 1}{a_\ell}.$$  

(2.3)

If the side lengths are rationally dependent, then there are degeneracies in the spectrum. While the corresponding eigenspaces will always be spanned by the functions of the form $\phi_n(q)$ a generic eigenfunction for a degenerate energy eigenvalue will not be of this form. In this work, we will restrict our attention to the basis $\{\phi_n(q)\}$ as this eigenbasis is singled out by separability even in the presence of degeneracies.

Our first goal will be to obtain the asymptotic mean value of $\sigma_n$ in a spectral interval $E_n \in [E, E + \Delta E]$ of width $\Delta E$ near the energy $E$

$$\langle \sigma_n \rangle_{[E,E+\Delta E]} = \frac{1}{N_{[E,E+\Delta E]}} \sum_{n \in \mathbb{N}^s} \sigma_n \chi_{[E,E+\Delta E]}(E_n).$$

(2.4)

Here, $N_{[E,E+\Delta E]}$ is the number of eigenfunctions with energies in the interval $E_n \in [E, E + \Delta E]$ and $\chi_{[E,E+\Delta E]}(\alpha)$ is the characteristic function of this interval.

We will be interested in the asymptotic behaviour as $E \to \infty$ such that $N_{[E,E+\Delta E]} \to \infty$ at the same time. The choice $\Delta E = gE^{1/4}$ for some constant $g > 0$ satisfies the above requirement. We will
see that this interval is sufficiently small so that all leading asymptotic contributions to spectral averages will not depend explicitly on \( g \)—and the same is true for all correction terms that we will present explicitly here. The standard tool for extracting asymptotic behaviour in this setting is Poisson’s summation formula. We will apply it in the form

\[
\sum_{n=1}^{\infty} f(n) = \int_{0}^{\infty} f(n) \, dn - \frac{1}{2} f(0) + 2 \sum_{M=1}^{\infty} \int_{0}^{\infty} f(n) \cos(2\pi Mn) \, dn,
\]

which is valid for sufficiently well-behaved functions \( f(n) \) (such that all sums and integrals converge). Poisson summation allows us to find the asymptotic behaviour of \( N_{E,E+\Delta E} = N(E + \Delta E) - N(E) \), where

\[
N(E) = \sum_{n \in \mathbb{N}^s} \theta(E - E_n) = N_{\text{Weyl}}(E) + N_{\text{osc}}(E)
\]

is the spectral counting function and \( \theta(x) \equiv \chi([x, \infty)) \) is the Heaviside function. In the second line, we have written \( N(E) \) as a sum of a smooth monotonically increasing function \( N_{\text{Weyl}}(E) \) and an oscillating function \( N_{\text{osc}}(E) \). Application of the Poisson summation formula reveals that \( N(E) \) is asymptotically dominated by the smooth part

\[
N_{\text{Weyl}}(E) = \frac{\xi_s V}{2^s \pi^s} E^{s/2} - \frac{\xi_{s-1} S}{2^{s+1} \pi^{s-1}} E^{(s-1)/2} + O(E^{(s-2)/2}),
\]

which is known as Weyl’s law. Here, \( \xi_s = \pi^{s/2}/\Gamma(s/2 + 1) \) is the volume of the \( s \)-dimensional unit ball and \( S = 2V \sum_{\ell=1}^{s} 1/\ell_0 \) is the \( (s-1) \)-dimensional volume of the surface of the cuboid.

The first term in (2.8) gives the leading growth of the number of states with increasing energy. The second term is the leading correction—a boundary effect as can be seen from the appearance of the surface volume. Each oscillating contribution is of the order of \( O(E^{(s-1)/4}) \) and thus asymptotically smaller than the boundary correction.

Weyl’s law implies the estimate

\[
N_{E,E+\Delta E} = \frac{\xi_s V}{2^s \pi^s} E^{s/2} \frac{\Delta E}{2E} \left( 1 - \frac{(s-1)\pi \xi_{s-1} S}{2s\xi_s V} E^{-1/2} + O(E^{-3/4}) \right),
\]

which includes the effect of the boundary in the leading correction. Here, \( \Delta E = gE^{1/4} \) has been used to give the order of further corrections, which will be neglected in the sequel.

We may now apply Poisson summation to find the mean value of the rescaled nodal volume (figure 1)

\[
\langle \sigma_n \rangle_{E,E+\Delta E} = \frac{2\xi_{s-1}}{\pi \xi_s} \left( 1 - \beta_s \frac{S}{V} E^{-1/2} + O(E^{-3/4}) \right),
\]

where

\[
\beta_s = \frac{(s-1)\pi \xi_{s-2}}{2s\xi_{s-1}} + \frac{\pi \xi_s}{4\xi_{s-1}} - \frac{(s-1)\pi \xi_{s-1}}{2s\xi_s}.
\]

The three terms in the above expression for \( \beta_s \) have different origins in the asymptotic expansion of the Poisson sum. The nodal volume \( \sigma_n \) contains terms proportional to \( n_\ell - 1 \). The terms proportional to \( n_\ell \) give the leading term in (2.10) and the first term of \( \beta_s \). The unit shift leads to the second term in \( \beta_s \) and the third term comes from the boundary correction to the spectral counting function. Note that \( \beta_s \) is a positive constant for any \( s \) and that it decays as the dimension grows.
Figure 1. (a–d) Mean (main panels) and variance (insets) of the rescaled nodal volume as a function of energy. Data points are averages over \( \approx 10^6 \) eigenstates in an interval around the given energy. (Online version in colour.)

Analogously one may calculate the variance which we only give to leading order (figure 1)

\[
\text{Var}(\sigma_n)[E, E+\Delta E] = \frac{1}{\pi^2} + \frac{4(s-1)\xi_s - 2}{s^2\pi^2\xi_s} - \frac{4s^2 - 1}{\pi^2s^2\xi_s} + O(E^{-1/2}). \tag{2.12}
\]

Higher moments can be obtained in a similar way. Alternatively, one may just consider the limiting distribution

\[
P_s(\sigma) = \lim_{E \to \infty} \langle \delta(\sigma - \sigma_n) \rangle[\bar{E}, E+\Delta E] \tag{2.13}
\]

for which one may derive the formal expression

\[
P_s(\sigma) = \frac{1}{s\xi_s} \int d^{s-1} \Omega \delta \left( \sigma - \frac{\sum_{\ell=1}^s |\epsilon_{\ell}|}{\pi} \right), \tag{2.14}
\]

where the integral is over the unit \((s-1)\)-dimensional sphere and \( \mathbf{e} = (\epsilon_1, \ldots, \epsilon_s) \) is a point on the sphere (i.e. \( \mathbf{e}^2 = 1 \)). It is straightforward to see that these limiting distributions vanish outside the interval \( \sigma \in [1/\pi, \sqrt{s}/\pi] \). For low dimensions, (2.14) can be given by direct integration

\[
P_2(\sigma) = \begin{cases} \frac{4}{\sqrt{2 - \pi^2\sigma^2}} & \text{for } \sigma \in \left[\frac{1}{\pi}, \frac{\sqrt{2}}{\pi}\right] \\ 0 & \text{else} \end{cases} \tag{2.15}
\]
Figure 2. (a–d) Nodal volume distributions for an \( s \)-dimensional cuboid. Full lines show histograms obtained from \( 10^6 \) highly excited eigenstates. For \( s = 2 \) and \( 3 \), (a) and (b) also show the limiting distributions (2.15) and (2.16) as a dashed line. (Online version in colour.)

and

\[
P_3(\sigma) = \begin{cases} 
\frac{4}{\sqrt{3}} \left( \frac{\pi}{2} + \arctan \left( \frac{\pi \sigma}{\sqrt{6 - 3\pi^2\sigma^2}} \right) \right) \\
\arctan \left( \frac{\pi \sigma - 3\sqrt{2} - \pi^2\sigma^2}{\sqrt{6 + 6\pi \sqrt{2 - \pi^2\sigma^2}}} \right) \\
- \arctan \left( \frac{\pi \sigma + 3\sqrt{2} - \pi^2\sigma^2}{\sqrt{6 - 6\pi \sqrt{2 - \pi^2\sigma^2}}} \right)
\end{cases}
\]

for \( \sigma \in \left[ \frac{1}{\pi}, \frac{\sqrt{2}}{\pi} \right] \), \( \sigma \in \left[ \frac{\sqrt{2}}{\pi}, \frac{\sqrt{3}}{\pi} \right] \)

For larger dimensions, direct integration of the expression (2.14) may be performed with increasing effort. In figure 2, histograms of these distributions for a finite energy interval are shown that illustrate how the distribution changes with increasing dimension.
Let us conclude this section with a few observations about this example which may be generalized to a wider class of separable systems. Apart from points of high symmetry separable wave functions all have a similar local chequerboard structure. One may then expect that the qualitative behaviour of the mean nodal volume (including the boundary effect), the variance and the limiting distribution of nodal volumes will be very similar. We conjecture that the following features of the limiting distribution are universal:

— the limiting distribution has compact support $\sigma \in [\sigma_{\text{min}}, \sigma_{\text{max}}]$ with $\sigma_{\text{min}} > 0$ (this is consistent with Yau's conjecture);
— near $\sigma_{\text{min}}$ the behaviour is $P(\sigma) = c_1(\sigma - \sigma_{\text{min}})^{s-2}\theta(\sigma - \sigma_{\text{min}})$; and
— near $\sigma_{\text{max}}$ the behaviour is $P(\sigma) = c_2(\sigma_{\text{max}} - \sigma)^{s-3}/2\theta(\sigma_{\text{max}} - \sigma)$ (see [12] for an analogous singularity in limiting distributions of nodal counts).

3. The chaotic case: nodal volume statistics of boundary-adapted random waves

Let us now consider nodal volume statistics for $s$-dimensional random waves as proposed by Berry [21]. These are models for wave functions in chaotic billiards. In order to account for boundary corrections to the mean nodal volume we use boundary-adapted random waves following closely the analysis presented by Berry in [13], where the mean nodal volume for random waves in $s = 2$ dimensions which satisfy a Dirichlet boundary condition along an infinite line was calculated (see also [14–16]). In the mathematical literature, similar approaches have been used to study statistical properties of nodal volumes on tori [17, 18] and spheres [19, 20]. In these cases, random-wave models have been constructed in terms of random superpositions of degenerate eigenfunctions and rigorous results on expected nodal volumes and on the fluctuations were obtained. The latter are largely consistent with the results obtained by Berry for planar random waves for appropriate choices of eigenspaces.

Let us now construct a Gaussian random-wave model whose realizations are solutions of the $s$-dimensional free stationary Schrödinger equation (1.1) on the Euclidean space $\mathbb{R}^s$ at energy $E = k^2$ with a Dirichlet condition $\Phi = 0$ on the hyperplane $x_s = 0$ (where $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$ are Cartesian coordinates). We will consider solutions in the half-space $x_s > 0$ and thus refer to the plane $x_s = 0$ as the boundary. It will be convenient to use ‘dimensionless’ (rescaled) coordinates, so we define $R = (R_1, \ldots, R_s)$, where $R_i = kx_i$.

Let us first define the standard random waves (without any boundary conditions) by

$$u(R) = R e^{\sqrt{2/N} \sum_{j=1}^{N} e^{iR \cdot n_j + i\phi_j}}, \quad (3.1)$$

where $n_j$ are uniformly distributed on a unit $(s - 1)$-sphere and the phases $\phi_j$ are uniformly distributed on $[0, 2\pi)$. The Gaussian random-wave model is achieved in the formal limit $N \to \infty$. The random waves (3.1) do not obey any boundary conditions. Dirichlet boundary conditions at the boundary $R_s = 0$ can be implemented in a straightforward way by anti-symmetrization with respect to the boundary

$$\Phi(R) = \frac{1}{\sqrt{2}} (u(R) - u(\bar{R})), \quad (3.2)$$

where $\bar{R} \equiv R - 2R_s e_s$, with $e_s = (0, \ldots, 0, 1)$, is the unit vector in direction $x_s$. It is expected that the effect of the boundary at $x_s = 0$ becomes weaker when $x_s$ becomes larger. We have chosen the normalization constant in (3.2) so that $\langle \Phi^2 \rangle \to 1$ as $R_s \to \infty$, where $\langle \cdot \rangle$ refers to the average over random waves.
For a given region $G$ in the half-space $x_s > 0 \ (R_s > 0)$, we may now write the expected rescaled nodal volume as

$$
\langle \sigma_G \rangle = \frac{\int_G \langle \delta(\Phi(kx)) \mid \nabla_x \Phi(kx) \rangle \, d^s x}{k \int_G \, d^s x} = \frac{\int_{kG} \langle \delta(\Phi(R)) \mid \nabla_R \Phi(R) \rangle \, d^s R}{\int_{kG} \, d^s R} = \frac{\int_{kG} \rho(R) \, d^s R}{\int_{kG} \, d^s R},
$$

(3.3)

where $kG = \{kx : x \in G\}$ is the rescaled region and we have introduced the (expected) nodal density

$$
\rho(R) = \langle \delta(\Phi(R)) \mid \nabla_R \Phi(R) \rangle.
$$

(3.4)

Note that the random-wave model is translation invariant with respect to translations parallel to the boundary $R_s = 0$ such that the nodal density only depends on the distance $R_s$ from the boundary

$$
\rho(R) = \rho(R_s).
$$

(3.5)

As $N \to \infty$, the random-wave model becomes a Gaussian process such that any probability distribution involving $\Phi$ and its derivatives $\partial_i \Phi = \partial \Phi / \partial R_i$ at a point $R$ is a multi-variate Gaussian. For the present purpose, we need $P(\Phi, \partial_1 \Phi, \ldots, \partial_s \Phi)$, and more specifically $P(\Phi = 0, \partial_1 \Phi, \ldots, \partial_s \Phi)$.

The relevant variances and cross-correlations can be calculated from the known two-point correlator of the standard random-wave model, which is given by

$$
\langle u(R_1)u(R_2) \rangle = 2^{(s-2)/2} \Gamma \left( \frac{s}{2} \right) \frac{J_{(s-2)/2}(|R_1 - R_2|)}{|R_1 - R_2|^{(s-2)/2}}.
$$

(3.6)

The lengthy but straightforward calculation of the nodal density of the boundary-adapted random-wave model can be performed by generalizing Berry’s two-dimensional calculation [13]. We refer to [22] for details of the calculation which leads to the nodal density

$$
\rho(R_s) = \rho_{\text{bulk}} \sqrt{\frac{sD_{R_1}}{B}} \Gamma \left( \frac{1}{2}, \frac{1}{2} \frac{s}{2}, \frac{s}{2}, M \right),
$$

(3.7)

where

$$
\rho_{\text{bulk}} = \langle \delta(\Phi(R)) \mid \nabla_R u(R) \rangle = \frac{\Gamma(s + 1/2)}{\sqrt{\pi} \Gamma(s/2)}
$$

(3.8)

is the constant nodal density of the standard random-wave model without boundary (below we will show that $\rho(R_s) \to \rho_{\text{bulk}}$ as $R_s \to \infty$) and $F(a, b; c; x) = {}_2 F_1(a, b; c; x)$ is a hypergeometric function [23]. Nodal density (3.7) depends on the distance $R_s$ from the boundary via the covariance functions

$$
B \equiv \langle \Phi(R)^2 \rangle = 1 - \Gamma \left( \frac{s}{2} \right) \frac{J_{(s-2)/2}(2R_s)}{R_s^{(s-2)/2}},
$$

(3.9)

$$
D_{R_1} \equiv \langle \partial_1 \Phi(R)^2 \rangle = \frac{1}{s} - \Gamma \left( \frac{s}{2} \right) \frac{J_{s/2}(2R_s)}{R_s^{s/2}},
$$

(3.10)

$$
D_{R_s} \equiv \langle \partial_s \Phi(R)^2 \rangle = \frac{1}{s} + \Gamma \left( \frac{s}{2} \right) \frac{J_{s/2}(2R_s) - 2R_s J_{(s+2)/2}(2R_s)}{2R_s^{s/2}},
$$

(3.11)

and

$$
K \equiv \langle \Phi(R) \partial_1 \Phi(R) \rangle = \Gamma \left( \frac{s}{2} \right) \frac{J_{s/2}(2R_s)}{R_s^{(s-2)/2}},
$$

(3.12)
where \( f_n(x) \) is the \( n \)th Bessel function and the abbreviation
\[
M \equiv 1 - \frac{BDR_s - K^2}{BDR_1}.
\] (3.13)

Let us now consider the asymptotic behaviour of the nodal density (3.7) as \( R_s \to \infty \). Using the known asymptotic behaviour of Bessel functions and for the hypergeometric function one then obtains \( \rho(R_s) \to \rho_{\text{bulk}} \) with the leading-order smooth and oscillatory corrections given by (see [22] for details of the calculation)
\[
\frac{\rho(R_s)}{\rho_{\text{bulk}}} = 1 + C_{s}^{\text{osc}} R_s^{-(s-1)} + C_{s}^{\text{osc}} \frac{\cos(2R_s - ((s - 1)/4)\pi)}{R_s^{(s-1)/2}} + O(R_s^{-(s-1)}),
\] (3.14)

where
\[
C_{s}^{\text{osc}} = \frac{(s - 1) \Gamma(s) \Gamma(s/2)}{2^{s+2} \pi^{(s+2)/2}}
\] (3.15)

and
\[
C_{s}^{\text{osc}} = \frac{\Gamma(s/2)}{\sqrt{\pi}}.
\] (3.16)

The oscillatory part in (3.14) decays much more slowly than the smooth corrections. For nodal volumes one needs to integrate and we will see that the smooth correction will dominate over the oscillatory part. Note that (3.14) has oscillatory terms of the order of \( O(R_s^{-(s-1)}) \) which are formally of the same order as the smooth correction.

On the boundary \( R_s = 0 \) one finds
\[
\frac{\rho(0)}{\rho_{\text{bulk}}} = \frac{1}{2} \frac{\sqrt{s(s - 1)} \Gamma(s/2)^2}{\sqrt{s + 2} \Gamma((s + 1)/2)^2} < 1,
\] (3.17)

which is consistent with the intuitive expectation that a Dirichlet boundary condition will lead to a suppression of the nodal density near the boundary. This intuition is based on two facts: (i) our definition of the nodal volume excludes the boundary and (ii) generically nodal hypersurfaces intersect with the boundary in right angles.

We can now come back to the estimate of the rescaled nodal volume inside a given bounded region. For the standard random-wave model, the rescaled nodal volume is equal to the constant nodal density \( \langle \sigma_G \rangle = \rho_{\text{bulk}} \). For the boundary-adapted random-wave model, there are corrections which become stronger close to the boundary. To be specific let \( G \) be a cylindrical region in \( \mathbb{R}^s \) of height \( a \) such that the bottom is a connected bounded \( (s - 1) \)-dimensional region in the hyperplane \( x_s = 0 \). The volume of \( G \) is \( V = a S \), where \( S \) is the \( (s - 1) \)-dimensional hypervolume of the bottom. The rescaled nodal volume is
\[
\langle \sigma_G \rangle = \frac{S}{V_k} \int_0^{ka} \rho(R_s) dR_s = \rho_{\text{bulk}} \left( 1 + \frac{S}{V_k} \int_0^{ka} \frac{\rho(R_s) - \rho_{\text{bulk}}}{\rho_{\text{bulk}}} dR_s \right).
\] (3.18)

In the high-energy limit \( E = k^2 \to \infty \), the rescaled nodal density converges to \( \rho_{\text{bulk}} \). Indeed, \( \rho(R_s) \) is a bounded function that converges to \( \rho_{\text{bulk}} \) as \( R_s \to \infty \) such that the integral \( \int_0^{ka} ((\rho(R_s) - \rho_{\text{bulk}}) / \rho_{\text{bulk}}) dR_s \) is at most order \( o(k) \) and the correction term is at most \( O(1) \) for \( k \to \infty \). The form of the leading-order correction can be obtained from the asymptotic expansion of the nodal density (3.14). In the two-dimensional case one finds
\[
\int_0^{ka} \frac{\rho(R_s) - \rho_{\text{bulk}}}{\rho_{\text{bulk}}} dR_s \sim -I_2 + C_2^{\text{osc}} \int_1^{ka} R_2^{-1} dR_2
\] (3.19)
\[
\sim -I_2 + C_2^{\text{osc}} \log(ka),
\] (3.20)

where \( I_2 = \int_0^{\infty} (\rho(R_2) / \rho_{\text{bulk}} - 1 - C_2^{\text{osc}} \theta(R_2 - 1) R_2^{-1}) dR_2 \) is a constant term. For any higher dimension \( s \geq 3 \), we may define the constant
\[
I_s = -\int_0^{\infty} \left( \frac{\rho(R_s)}{\rho_{\text{bulk}}} - 1 \right) dR_s.
\] (3.21)
Altogether, we obtain

\[
\langle \sigma \rangle_G = \begin{cases} 
\rho_{\text{bulk}} \left( 1 - \frac{S \log k}{V} \frac{1}{32\pi k} + O \left( \frac{1}{k} \right) \right) & \text{for } s = 2, \\
\rho_{\text{bulk}} \left( 1 - \frac{S I_s}{V} + O \left( \frac{1}{k^{3/2}} \right) \right) & \text{for } s \geq 3. 
\end{cases} \tag{3.22}
\]

By numerical integration one obtains the coefficients \( I_3 \approx 0.758 \) and \( I_4 = 0.645 \).

Let us also mention that Berry has shown that \( \langle \sigma^2 \rangle - \langle \sigma \rangle^2 = O(\log k/k) \) \cite{berry1998random} in \( s = 2 \) dimensions, which implies that the distribution of nodal volumes \( P(\sigma) \) for a finite energy interval is very narrow and converges to

\[
P(\sigma) = \delta(\sigma - \rho_{\text{bulk}}) \tag{3.23}
\]

as \( E = k^2 \rightarrow \infty \). The same scaling also applies to a random-wave model on the 2-sphere as shown by Wigman \cite{wigman2012resonances}. The latter work by Wigman also states that for \( s \geq 3 \) one may show that the rescaled nodal volume variance for the \( s \)-sphere is bounded by \( O(k^{-(s-3)}) \). This implies that (3.23) applies to all dimensions for random waves on spheres. As curvature effects should not change these scalings one expects the limiting distribution (3.23) also in the case of the Euclidean random waves with boundary considered here.

At the end of this section, let us come back to more general manifolds with chaotic ray dynamics. Berry has introduced the standard Gaussian random-wave model without boundary as a model for wave functions in this case and he also proposed to include boundary effects with the boundary-adapted random-wave model. In \cite{berry1998random}, he conjectured that for \( s = 2 \) the \( \log k/k \)
corrections of the boundary-adapted random-wave model should also apply to chaotic billiards. Here, we extend his conjecture to arbitrary dimensions, i.e. the asymptotic behaviour of the nodal volume is described by (3.22) including the leading-order correction terms and no free parameters as \( V \) should be replaced by the volume of the manifold \( M \) and \( S \) is the hypervolume of its boundary. For higher-order corrections one may need to include curvature effects which are not represented in the random-wave model. In the case \( s = 2 \) terms of order \( O(k^{-3}) \) in the nodal volume of random waves in a cylindrical region explicitly contain the height \( a \), which is not well defined when one tries to translate the result to general billiards \cite{berry1998random}.

4. Signatures of wave chaos and integrability

In the previous two sections, we have given a detailed account of nodal volume statistics for two paradigmatic systems in arbitrary dimension. The \( s \)-dimensional cuboid is a paradigm for a regular shape with separable wave functions while planar random waves are a model for wave functions on irregular shapes with chaotic ray dynamics. In both cases, we have found limiting distributions for the rescaled nodal volume. The limiting distributions in the two paradigms have a very different character: for random waves we find a delta function while the cuboid’s limiting distribution has a finite support. It is consistent with Berry’s random-wave conjecture that the distribution for all irregular shapes will be a delta function at \( \sigma = \rho_{\text{bulk}} \) (see (3.8)), which only depends on the dimension and nothing else. For regular shapes other than the cuboid we may refer to the analogous calculations for nodal count distributions \cite{chamberland2012resonances, beigman2015resonances}, which showed that many features such as the types of singularities near the upper and lower ends of the support are universal. We conjecture that the same is true for the nodal volume distributions studied here though the actual values for the upper and lower ends of the support may be system dependent. One clear signature of irregular versus regular shapes is the behaviour of the fluctuations of rescaled nodal volumes as the energy increases. For irregular shapes, one expects that the variance decreases while it remains finite and bounded for regular shapes.

Another interesting difference between regular and irregular shapes in two dimensions is the different nature of the boundary correction to expected rescaled nodal volumes—these are \( 1/k \) for the cuboid (and, conjecturally, for other regular shapes with separable wave functions) and \( \log k/k \) for the random-wave model. It has been shown by numerical computations that the
boundary-adapted random-wave model gives an accurate account of the nodal density near a boundary [15] and it was conjectured by Berry that this may be seen in any chaotic billiard.

Let us also note that the average rescaled nodal volume for the cuboid is always larger than the one for a random wave. This is expected as the nodal surfaces of a separable function generally intersect in a chequerboard structure while nodal intersections are avoided in random waves [24]—this effective repulsion of nodal surfaces leads to the decrease in the expected nodal volume. Again, the same decrease can be expected for eigenfunctions in irregular shapes when compared with separable eigenfunctions of a regular shape.

References