We study a semiclassical asymptotics of the Cauchy problem for a time-dependent Schrödinger equation on metric and decorated graphs with a localized initial function. A decorated graph is a topological space obtained from a graph via replacing vertices with smooth Riemannian manifolds. The main term of an asymptotic solution at an arbitrary finite time is a sum of Gaussian packets and generalized Gaussian packets (localized near a certain set of codimension one). We study the number of packets as time tends to infinity. We prove that under certain assumptions this number grows in time as a polynomial and packets fill the graph uniformly. We discuss a simple example of the opposite situation: in this case, a numerical experiment shows a subexponential growth.

1. Introduction

This paper is devoted to mathematical problems that arise in the study of asymptotic solutions of equations of quantum mechanics on singular spaces. Popular examples of such spaces are metric and decorated graphs. Differential equations on metric graphs have been studied with growing interest in the last 30 years, and we are unlikely to present here even a brief review of the results. For details see, for example, [1,2] and references therein. Differential operators on decorated graphs (hybrid spaces) were intensively studied since the beginning of 1980 (see [3–5] and references therein).

Questions closely related to some topics presented in this paper, namely analysis of degeneracy classes of the
periodic orbits on metric graphs, have been discussed in a number of works. The leading coefficient of the asymptotics of the number of degeneracy classes has been studied in [6], cf. our theorem 2.3. For binary directed graphs with at most six vertices, the asymptotics of the number of cycles in a degeneracy class was analysed in [7]. Gavish & Smilansky [8] have obtained an asymptotics for the average size of a degeneracy class for a fully connected graph. Let us note the recent article [9] and references therein. The total number of clusters of periodic orbits on directed graphs and the relationship of this object with random matrix theory were studied in [10,11]. Note that the equations of quantum mechanics on spatial networks are actively studied at the moment (e.g. [12,13]).

The authors of this paper have developed a version of the semiclassical theory on metric graphs and have obtained some results related to decorated graphs. In particular, an algorithm for constructing quantization rules (generalizing the known rules of Bohr–Sommerfeld quantization) in the case of metric graphs was formulated in [14]. Semiclassical asymptotics for a solution of the time-dependent Schrödinger equation with the initial function localized in a small neighborhood of a certain point on a metric graph was obtained in [15]; statistical properties of such solutions (Gaussian packets) were studied in [15–18]. In particular, it was proved that the number of packets grows in time in a polynomial way; the leading coefficient of the corresponding asymptotic formula was obtained and the distribution of packets on the graph was studied. Note that these results are connected with certain number-theoretic problems. In particular, certain statistical characteristics of the number of Gaussian packets can be expressed in terms of numbers of integral points in polyhedra.

Further presentation of the material is organized as follows. In the next two subsections, we introduce the necessary terms and definitions. In §2, we state the results for metric graphs. The third section describes the dynamics of a generalized Gaussian packet on a decorated graph. In the last section (§4), we discuss the statistical properties of the solutions on decorated graphs. Namely, we prove that under certain conditions the number of packets grows in time as polynomial and present an example of the opposite situation, providing subexponential growth. Theorems that do not have references in their statements are new and have not been published before.

(a) Terms and definitions: metric graphs

Recall (see [1,2] and references therein) that a graph is called metric if each edge is considered as a parametrized curve with positive length. We denote a metric graph by $\Gamma$, its edges by $\gamma_j$ and vertices by $a_j$. The set of all edges adjacent to the vertex $a$ we denote by $\Gamma(a)$. We consider only finite metric graphs. Let $E$ and $V$ stand for the number of edges and the number of vertices, respectively.

Let $Q$ be an arbitrary real-valued continuous function on $\Gamma$, smooth on the edges. Let $Q_j$ be a restriction of $Q$ to the $j$th edge. Consider a direct sum $\hat{H}_0 = \bigoplus_{j=1}^E (-\hbar^2/2)d^2/dz_j^2 + Q_j(z_j))$ with Neumann boundary conditions on each edge, where $z_j$ is the parameter on $\gamma_j$, and $h > 0$ is the semiclassical parameter (further we consider the limit $h \to 0$). The domain of $\hat{H}_0$ is $H^2(\Gamma) = \bigoplus_{j=1}^E H^2(\gamma_j)$, where $H^2(\gamma_j)$ stands for the second Sobolev space on $\gamma_j$.

**Definition 1.1.** The Schrödinger operator $\hat{H}$ is a self-adjoint extension of the restriction $\hat{H}_0|_L$, where $L = \{\psi \in H^2(\Gamma), \psi(a_j) = 0\}$.

**Remark 1.2.** One can explicitly specify coupling conditions that describe an arbitrary extension (e.g. [11]). For each endpoint of an arbitrary edge, consider the pair $\psi_j, h\psi_j$, computed at this endpoint and a vector $\xi = (u, v), u = (h\psi_1', \ldots, h\psi_{2E}'), v = (\psi_1, \ldots, \psi_{2E})$. Here, derivatives are computed in the incoming direction to the vertex and the quantities $h\psi_j, \psi_j$ correspond to the $j$th endpoint. Let us consider a standard skew-Hermitian form $[\xi^1, \xi^2] = \sum_{j=1}^{2E} (u_j^1 v_j^2 - v_j^1 u_j^2)$ in $\mathbb{C}^{2E} \oplus \mathbb{C}^{2E}$ and fix the Lagrangian plane $\Lambda$. The coupling boundary conditions one can take in the form $\xi \in \Lambda$, or, equivalently,

$$-i(I + U) u + (I - U) v = 0,$$
where $U$ is a unitary matrix and $I$ is an identity matrix. Furthermore, we consider only local coupling conditions, $A = \bigoplus_{j=1}^{V} A_j$, where $A_j$ is defined for each vertex of $\Gamma$ separately (note that local conditions preserve the graph structure of $\Gamma$, whereas general ones connect all vertices, and therefore do not distinguish $\Gamma$ and a graph with only one vertex and a number of loops).

Boundary conditions of this form with an assumption of continuity of function $\psi(x, t)$ are called **natural** or Kirchhoff (see [1]). Explicitly, this means that there is an additional condition: the sum of one-sided derivatives of $\psi$ equals zero.

A **time-dependent Schrödinger equation on the graph $\Gamma$** is an equation of the form

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi. \quad (1.1)$$

We choose initial conditions that have the form of a narrow packet localized near the point $z_0$, which lies on the $k$th edge of the graph,

$$\psi(z, 0) = h^{-1/4}K(z_k) \exp \left( \frac{iS_0(z_k)}{\hbar} \right)$$

and

$$S_0(x) = \omega(z_k - z_0)^2 + p_0(z_k - z_0), \quad (1.2)$$

where $z_0 \in \gamma_k, p_0 \in \mathbb{R}$ and $\omega$ is complex, with $\text{Im}(\omega) > 0; K(z_j)$ is a cut-off function supported on the edge $\gamma_k$, and $K = 1$ in the vicinity of $z_0$. Factor $h^{-1/4}$ is introduced to ensure that the initial function $\psi(z_k, 0)$ is of order one in $L^2(\Gamma)$-norm. Owing to the positivity of the imaginary part of $\omega$, the initial function is localized in a small neighbourhood of $z_0$: $\psi(z_0, 0) = O(h^{\infty})$, with $|z_k - z_0| \geq \delta > 0$ ($\delta$ is independent of $h$). For simplicity, we assume that $\frac{1}{2}p_0^2 + Q(z_0) > Q(z)$, for all $z \in \Gamma$, which guarantees that there are no turning points (e.g. [19,20]) on $\Gamma$. Note that the presence of turning points leads to a change of the graph $\Gamma$: one has to cut $\Gamma$ by the turning points and consider the connected component of the cut graph, which contains $z_0$.

An asymptotic solution of the Cauchy problem (1.1) and (1.2) is described in [15,17] and the explicit formulae are given therein.

**Theorem 1.3 (see [15,17]).** Solution of Cauchy problem (1.1) and (1.2) for $t \in [0, T]$ ($T$ does not depend on $h$) is given by the formula

$$\psi = \sum_{j=1}^{N(t)} h^{-1/4} \varphi_j(z_j, t) e^{iS_j(z_j, t)/\hbar} + O(\sqrt{\hbar}), \quad (1.3)$$

with $S_j(z_j, t) = S_j^0(t) + p_j(t)(z_j - Z_j(t)) + W_j(t)(z_j - Z_j(t))^2$, where $p_j(t)$ and $Z_j(t)$ are solutions of the Hamiltonian system

$$\dot{z} = H_p, \quad \dot{p} = -H_z \quad \text{and} \quad H = \frac{1}{2}p^2 + Q.$$

These solutions are defined by initial conditions on the selected edge $Z_k(0) = z_0$, $P_k(0) = p_0$ and natural conditions in vertices: each trajectory coming to the vertex at a certain instant of time produces immediately trajectories emitted to all the edges adjacent to this vertex with equal momenta $p$. Functions $\varphi_j(z_j, t)$, $S_j^0(t)$ and $W_j(t)$ are explicitly expressed in terms of the solutions of the Hamiltonian system, $\text{Im} W_j(t) > 0$. By $z_j$, we denote coordinates on the edges of the graph; the index corresponds to the number of Gaussian packets and not to the number of the edges (so coordinates $z_j$ can be the same for different values of $j$).

Each term in sum (1.3) is localized in a small neighbourhood of the point $Z_j(t)$. Here, we assume that all terms that are localized in the same point $Z_j(t)$ form one **Gaussian packet**. Later, by $N(t)$ we will denote the number of such packets, i.e. the number of summands in (1.3), localized in different points.

In [15,17], scattering of a Gaussian packet on a vertex of a metric graph is described in detail. Namely, it is proved that each packet entering a vertex of degree $m$ generates $m$ packets propagating along the edges adjacent to the vertex. The momenta of the packets (denoted above as $P_j$) at the instant of scattering are equal to each other and the amplitudes are defined by the
coupling conditions in the vertex. In the case of the Kirchhoff conditions, the amplitude is divided between reflected and transmitted packets as \((2 - m)/m\) for the reflected packet and \(2/m\) for each of the \(m - 1\) transmitted ones.

We study asymptotics of \(N(t)\) as \(t \to \infty\). Note that this problem differs from the problem of describing the asymptotic solution of the Schrödinger equation at \(t \to \infty\), as the error estimation \(O(\sqrt{h})\) is valid only for finite times. This means that we first pass to the limit \(h \to 0\), and then to the limit \(t \to \infty\).

Let \(t_i\) stand for \(j\)th edge travel time, i.e. the time at which the trajectory of the corresponding Hamiltonian system passes an edge. The initial function is fixed. An edge travel time is an analogue of the length of an edge.

**Remark 1.4.** The travel time of any edge of the graph depends only on the initial data and is the same for any Gaussian packet on each fixed edge.

**Definition 1.5.** Let us denote by \(N_{a \to d}(t)\) the number of moments when packets came out of a fixed vertex \(a\) on a fixed edge \(ad\), which have occurred by the time \(t\). Later it will be shown that

\[ N_{a \to d}(t) = R t^E + o(t^E). \]

The number \(R\) is called a radiation coefficient.

(b) **Terms and definitions: decorated graphs**

A *decorated graph* is a topological space, obtained from a metric graph via replacing vertices by smooth manifolds. More precisely, consider a finite number of smooth complete Riemannian manifolds \(M_1, \ldots, M_V\), \(\dim M_k \leq 3\), and a number of segments \(\gamma_1, \ldots, \gamma_E\), endowed with regular parametrization. For each endpoint \(q\) of an arbitrary segment \(\gamma_i\), fix a point \(\tilde{q}\) on one of the manifolds \(M_k\); we assume all points \(\tilde{q}\) to be distinct. A decorated graph \(\Gamma_d\) is a quotient space of the disjoint sum \(\bigcup_{i=1}^V M_k \bigcup \gamma_j\) by the equivalence \(q \sim \tilde{q}\). This is a topological space built from manifolds and segments; contraction of all the manifolds into points transforms it into a metric graph.

The Schrödinger equation on a decorated graph is defined as follows (see [11] for detailed explanation; the original ideas were presented in [4,5]).

Let \(Q\) be an arbitrary real-valued continuous function on \(\Gamma_d\), smooth on the edges. Let \(Q_i\) and \(Q_k\) be restrictions of \(Q\) to \(\gamma_j\) and to \(M_k\), respectively. Consider a direct sum

\[ H_0 = \bigoplus_{i=1}^E (-h^2/2)d^2/dz_j^2 + Q_i) \bigoplus_{k=1}^V (-h^2/2)\Delta_k + Q_k \]

with domain \(H^2(\Gamma) = \bigoplus_{j=1}^E H^2(\gamma_j) \bigoplus_{k=1}^V H^2(M_k)\). Here, \(d^2/dz_j^2\) is an operator of the second derivative on \(\gamma_j\) with respect to a fixed parametrization with Neumann boundary conditions, and \(\Delta_k\) is the Laplace–Beltrami operator on \(M_k\).

**Definition 1.6.** The Schrödinger operator \(\hat{H}\) is a self-adjoint extension of the restriction \(\hat{H}_0|_L\), where \(L = \{\psi \in H^2(\Gamma), \psi(q_i) = 0\}\).

The domain of the operator \(\hat{H}\) contains functions with singularities in the points \(q_i\). Namely, let \(G(x, q, \lambda)\) be the Green function on \(M_k\) (integral kernel of the resolvent) of \(\Delta\), corresponding to the spectral parameter \(\lambda\). This function has the following asymptotics as \(x \to q_i\): \(G(x, q, \lambda) = F_0(x, q) + F_1\), where \(F_1\) is a continuous function and \(F_0\) is independent of \(\lambda\) and has the form

\[ F_0 = \begin{cases} \frac{-c_2}{2\pi} \ln \rho, & \text{dim } M = 2; \\ \frac{c_3}{4\pi \rho}, & \text{dim } M = 3; \end{cases} \]

Here, \(c_j(x, q)\) is continuous, \(c_j(q, q) = 1\), and \(\rho\) is the distance between \(x\) and \(q\). The function \(\psi\) from the domain of the operator \(\hat{H}\) has the following asymptotics as \(x \to q_i\): \(\psi = o_j F_0(x) + b_j + o(1)\). Now for each endpoint of the segment (i.e. for each point \(q_i\)) consider a pair \(\psi(q), h\psi'(|q)\) and a vector \(\xi = (u, v), u = (h\psi'(q_1), \ldots, h\psi'(q_{2E}), \alpha_1, \ldots, \alpha_{2E}), v = (\psi(q_1), \ldots, \psi(q_{2E}), h\beta_1, \ldots, h\beta_{2E})\).
Consider a standard skew-Hermitian form $[\xi^1, \xi^2] = \sum_{j=1}^{4E} (u_j^1 v^2_j - v_j^1 u_j^2)$ in $\mathbb{C}^{4E} \oplus \mathbb{C}^{4E}$. Let us fix the Lagrangian plane $\Lambda \subset \mathbb{C}^{4E} \oplus \mathbb{C}^{4E}$. An arbitrary self-adjoint extension $H$ is defined by the coupling conditions $\xi \in \Lambda$ or equivalently $-i(U + iU)u + (i - U)v = 0$, where $U$ is a unitary matrix defining $\Lambda$ and $I$ is an identity matrix. Physically, it is more natural to consider local coupling conditions, $\Lambda = \bigoplus_q \Lambda_q$, where $\Lambda_q \subset \mathbb{C}^4$ is defined for each point $q$ separately.

In §3, we will consider the Cauchy problem (1.1) and (1.2), where $\hat{H}$ is the Schrödinger operator on a decorated graph, and function (1.2) is localized on a segment $(z_0 \in \gamma_j$ for some $j)$.

2. Results for metric graphs

In this section, we consider local boundary conditions and assume that coupling Lagrangian planes are in general position. We need this condition to ensure that, if $k$ Gaussian packets come at the same moment of time to a vertex of a valence $v$, then $v$ packets start to move over all edges adjacent to this vertex.

In [17], it is proved that the number of Gaussian packets on an arbitrary compact finite graph has the following asymptotics:

$$N(t) = Ct^{E-1} + o(t^{E-1}),$$

where $E$ is the number of edges. This result was obtained by reducing the problem of counting the number of packets to the calculation of the number of lattice points in an expanding simplex. Roughly speaking, the main idea of such reduction is as follows. Consider instants of time when the number of packets can change. This can occur only when a certain packet enters a vertex; and function (1.2) is localized on a segment $(z_0 \in \gamma_j$ for some $j)$.

Further results (for example, a theorem about uniformity of distribution and an explicit formula for $C$) are based not only on the approximation for the number of lattice points in a polytope by its volume, but also on the rather non-trivial number-theoretic analysis presented in [22]. That is why these results are not true for all, but only for almost all, of the edge travel times.

**Theorem 2.1 (Uniformity of distribution, see [18]).** Consider a finite connected graph $\Gamma$. Consider a segment of the travel time $\tau$ on an arbitrary edge. Let $N_\tau(t)$ be a number of packets on this segment at the time $t$. Then for almost all incommensurable (i.e. linearly independent over $\mathbb{Q}$) $t_1, \ldots, t_E$,

$$\lim_{t \to \infty} \frac{N_\tau(t)}{N(t)} = \frac{\tau}{\sum_{j=1}^{E} t_j}.$$

Thus, the distribution of the number of packets is asymptotically uniform.

**Proof.** Let us choose on any edge with travel time $t_j$ a segment $dg$ with travel time $\tau$. Let us find $N_\tau(t)/N(t)$. We know [17] that $N(t) = Ct^{E-1} + o(t^{E-1})$. Let us find $N_\tau(t)$. As the number of packets changes only in vertices and there are no turning points, then

$$N_\tau(t) = N_{\rightarrow d}(t) - N_{\rightarrow d}(t - \tau) + N_{\rightarrow g}(t) - N_{\rightarrow g}(t - \tau). \quad (2.1)$$

Here, $N_{\rightarrow d}(t)$ stands for the number of moments when packets arrived at the segment from the point $d$.

It is clear that the number of moments when packets arrived at the point $d$ by the time $t$ equals the number of moments when packets came out of the nearest vertex $a$ by the time $t - T_1$. Here, $T_1$ is the travel time from $a$ to $d$. By $N_{a \rightarrow d}(t)$ we denote the number of moments when packets came from $a$ to $d$. 


We need to know asymptotics of the number of moments when packets came out of a vertex $a$. Packets can come out of the vertex only at times that are linear combinations (with non-negative integer coefficients) of edge travel times.

The number of release moments (when at least one packet comes out of the vertex $a$) is described by the number of sets $(n_j)$ satisfying inequalities of a kind

$$ n_1 t_{i_1} + \cdots + n_m t_{i_m} \leq t, \quad (2.2) $$

where $t_j$ is the travel time of the $j$th edge.

As the leading part of the asymptotics of the number of moments is defined by the volume of a simplex defined by (2.2), events with maximal numbers of summands happen more often. In other words, packets that have arrived at our vertex should have visited all edges, i.e.

$$ N_{a\rightarrow d}(t) = R^a t^E + o(t^E). \quad (2.3) $$

For almost all $t_1, \ldots, t_E$, the estimation can be improved [22]. There exists $K^a$ such that $N_{a\rightarrow d}(t) = R^a t^E + K^a t^{E-1} + o(t^{E-1})$. Let us show that $R^a$, which is called a radiation coefficient, does not depend on the choice of vertex. Consider vertices $a$ and $b$. There exists a path connecting $a$ and $b$. Let $\delta$ be its travel time. Any packet coming out from $a$ to $d$ over time that does not exceed $2\delta$ generates at least one packet that comes out from $b$ to $d'$. This is correct for packets coming out from $b$. We obtain inequalities: $N_{a\rightarrow d}(t + 2\delta) \geq N_{b\rightarrow d'}(t)$ and $N_{b\rightarrow d'}(t + 2\delta) \geq N_{a\rightarrow d}(t)$. We know that $N_{a\rightarrow d}(t) = R^a t^E + o(t^E)$ and $N_{b\rightarrow d'}(t) = R^b t^E + o(t^E)$. Thus, $R^a t^E + o(t^E) = R^b t^E$. Hence, $R^a = R^b$.

Let us modify the expression for $N_\tau(t)$:

$$ N_\tau(t) = R(t - T_1)^E + K^a(t - T_1)^{E-1} - R(t - T_1 - \tau)^E - K^a(t - T_1 - \tau)^{E-1} $$

$$ + R(t - T_2)^E + K^b(t - T_2)^{E-1} - R(t - T_2 - \tau)^E - K^b(t - T_2 - \tau)^{E-1} + o(t^{E-1}) $$

$$ = 2ER\tau t^{E-1} + o(t^{E-1}). $$

Thus, we obtain

$$ \frac{N_\tau(t)}{N(t)} \rightarrow \frac{2ER}{C} \tau. \quad (2.4) $$

It remains to show that the coefficient in front of $\tau$ has the required form.

We consequently take edges as $d_g$, and then sum the obtained expressions:

$$ 1 = \sum_{j=1}^{E} \frac{N_j(t)}{N(t)} \rightarrow \frac{2ER}{C} \sum_{j=1}^{E} t_j. \quad (2.5) $$

Hence,

$$ C = 2ER \sum_{j=1}^{E} t_j. \quad (2.6) $$

The proof is complete. ■

**Corollary 2.2 (Relation between coefficients $C$ and $R$).** The leading coefficient for the number of packets $C$ and the radiation coefficient $R$, for almost all edge travel times, are related in the following manner:

$$ C = 2ER \sum_{j=1}^{E} t_j. \quad (2.7) $$

**Theorem 2.3 (Leading coefficient of the number of packets, see [18]).** Consider a finite connected compact graph $\Gamma$. For almost all incommensurable numbers $t_1, \ldots, t_E$, the leading coefficient has the following form:

$$ C = \frac{1}{2^{E-2}(E-1)!} \sum_{j=1}^{E} t_j \prod_{j=1}^{E} t_j. \quad (2.8) $$

The proof is based on (2.7) and the following lemma.
Lemma 2.4. Let us consider a finite connected graph with incommensurable edge travel times \( t_i (i = 1, \ldots, E) \) and denote by \( \beta \) the number of independent cycles. Let \( B \) be an arbitrary vertex. Then for almost all edge travel times

\[
R = \frac{2^\beta}{2^E E! \prod_{j=1}^{E} t_j} = \frac{1}{2^{V-1} E! \prod_{j=1}^{E} t_j}.
\]

Proof. In order to understand how many moments occurred when packets came out of the vertex \( B \) by the time \( t \), we must consider the set of all possible paths of packets leading to that vertex. It is sufficient to consider only trajectories of the packets that travelled upon all edges. Only those paths give us the leading coefficient. That can be shown in the same way as shown in the proof of theorem 2.1. Let us divide the set of all possible paths into convenient subsets. Let \( A \) be an initial vertex. For each path that starts from \( A \), we can calculate the number of transitions over a fixed edge. This number can be odd or even. So we construct a ‘code of the path’: a sequence of 0s or 1s, each of the coefficients is the parity of passages over the corresponding edge. It is clear that the code does not change under continuous deformations of the path on the graph. Let us find the number of all possible codes. Consider cross connections, i.e. edges that are not in the spanning tree. The parity of passages on the cross connections defines a path’s homotopy class. Thus, the number of possible codes equals \( 2^\beta \). In other words, for a set of all possible codes \( C \), we have \(|C| = |H^1(\mathbb{Z}_2)| = 2^\beta \).

Let \( D \) be the set of all times that do not exceed \( t \) and within these times at least one packet arrives at \( B \). We should find \(|D|\).

Each time \( t \in D \), we can associate with some path from \( A \) to \( B \) (this path is not unique, but the code of such a path is unique). Thus, for each \( t \in D \) we can construct the code (it is the code of the corresponding path). Now \( D \) can be divided into disjoint unions of times with equal codes:

\[
D = \bigcup_{c \in C} D_c.
\]

If code \( c = (c_1, \ldots, c_E), c_i \in \{0, 1\} \), then

\[
D_c = \left\{ T = \sum_{i=1}^{E} t_i(c_i + 2n_i) \mid n_i \in \mathbb{N} \cup \{0\} \mid T \leq t \right\}.
\]

At every such time, at least one packet arrives at the vertex \( B \). It is well known that the number of such times asymptotically equals the volume of a corresponding simplex:

\[
|D_c| = \frac{t^E}{2^E E! t_1 \cdots t_E} + o(t^E).
\]

Finally, we sum this over all possible codes. Thus,

\[
|D| = \frac{2^\beta t^E}{2^E E! t_1 \cdots t_E} + o(t^E).
\]

Formula (2.8) follows from the lemma after application of the Euler relation \( \beta = E - V + 1 \) (see, for example, [23]).

3. Dynamics of generalized Gaussian packets on decorated graphs

Now let \( \Gamma_d \) be a decorated graph and consider the Cauchy problem (1.1) and (1.2). We suppose that the point \( z_0 \) belongs to the segment of \( \Gamma_d \) (not to the manifold) and that the initial energy is large enough: \( \frac{1}{2} P_{z_0}^2 + Q(z_0) > Q(z) \), for all \( z \in \Gamma_d \). First, we describe what happens at the time of scattering.

(a) Scattering on manifold

Let \( \Gamma_d \) be a half-line, connected with a manifold \( M \) in a single point \( q \). Let \( t_0 \) be the instant of scattering (i.e. the time when the trajectory of the classical Hamiltonian system on the half-line reaches \( q \)). Consider the sphere in \( T_q^* M: L_0 : |p| = |P(t_0)| \). Consider the flow \( g_t \) of the classical
Hamiltonian system on \( M \) with the Hamiltonian \( H = \frac{1}{2} |p|^2 + Q \) and let \( L_t \) be the shifted sphere \( L_0: L_t = g_t L_0 \).

**Theorem 3.1.** For a certain time interval \( t \in (t_0, t_0 + \varepsilon) \), the solution of the Cauchy problem (1.1) and (1.2) has the form
\[
\psi = \begin{cases}
A(t) e^{iS(z, t)/\hbar}, & z \in \mathbb{R}_+, \\
KL_t[B(x, t)], & x \in M, \\
+ O(\sqrt{\hbar}).
\end{cases}
\]

Here, \( S(z, t) \) has the same form as in §1 (see theorem 1.3), \( KL_t \) is the Maslov canonical operator on isotropic manifold \( L_t \) with complex germ \([19,24]\) and functions \( A \) and \( B \) can be expressed explicitly in terms of the coupling matrix \( U \).

The proof is based on the study of the behaviour of the function \( KL_t[B] \) as \( t \to t_0^+ \); cumbersome analysis shows that this function can be matched to the exponent in such a way that coupling conditions are fulfilled.

**Remark 3.2.** As \( \hbar \to 0 \), the support of the function \( \psi \) tends to \( \pi(L_t) \), where \( \pi : T^* M \to M \) is the natural projection. In general, position \( \psi \) is localized near the surface of codimension one; we call the function \( KL_t[B] \) a generalized Gaussian packet near the hypersurface. The set \( \pi(L_t) \) is called the support of the generalized Gaussian packet. Note that the corresponding classical object is not a single particle, but a surface filled by the particles emitted from the point of gluing. The classical momenta of all these particles have the same absolute value \( |P(t_0)| \) equal to the momentum of the incoming particle while the direction of the momentum is arbitrary. For example, consider a two-dimensional sphere connected at a single point to a half-line and let the potential \( Q \) vanish. Then, the particle entering the point of gluing produces on the sphere the function that is localized near the circle formed by the endpoints of geodesics emitted from the point of gluing with any initial directions.

**Remark 3.3.** Let \( \Gamma_d \) be an arbitrary decorated graph. During some time (neighbourhood of the instant \( t_0 \)), the solution will have the same form as described in the previous theorem. After some time, the support of the generalized Gaussian packet reaches some gluing point \( q \) (it can coincide or not coincide with the point of the first scattering). At that time, the packet produces one packet propagating along the segment, glued at the point \( q \), and another propagating inside the manifold. Then one of these packets reaches a certain point of gluing and produces the next two packets, etc. It is easy to see that for an arbitrary time \( t \) the number of packets localized on the segments of \( \Gamma_d \) (not on the manifolds) coincides with the number of packets localized on some selected edges of a new graph \( \tilde{\Gamma} \). Namely, the vertices of \( \tilde{\Gamma} \) correspond to the gluing points of \( \Gamma_d \). The edges correspond to the times \( t_j \) of passage of the trajectories of the classical Hamiltonian system along the segments of \( \Gamma_d \) and between gluing points on the manifolds. Two vertices are connected by the edge if there exists a trajectory, connecting these points in \( \Gamma_d \) during the corresponding time. We select the edges of \( \tilde{\Gamma} \) corresponding to the segments of \( \Gamma_d \). So, in order to study the statistics of the number of such packets, we can use the results of §2.

4. **Statistics of generalized Gaussian packets on decorated graphs**

Let \( \Gamma_d \) be a decorated graph with finite number of edges. For arbitrary finite \( t \), a solution has the form \( \psi = \sum \psi_j + O(\sqrt{\hbar}) \), where \( \psi_j \) are generalized Gaussian packets. Let \( N(t) \) be the number of packets localized on the segments \( \gamma_j \) of \( \Gamma_d \) (not on the manifolds). Let \( t_j \) be times of passage of the trajectories of the classical Hamiltonian system along the edges of the graph and between gluing points on the manifolds.

We will at first assume that there is a finite number of times \( t_1, \ldots, t_M \).

**Theorem 4.1.** Let \( t_j \) be linearly independent over \( \mathbb{Q} \). Then for almost all \( t_1, \ldots, t_M \) and for almost all coupling Lagrangian planes \( A_j \),
\[
N(t) = Ct^{M-1} + o(t^{M-1}),
\]
as \( t \to \infty \).
Proof. The calculation of \( N(t) \) is equivalent to the calculation of the number of Gaussian packets on a new metric graph \( \tilde{\Gamma} \), corresponding to selected edges (see remark 3.2). As the number of the travel times defined by trajectories of the classical Hamiltonian system between gluing points on the manifolds is finite, this new graph is finite too. Clearly, the number of edges of \( \tilde{\Gamma} \) equals \( M \). Thus, the theorem will follow from the results presented in [17] and theorem 2.1. ■

**Theorem 4.2.** For almost all incommensurable \( t_j \) and for almost all coupling Lagrangian planes \( L_j \),

\[
C = \sum_{\text{edges}} t_j / 2^{2E - 2} (M - 1)! \prod_{j=1}^{M} t_j.
\]

Here \( 2E \) is the number of points of gluing \( q_j \) (\( E \) is the number of segments \( \gamma_j \)), and \( M \) is the number of times \( t_j \).

Proof. The result follows from theorems 2.1 and 2.3. Namely, owing to theorem 2.3, the number \( \tilde{N}(t) \) of all packets on \( \tilde{\Gamma} \) has the form

\[
\tilde{N} = \frac{\sum_{j=1}^{M} t_j}{2^{2E - 2} (M - 1)! \prod_{j=1}^{M} t_j} t^{M-1} + o(t^{M-1}).
\]

However, we are interested not in all packets but in those localized on the selected edges (which correspond to the segments of the initial decorated graph—see remark 3.2). Owing to theorem 2.1, the number \( N(t) \) of these packets differs from \( \tilde{N}(t) \) asymptotically by the factor

\[
\frac{\sum_{\text{edges}} t_j}{\sum_{\text{all}} t_j},
\]

which leads to the statement of the theorem. The Lagrangian planes should be in the general position to ensure that in every vertex of a new metric graph, if \( k \) packets come simultaneously to a vertex of a valence \( v \) of the new graph \( \tilde{\Gamma} \), then \( v \) packets start to move over all edges. ■

The last result in this subsection is devoted to the distribution of the number of generalized Gaussian packets. Let \( \delta \) be a segment on an arbitrary edge of \( \Gamma_d \). Let \( N_{\delta} \) be the number of packets, located on \( \delta \).

**Theorem 4.3.** For almost all incommensurable \( t_j \) and for almost all coupling Lagrangian planes \( L_j \),

\[
\lim_{t \to \infty} \frac{N_{\delta}(t)}{N(t)} = \frac{t_{\delta}}{\sum_{j} t_{j}}.
\]

Proof. The result follows from theorem 2.1 applied to \( \tilde{\Gamma} \). It means that the distribution of the number of generalized Gaussian packets on a decorated graph tends to a uniform distribution as \( t \to \infty \). ■

(a) An example with infinite number of times.

In this section, we will consider an example of the decorated graph with an infinite set of travel times.

Let \( \Gamma_d \) be a decorated graph, constructed from a circular cylinder and an edge. We fix two points on the cylinder and glue endpoints of an edge to them. Let the potential \( Q(z) \) be a zero function.

Let \( t^* \) be the length of the attached edge, \( t_0 \) be the distance between two points of gluing and \( 2\pi R \) be the length of the circle.

The number of travel times is infinite in this situation, because there is an infinite number of geodesics connecting two points of gluing with different passage times \( t_k = \sqrt{t_0^2 + (2\pi RK)^2} \). The paths on the surface differ by the number \( k \) of turns around the cylinder.

**Statement 4.4.** The number of packets on the attached edge of this decorated graph grows faster than any polynomial of \( t \).
Figure 1. Time evolution of $\ln(\ln(N(t)))/\ln(t)$. Results of the computer experiment for $t^* = \sqrt{2}$, $t_0 = 1$ and $R = 1$.

Proof. Consider a metric graph, consisting of two loops of length $2\pi R$ and two vertices connected by an infinite number of edges: the first one of length $t^*$ and the others of length $t_k$. The number of packets on the initial decorated graph that are situated on the attached edge (not on the cylinder) is equal to the number of packets on the first edge of the described metric graph. Let us fix the arbitrary integer $M$ and exclude from the metric graph all the edges except for the first $M$ ones. Evidently, the considered number of packets is at least the number of packets on this reduced graph situated on the first edge. Owing to the results of §2, the last number is of order $t^{M-1}$. So, the number of packets on the attached edge of the decorated graph is at least $Ct^{M-1}$ for an arbitrarily large $M$.

A numerical experiment was carried out in cooperation with O. V. Sobolev. An initial decorated graph was replaced by an infinite metric graph, only the final part of which is occupied at any finite time. As we know all edge travel times, we can describe the dynamics on such a graph discretely. That is, the number of packets changes only at some fixed times, according to certain rules. To model this process, we wrote a program in C++. The experiment demonstrates that the number of packets grows as $e^\alpha$ with $\alpha$ close to $1/2$, as $t$ goes to infinity, for $t^* = \sqrt{2}$, $t_0 = 1$ and $R = 1$. Note that the plot of $\ln(\ln(N(t)))/\ln(t)$, shown in figure 1, converges to a number close to $1/2$ at large $t$.

Acknowledgements. The authors are grateful to A. A. Tolchennikov, M. M. Skriganov, N. G. Moschevitin, P. B. Kurasov, U. Smilansky and O. V. Sobolev for useful discussions and interest in their work.

Funding statement. This work was supported by a grant from the Russian Government for scientific research under the supervision of leading scientists in M. V. Lomonosov Moscow State University, contract no. 11G.34.31.0054, and by the grant ‘The National Research University Higher School of Economics’ Academic Fund Program in 2013–2014, research grant no. 12-01-016. The work was also done with partial financial support of grant nos. MK-4255.2012.1, RFFI 12-01-31235, 11-01-00973, 11-01-12058-2011, NSh-1410.2012.1 and state contract nos. 14.B37.21.0370 and 14.740.11.0794.

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