Between the study of small finite frameworks and infinite incidentally periodic frameworks, we find the real materials which are large, but finite, fragments that fit into the infinite periodic frameworks. To understand these materials, we seek insights from both (i) their analysis as large frameworks with associated geometric and combinatorial properties (including the geometric repetitions) and (ii) embedding them into appropriate infinite periodic structures with motions that may break the periodic structure. A review of real materials identifies a number of examples with a local appearance of ‘unit cells’ which repeat under isometries but perhaps in unusual forms. These examples also refocus attention on several new classes of infinite ‘periodic’ frameworks: (i) columns—three-dimensional structures generated with one repeating isometry and (ii) slabs—three-dimensional structures with two independent repeating translations. With this larger vision of structures to be studied, we find some patterns and partial results that suggest new conjectures as well as many additional open questions. These invite a search for new examples and additional theorems.

1. Introduction

If you hold a finite piece of a material or analyse a built structure, you may ‘see’ it as a finite structure that locally has ‘repetitive’ structure with local transformations which are isometries. It looks like part of an infinite periodic structure. For example, a crystal—which locally has a crystal symmetry that can be extended (mathematically) to an infinite framework on a lattice extending in three independent directions.

Alternatively, a model builder may try to build a physical model to ‘represent’ a theoretically infinite periodic structure and figure out what mechanical
behaviour will be captured in a finite model. This model builder faces a challenging problem: the model loses the implicit constraints that enforce periodic behaviour of the mathematical model. Rather than having a strong comparison with periodic frameworks with a forced lattice structure for its motions [1–3], these physical models are closer to ‘incidentally periodic frameworks’, where the infinite framework starts in a periodic configuration, but the permitted motions of the infinite framework may break the periodicity [4,5].

If a collection of finite structures are rigid (or infinitesimally rigid) and a sequence of them grows to cover the whole infinite structure in the limit, then the infinite structure is also rigid (infinitesimally rigid) without any periodic conditions—something that is now called ‘sequential rigidity’ in the literature [4,5]. We will offer some relevant combinatorial and geometric examples of such sequential rigidity, as well as some examples of incidentally periodic infinitesimally rigid structures which have no rigid sequence of substructures. We will also offer some conjectures and questions about both the rigidity of fragments and the incidental rigidity of the infinite structures which happened to have started off as periodic.

The overarching goal is to understand finite fragments which show some repetitions. As finite structures, they are not ‘generic’—even if the framework of an initial unit cell is generic, the repetition provides additional geometric conditions that may, or may not, change their rigidity. As a path of entry, we look at several key classes of examples, and their connections to a range of other structures, including both periodic frameworks with a forced lattice structure in the motions and the less-studied incidentally periodic structures, with infinitesimal and finite motions which are not forced to be periodic. The following list indicates the sections where these are analysed:

§2 1-periodic columns: infinite structures in three dimensions with one direction of screw (or translational) periodicity and a finite number of vertices in each unit cell, and a finite number of edges attached to each unit cell of the periodic column, both for incidental motions and for motions forced to be periodic, permitting variation of the screw;

column fragments: finite frameworks in three dimensions which are noticed to have one direction of initial screw repetition (or a translation) and can be embedded as part of a periodic column with this screw, with a representative of every class of edges in the periodic column (figure 1a);

§3 2-periodic slabs: infinite structures in three dimensions with two directions of translational periodicity (but with a variable lattice), both with motions which may break the periodicity and with motions forced to maintain the periodic structure, but not the plane lattice of the translations;

slab fragments: finite structures in three dimensions with two directions of initial translational repetition, embedded as part of a periodic slab with a representative of every class of edges (figure 1b);
§4 3-periodic crystals: infinite frameworks which are 3-periodic directions, which are either incidentally periodic or forced to retain the periodic structure but with a varying lattice; crystal fragments: finite fragments with three independent directions of initial translational repetition, and embedded in a periodic crystal with a representative of every class of edges of the periodic crystal (figure 1c); and

§5 In the final section, we survey some further directions to work.

Over the past several years, we have investigated a wide variety of examples, both in models and in the literature, that fit somewhere in this larger programme. The selected examples offer suggestive behaviours and raise key unanswered questions. For this paper, we focus primarily on examples where the fragment is either rigid, or has a restricted space of motions (of small dimensions, as in the cylindrical tubes (§2) or motions restricted to location (rigid or almost rigid in the core, but flexible on the boundaries).

We focus on fragments related to periodic structures in 3-space, with flexible lattices or just incidental periodicity. There is a related theory of periodic structures connected with fixed lattices—and a corresponding theory of their fragments with a rigid substructure fixing the lattice, which we call ‘coatings of a rigid substrate’. This theory will be the subject of a future paper.

There are also good analogues in two dimensions in 1-periodic strip frameworks, where some of the properties can be illustrated simply, and more of the theory of periodic frameworks has theorems providing necessary and sufficient conditions. Strip patterns also create surprises for incidentally periodic structures, as has already been observed by Owen & Power [4].

In all cases, the final goal is to understand the finite fragments which we encounter in real structures, with their entwined local geometry and combinatorial properties. The sources of information include the finite geometric and combinatorial theory of rigidity for finite frameworks, the combinatorial and geometric theory of forced periodic motions of periodic frameworks, and the developing theory of incidentally periodic infinite frameworks without forced periodic constraint on their motions. The challenge is to find ways to extract maximal information from all these associated structures to inform our analysis in essential ways. We hope that our survey of patterns and our selection of some key examples will stimulate others to join in further study of these intriguing and important topics.

2. Periodic columns and column fragments

We start with a simple type of three-dimensional structure—a column which shows a single direction of repetition (figure 2) with a translation or screw motion which takes part of the structure onto other parts of the structure. Such structures have an interesting and accessible theory, both as finite fragments with embedded isometry and as infinite periodic structures with one direction of periodic transformation. Such finite columns have some interesting applications in nature (beta barrels, tubulin [6]) and in structural design [7–9].

Because these samples of columns model some patterns which reappear for slabs and crystals in the following sections, we will present more detail here and shorten the analogous presentation for slabs and crystals. The outline of this section is: §2a—definition of a periodic column; §2b orbit rigidity matrix of a periodic column; §2c definition of fragments for columns; §2d key examples and §2e summary on incidentally periodic columns.

(a) Definition of a periodic column

The following is adapted from the usual definition of a periodic framework in 3-space [1–3].

The pair \((G, \Gamma)\) is a 1-periodic graph if \(\tilde{G} = (\tilde{V}; \tilde{E})\) is a simple infinite graph with finite degree at every vertex, and \(\Gamma \subset \text{Aut}(\tilde{G})\) is a free abelian group of rank 1, which acts without fixed points and has a finite number of vertex orbits (and therefore a finite number of edge orbits). \(\Gamma\) is isomorphic to \(Z\) and we can identify a group generator \(\gamma\).
A columnar periodic placement of a 1-periodic graph \( \langle \tilde{G}, \Gamma \rangle \) is a pair \((\tilde{p}, S)\) where \(\tilde{p} : V \rightarrow \mathbb{R}^3\) and \(S\) is a non-zero screw transformation in 3-space. We associate the generator \(\gamma\) with \(S\) and extend this to a map of \(\Gamma\) to the free abelian group generated by composition or iteration, of \(S\), with \(n\gamma\) mapping to \(S^n\). We require that \(\tilde{p}(n\gamma(v)) = S^n(\tilde{p}(v))\) for every vertex \(v\) of \(\tilde{G}\). For convenience, we assume that the translational component of the screw is along the \(z\)-axis. The action of the screw can then be written as \(S(p_i) = (r \times p_i + s)\), where \(r\) and \(s\) are vectors along the \(z\)-axis. With this notation, \(S^n(p_i) = (nr) \times p_i + ns\).

A 1-periodic graph \( \langle \tilde{G}, \Gamma \rangle \) together with a columnar periodic placement \((\tilde{p}, S)\) forms a 1-periodic column \(C = (\langle \tilde{G}, \Gamma \rangle; \tilde{p}, S)\). An infinitesimal motion of a periodic column is an assignment of velocities \(w : V \rightarrow \mathbb{R}^3\) such that for each edge \((i, j) \in \tilde{E}, (p(i) - p(j)) \cdot (w(i) - w(j)) = 0\). These constraints represent the first-order form of the requirement that the lengths of the edges are preserved.

(b) Orbit graph and rigidity matrix of a periodic column

To describe the 1-periodic graph, the 1-periodic column and the periodic infinitesimal motions in a compact form, we can refocus on representatives of the orbits of the vertices and edges under the action of the group \(\Gamma\). This is a multi-graph formed with one representative from each equivalence class of vertices, and one representative of each equivalence class of edges. First select standard representatives of the vertex classes \(V_O\)—the copy of the vertex in the slice \(0 \leq z < t\), where \(t\) is the translational component of the screw along the \(z\)-axis. The representatives of the equivalence classes of edges are then the edges \(E_O\) within that slab and those going from the slab ‘up’ to another copy of a vertex. If the edge joining the equivalence class of \(p_i\) and \(p_j\) from \(V_O\) goes up \(m_{ij}\) slabs, from \(p_i\) to \(S^{m_{ij}}p_j\), then we give the edge representative the label \((i, j; m_{ij})\). With these gains, we have a unique label for each equivalence class of edges [2,10].

Combined this gives an orbit graph \(\langle G_O, m \rangle\) and an orbit framework \(((G_O, m); p_O, S)\), where \(p_O\) is the restriction of \(p\) to \(V_O\). Note that the orbit graph may include loops (with non-zero gain; figure 3b).

As a useful subclass of infinitesimal motions, we are interested in infinitesimal motions which respect the periodic structure—but permit the screw to change both its translational and rotational components. We start with the length constraint on an edge \((i, j; mz_{ij}) (p_i - ((nr) \times p_i + ns)) = c_{ij,mz_{ij}}\) and consider all of the positions and screw as functions of time \(t\), we linearize it by taking the derivative w.r.t. \(t\):

\[
(p_i - ((nr) \times p_j + ns) \cdot (w_i - ((nr) \times p_j + (nr) \times w_j + ns'))) = 0.
\]

A periodic infinitesimal motion of a periodic column is an infinitesimal motion of the periodic column \((p(i) - p(j)) \cdot (w(i) - w(j)) = 0\) with the added property that there are two added variables \(r'\) and...
Iterated, this means that \(w_R = w_P O\), where the orbit rigidity matrix captures all the constraints for forced periodic motions as figure 3 shows.

By the T-gain procedures of [2], the corresponding rigidity matrix has the same rank. In fact, one can take any set of representative vertices, which are connected by a spanning tree, and create an analogous set of representatives of the 1-periodic orbit rigidity matrix. In fact, one can take any set of representative vertices, which are connected by a spanning tree, and create an analogous set of representatives of the 1-periodic orbit rigidity matrix. In fact, one can take any set of representative vertices, which are connected by a spanning tree, and create an analogous set of representatives of the 1-periodic orbit rigidity matrix. The other choice of the standard representatives is relatively arbitrary (just translate the column vertically) but all choices give isomorphic spaces of infinitesimal motions with their forced infinitesimal motions.

For the minimal orbit graph in figure 3c with \(\varepsilon_0 = 3\nu_0 - 2\), the 1-periodic orbit matrix confirms periodic infinitesimal rigidity. However, the infinite framework is not infinitesimally rigid (figure 3d). The other infinitesimal motions of the column are not periodic with the given period, but are not captured by the orbit matrix. This orbit count is a lower bound—but we will develop a tighter lower bound on the edges of an infinitesimally rigid framework also has a 4-space of trivial infinitesimal motions—the 3-space of translations and the z-axis of the translation or screw, assuming that there are at least three non-collinear vertices, the necessary condition for infinitesimally periodic rigidity is \(\text{rank}(R_{(G_{O,m}}(p_O, S) ≥ 3\nu_0 - 2\).

Our choice of the standard representatives is relatively arbitrary (just translate the column vertically) but all choices give isomorphic spaces of infinitesimal motions with their forced periodic rigidity matrices. In fact, one can take any set of representative vertices, which are connected by a spanning tree, and create an analogous set of representatives of the 1-periodic orbit rigidity matrix. By the T-gain procedures of [2], the corresponding rigidity matrix has the same rank.

For the minimal orbit graph in figure 3c with \(\varepsilon_0 = 3\nu_0 - 2\), the 1-periodic orbit matrix confirms periodic infinitesimal rigidity. However, the infinite framework is not infinitesimally rigid (figure 3d). The other infinitesimal motions of the column are not periodic with the given period, and therefore are not captured by the orbit matrix.
Figure 4. A single finite framework (a), may fit several different screw motions (b–d) to become a subframework of a periodic column. (c) Uses two alternative displays of the screw - as the composition of a translation and a rotation, or as the induced translation of a vertex. (Online version in colour."

(c) Definition of fragments for 1-periodic columns

Our interest in the periodic columns comes from our interest in finite structures which are ‘seen’ as substructures of these periodic columns. However, we want substructures which are ‘extended enough’ to let us identify the screw motion and to develop an entire periodic graph by repeatedly applying the screw motion to the initial structure. We start with an initial finite framework $F = (U, D)$ and a configuration $p : U \rightarrow \mathbb{R}^3$ with $p(i) = p_i \neq p_j$ if $\{i, j\} \in E$. The framework is written as $(F, p)$ (figure 4).

Formally, a fragment $\mathcal{F}$ of a $1$-periodic column $\mathcal{C} = (\tilde{G}, \Gamma); \tilde{p}, S$ is an injective map $f$ from the finite framework $F$ into the vertices, edges and placement of the periodic column. To make this fragment non-trivial, we require that image of the vertices and edges contains representatives of all the equivalence classes of vertices and edges. Finally, we require that in the embedding there are at least three non-collinear vertices $p_1, p_2$ and $p_3$ with $S(p_1), S(p_2) \text{ and } S(p_3)$ also in the embedding. This labelled overlap guarantees that the screw is unique for the labelling and we will see below that this overlap is also sufficient for exploring sequential rigidity.

We are interested in relatively large, and growable, non-trivial fragments, which must span multiple copies of some, or all, of the representatives of the vertex classes (figure 4). Given a non-trivial fragment, with the identified screw $S$ from the column, the $S$-closure of $\mathcal{F}$ is the infinite periodic framework generated by repeated applications of $S$ and $S^{-1}$ to the vertices and edges of $\mathcal{F}$, written as $(\mathcal{F}, S)$. The $S$-closure of non-trivial fragment $\mathcal{F}$ is the entire 1-periodic column $\mathcal{C} = (\tilde{G}, \Gamma); \tilde{p}, S$.

In fact, just giving a screw $S$ and an $S$-compatible labelling of the vertices of a finite framework is sufficient to generate a screw completion and a unique periodic framework. A labelling is $S$-compatible for a screw $S$ if whenever a pair of vertices is labelled as connected by the screw, then geometrically the isometry is correct. We also ask that the labelling contains at least three non-collinear vertices $p_1, p_2$ and $p_3$ along with $S(p_1), S(p_2)$ and $S(p_3)$. This is sufficient to make the screw unique, given the labelling. There is clearly an induced $S$-closure from this labelling and the original framework is a non-trivial fragment of this infinite framework.

For example, when we review the examples in figure 4, we see that the labelling with the screw motions is sufficient to create an $S$-closure as a periodic column. With three different screw motions, we have three different periodic columns, with different periodic groups $\Gamma$ on the same shared infinite graph.

Figure 4d shows a shift to a subgroup of the initial $\Gamma$, doubling the period to $S^* = S^2$. If we look back to figure 3c,d, we see that such a doubling reveals a new periodic column with the same geometric graph, but with new periodic infinitesimal motions (figure 4c,d). This doubling will also shift the orbit counts $e_O = 3v_O - 2$, to a revised count on the new orbit graph of $e_O = 3v_O - 4$, so that the revised count alone predicts that there will now be non-trivial periodic infinitesimal
motions. More generally, any count with $e_O < 3v_O$ will predict infinitesimal motions as soon as we expand to triple the period. Thus, $e_O = 3v_O$ is the minimal orbit count if we want the infinite framework to be infinitesimally rigid for all choices of subgroup of $\Gamma$—what is called supercell periodic rigidity [5]. This is clearly a necessary requirement for infinitesimal rigidity of the 1-periodic column.

More generally, when we have several compatible screws labelling and the corresponding $S$-closures which are not just taking subgroups of $\Gamma$, the infinite framework will have a larger group of structure-preserving screws, closed under composition. In the examples of figure 4a,b, we see additional screws because the initial framework (and the infinite framework of the $S$-closures) had a rotational symmetry—and the composition of $S^{-1}S'$ is such a rotational symmetry of the whole infinite framework.

We are primarily interested in the incidental infinitesimal motions of the same 1-periodic infinite frameworks $(\langle \tilde{G}, \Gamma \rangle; \tilde{p}, S)$. Here, we no longer require that the velocities preserve the periodic structure. Such motions may also be found in the fragments (see figure 3 and corollary 2.1). We still have the orbit graph and its associated counts, but there is no simple matrix which captures the behaviour of these infinite frameworks [4,5]. We will see in conjecture 2.9 that the minimal count for incidental periodic rigidity of a column is still anticipated to be $e_p = 3v_p$.

**Proposition 2.1.** If a periodic column $(\langle \tilde{G}, \Gamma \rangle; \tilde{p}, S)$ has a non-trivial fragment $\mathcal{F}$ which is infinitesimally rigid (resp. rigid), then the column is infinitesimally rigid (resp. rigid).

**Proof.** Consider the infinitesimally rigid fragment $\mathcal{F}$ containing $p_1, p_2, p_3$ and $S(p_1), S(p_2), S(p_3)$ and the screw shifted framework $SF$ which is also infinitesimally rigid. These two frameworks share three non-collinear joints $S(p_1), S(p_2)$ and $S(p_3)$, so the combined framework is also infinitesimally rigid. When this is repeated with both $S$ and $S^{-1}$, we have a sequence of infinitesimally rigid frameworks which form the infinite 1-periodic column, in the limit. Any non-trivial infinitesimal motion of the 1-periodic column $(\langle \tilde{G}, S \rangle, \tilde{p})$ would have to restrict to a non-trivial motion of some finite framework in this sequence. We conclude that the periodic column is also infinitesimally rigid.

The identical argument applied with the assumption of rigidity of the fragment demonstrates the rigidity of the 1-periodic column.

**Corollary 2.2.** If a periodic column has a non-trivial infinitesimal motion (resp. motion), then every fragment has a non-trivial infinitesimal motion (resp. motion).

A column whose rigidity is demonstrated by such a sequence of finite rigid frameworks is called sequentially rigid [5,11]. We see in the examples below that there are rigid (and infinitesimally rigid) periodic columns which are not sequentially rigid.

Our final goal is to extract information from these various spaces of infinitesimal motions, and make predictions for the rigidity of large fragments $\mathcal{F}$ from all possible sources. Some of the analysis is combinatorial: what happens for almost all configurations $p$ with the required periodic or repetitive properties, often predicted by the counts of vertices and edges? Other analysis is geometric: what happens for a specific $p_O$, which may be singular in the algebraic variety? The examples in the next section will play a crucial role in this analysis.

**(d) Key examples of columns**

We select some key illustrative examples from the four examples in figure 5, and will explore some variants. Some animations of motions of these examples are displayed at http://wiki.iri.upc.edu/index.php/Symmetric_linkages.

**Example 2.3 (Triangulated triangular tube).** We have already considered the triangular tube—a triangulated triangular prism $(G, p)$ (figure 5a). A simple observation is that this is the graph of a convex triangulated sphere, with $e = 3v - 6$, for any length of fragment. As a finite fragment, it is a minimal infinitesimally rigid framework, by Alexandrov’s theorem [12].
The infinite framework formed by the $S$-closure for any of the screws in figure 4 is sequentially
infinitesimally rigid (and rigid). There are other singular screw motions of the bottom triangle
to top triangle within a single layer which will generate infinitesimally flexible fragments
(infinitesimally flexible non-convex octahedra as a single layer). This infinitesimal flexibility
extends to all numbers of layers. However, these are still rigid. The $S$-closure of these is also
infinitesimally flexible but sequentially rigid.

**Example 2.4 (Triangulated quadrilateral tube).** Consider a general finite fragment of a
triangulated quadrilateral tube $(G, p)$ (figure 5b,c). A simple observation of any length of such a
tube is that this is the graph of a convex triangulated sphere, with two edges (at the top and
bottom) missing. Therefore, the count of edges in any of these fragments is: $|E| = 3|V| - 8$, so
there is at least a two-dimensional space of non-trivial infinitesimal motions in every fragment.

When extended to infinite periodic frameworks, we have $v_O = 3v_O > 3v_O - 2$. So, we anticipate
that these will be infinitesimally rigid for periodic motions for generic choices of the vertices and
of $S$. We also anticipate that the infinite periodic framework will be incidentally rigid for generic
choices of the vertices and of $S$. This is indeed correct, for many, but not all values of $S$ as the
following geometric examples illustrate.

What is the range of behaviour in the fragmentary quadrilateral tubes as we change
the generating isometry $S$—and what is the change in behaviour of the corresponding
1-periodic tube?

**Example 2.5 (Vertical triangulated quadrilateral tube).** Consider the geometric tube in
figures 5b and 6, built on a plane parallelogram with $S$ as a simple translation. Note that there
are at least two vertices from the vertex orbits in all but figure 6c. From several pieces of prior
work, and direct observation, we know that

- The framework in the initial geometry is independent—so there are exactly two degrees
  of freedom and both extend to finite motions of the fragment.
- One motion is the simple plane flex of the $xy$-plane quadrilateral, extended in the
  $z$-direction for all lengths of tubes (figure 6b,c). This motion localizes the motion to the
  vertical edges—but involves all vertices.
- The periodic infinite column will inherit this motion, both with forced periodicity and
  with incidental periodicity.
- There is a second motion which moves all vertices and flexes the tube at all internal
  edges—with the velocities of the motion increasing with the distance up or down in
  at least one direction from a chosen central ring of the tube (which is similar to the
  quadrilateral of figure 6c). This flex is not periodic, for any period.
Figure 6. (a–c) Vertical tubes built on a spatial quadrilateral. (Online version in colour.)

Figure 7. (a–c) A tube with a $\frac{1}{8}$ screw twist has unusual motions. (Online version in colour.)

— This second motion will generate larger and larger infinitesimal velocities as the tube expands to the infinite periodic tube. The geometric infinite periodic tube has one degree of freedom, preserving the periodicity.

— All the motions of the tube fragment are parametrized by the motions of any one of the quadrilateral rings (figure 6c). This is related to the general observation that, with any one of these rings blocked (frozen), we know that the entire structure is infinitesimally rigid. In particular, if the tube fragment is pinned to the ground on one end it is infinitesimally rigid [13].

We now examine the same graph $G$ with different generating isometries $S$.

**Example 2.6 (Twisted triangulated quadrilateral tube).** Consider the quadrilateral tube with a screw isometry with a $\frac{1}{8}$ turn (figure 7). A detailed geometric analysis related to a single ring has been carried out by multiple authors [14,15]. This analysis confirms the following behaviour:

1. The single layer fragment (figure 7b) in the initial geometry is independent—so there are exactly two degrees of freedom and both extend to finite motions of the fragment.
2. Given a tube with more than one layer (figure 7a), the non-trivial motions separate into two components: one motion in each of the end rings, with no internal motion of vertices in the central core between these ends. http://wiki.iri.upc.edu/index.php/Symmetric_linkages.
(3) If a long tube fragment is pinned to the ground on one end, the framework remains flexible on the other end.

(4) From (2), we are able to conclude that the infinite 1-periodic framework is infinitesimally rigid as an incidentally 1-periodic framework. This is an extended form of almost sequential rigidity—but no finite fragment of this periodic cylinder will be infinitesimally rigid or rigid.

A non-obvious part of this analysis is the key role of the mirror symmetries of the fragments.

These two examples had special geometry and some of their behaviour is not ‘typical’. Which properties recur in most of the quadrilateral tubes as we continuously modify the screw $S$ from having no rotation (example 2.5) to having a $\frac{1}{4}$ turn (example 2.7)? The finite motions of a range of such quadrilateral tubes are animated at http://wiki.iri.upc.edu/index.php/Symmetric_linkages.

**Example 2.7 (General twisted triangulated quadrilateral tube).** When we place these two examples into a continuum with the same initial parallelogram in the $xy$-plane, and only vary $S$ (figure 8), we can fix initial values for the bottom parallelogram (the variables for the basic vertices) and generate a polynomial in the variables in $S$. As all infinitesimal rigidity properties of finite fragments are preserved by projective transformations [16,17], a vertical scaling of the entire structure has no impact—so only the relative twist of $S$ is significant. Moreover, the infinitesimal rigidity properties of periodic structures (the orbit matrix) are unchanged by affine transformations [2,18], so again, only the twist parameter matters.

We have the following additional observations:

(1) Almost all twists leave an independent fragment. In fact, all twists leave an independent fragment, so there is a 2-space of non-trivial finite motions.

(2) Almost all 1-periodic columns on this orbit graph are infinitesimally periodically rigid, because a polynomial (representing the rank of the rigidity matrix) which is non-zero for one $S$ (example 2.6) is non-zero for almost all values of $S$. Example 2.5 is one of the isolated failures.

(3) For almost all twists, there are non-trivial finite motions of all fragments which move all vertices and change the dihedral angle at all interior edges (an application of reasoning in Finbow-Singh et al. [19]).

(4) For almost all twists, the infinitesimal motions of one fragment can be extended to a larger fragment without changing the original velocities. This means that we can sequentially construct an infinitesimal motion of the 1-periodic cylinder which breaks the periodicity.

(5) In the same way, for almost all twists, the finite motions of a fragment can be extended to finite motions of an extended fragment, sequentially constructing a finite motion of the 1-periodic cylinder which breaks the periodicity.
For almost all twists, pinning down one end of a fragment will block all motions of the cylindrical fragment [13], so all motions must involve a non-trivial motion of both the end quadrilaterals.

We know that for a small change in twist from \( \frac{1}{8} \), the velocities in the interior will be smaller than the velocities of at least one of the ends. We conjecture that for almost all twists, the motions of long tube fragments increase towards at least one end. If this is correct, and if we bound the infinitesimal velocities in the 1-periodic cylinder, then there will only be trivial infinitesimal motions.

Example 2.8 (Symmetric triangulated 2\( k \)-gon tubes). For a regular 2\( k \)-gon tube, with a screw angle of \( 1/4k \), analysis of example 2.6 extends in an interesting way. By a simple count, any fragment is \( 2(2k - 3) \) short of being a triangulated sphere—and therefore has at least \( 2(2k - 3) \) infinitesimal degrees of freedom (all of which extend to finite motions).

A fragment of \( 2k - 3 \) layers, with a rigid base on one end, has \( 2k - 3 \) degrees of freedom (though it counts to have 0). The layer on the open end has \( (2k - 3) \)-degrees of freedom, the next layer in has \( (2k - 4) \)-degrees of freedom, on down till the layer at the rigid base now has one degree of freedom. This geometric singularity guarantees that if we take a fragment of length \( > 2(2k - 3) \), then the non-trivial motions will live on the two ends, and the remaining core will be infinitesimally rigid (and rigid), extending the analysis of example 2.6. Again the S-closure will be infinitesimally rigid, but no non-trivial fragment will have less than \( 2(2k - 3) \)-degrees of freedom.

As in examples 2.6 and 2.7, a small change in the twist will extend the infinitesimal motions across the whole tube, with larger motions on at least one end, and a form of stiffening in the middle.

A descriptive phrase for the observed (and generically conjectured) behaviour in examples 2.7 and 2.8 is that the tube becomes stiffer as the tube is extended. Similar behaviour has been observed in sample triangulated tubes with wider polygonal openings (of size \( k > 4 \), such as those of [7–9]) (figure 2d). Similar behaviour has also been observed in infinite (and finite) plane strips [4]. We conjecture that this increasing stiffness happens for a range of other initial graphs \( G \) for tubes which are rigid as incidentally 1-periodic columns with \( e_O = 3v_O \).

This increasing stiffness has also been observed by biochemists in protein microtubules [6] (figure 2c). Microtubules are key stiffening components of the cytoskeleton of cells [20]. These structures are so essential to cell function, including cell division (mitosis) that their formation is the target of several current cancer drugs. It has also been observed that microtubules change their length in response to external forces, and dynamically vary on a scale that a tube may form, fluctuate in length, and then disappear in 10 min—perhaps all to change their stiffness!

(e) Summary on incidentally 1-periodic columns

Consider the summary counts for 1-periodic columns in table 1. The table records how the counts shift from periodic rigidity to supercell periodic rigidity for columns, but may stay the same for incidental rigidity. The results of Power [21] for crystal frameworks suggest that the 1-periodic columns with the count \( e_O = 3v_O \) (what Powers calls the Maxwell Equilibrium count) will have a complex polynomial whose zeros test the supercell flexibility of crystals. We conjecture that if this polynomial is non-zero, putting the screw variables into this polynomial will show that for a given periodic rigid graph with \( e_O = 3v_O \), almost all choices of periodic placement and screw make the 1-periodic cylinder supercell rigid. With the same reasoning, we also conjecture that the supercell rigid frameworks form an open neighbourhood of periodic placements and choices of screw.

If we look again at the results of example 2.7, we see that it is an isolated screw that gives an infinitesimally rigid (and rigid) incidentally 1-periodic column, as all small changes become flexible. This demonstrates that in general supercell infinitesimal rigidity is not sufficient for
Table 1. Summary of minimal counts for 1-periodic and incidentally 1-periodic columns.

<table>
<thead>
<tr>
<th>class</th>
<th>forced period</th>
<th>example</th>
<th>supercell</th>
<th>example</th>
<th>incidental period</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>column</td>
<td>$e_0 = 3v_0 - 2$</td>
<td>example 2.3</td>
<td>$e_0 = 3v_0$</td>
<td>example 2.3</td>
<td>$e_0 = 3v_0$</td>
<td>example 2.6</td>
</tr>
</tbody>
</table>

incidental infinitesimal rigidity and this incidental infinitesimal rigidity is not open under small variations in the screw.

On the other hand, if we have an infinitesimally rigid non-trivial fragment of a 1-periodic column (example 2.3, then small changes in the screw and in vertices within the screw compatibility will preserve this infinitesimal rigidity of the fragments—and therefore the incidental infinitesimal rigidity of the periodically nearby 1-periodic columns. Is this sequential rigidity of the infinite 1-periodic column necessary for an infinite 1-periodic column to remain infinitesimally rigid under 1-periodic perturbations?

There could still be a subset property for incidentally rigid 1-periodic columns.

Conjecture 2.9. Given a 1-periodic column $⟨\tilde{G}, \Gamma⟩; \tilde{p}, S⟩$ which is incidentally infinitesimally rigid, then, there is some subset $\tilde{E}$ of edges with $\tilde{E}_O = 3v_O$, such that the induced infinite periodic graph $⟨\tilde{G}, \Gamma⟩; \tilde{p}, S⟩$ is incidentally 1-periodically infinitesimally rigid.

There is much more to be explored for these 1-periodic columns and their fragments.

3. Slabs: three-dimensional structures with two periodic directions

We now consider structures in 3-space which have the appearance of repetition under two independent (non-collinear) translations $t_1$ and $t_2$. For convenience, we use the term slab to name both the periodic frameworks and the fragments with such repetitions.

We will follow the pattern established for columns, though there is some added complexity and a few surprises.

(a) Definitions and basic theorems for slabs

We extend the definitions in §3c again following the patterns of [1–3].

The pair $⟨\tilde{G}, \Gamma⟩$ is a 2-periodic graph if $\tilde{G} = (V; \tilde{E})$ is a simple infinite graph with finite degree at every vertex, and $\Gamma \subset \text{Aut}(\tilde{G})$ is a free abelian group of rank 2, which acts without fixed points and has a finite number of vertex orbits (and therefore a finite number of edge orbits). $\Gamma$ is isomorphic to $Z^2$ and we can identify two group generators $\gamma_1$ and $\gamma_2$.

A slab periodic placement of a 2-periodic graph $⟨\tilde{G}; \Gamma⟩$ is a pair $(\tilde{p}; t_1, t_2)$ where $\tilde{p} : V \to \mathbb{R}^3$ and $t_1, t_2$ are independent translations in 3-space. We associate the generators $\gamma_1, \gamma_2$ with $t_1, t_2$ and extend this to a map of $\Gamma$ to the free abelian group. For convenience, we assume that the translations lie in the $xy$-plane.

Together a 2-periodic graph $⟨\tilde{G}; \Gamma⟩$ with a slab periodic placement $(\tilde{p}; t_1, t_2)$ forms a periodic slab $SI = (⟨\tilde{G}, \Gamma⟩; \tilde{p}; t_1, t_2)$. An infinitesimal motion of a 2-periodic slab is an assignment of velocities $w : V \to \mathbb{R}^3$ such that for each edge $(i,j) \in \tilde{E}$, $(p(i) - p(j)) \cdot (w(i) - w(j)) = 0$.

(b) Orbit graph and rigidity matrix of a 2-periodic slab

To describe the 2-periodic graph, the periodic slab and the 2-periodic infinitesimal motions in compact form, we refocus on the orbits of the vertices and edges under the action of the group $\Gamma$. First select standard representatives of the vertex classes $V_O$—the copy of the vertex in the slice $0 \leq x < t_1, 0 \leq y < t_2$. The representatives of the equivalence classes of edges are the edges $E_O$ within that slab, and those going from the slab to another copy of a vertex with larger $x$ and $y$. If the edge joining the equivalence class of $p_i$ and $p_j$ from $V_O$ goes from $p_i$ to $(m_{ij}t_1 + n_{ij}t_2) + p_j$ (with $m_{ij}, n_{ij} \geq 0$, then we give the edge representative the label $(i,j; m_{ij}, n_{ij})$. With these gains, we have a unique label for each equivalence class of edges [2,10].
Combined this gives an orbit multi-graph \((G_O, m)\), and an orbit framework \((\langle G_O, m \rangle, p_O, t_1, t_2)\), where \(p_O\) is the restriction of \(p\) to \(V_O\). The orbit graph may include loops (with non-zero gain) (figure 3b).

We will assume, for simplicity, that \(t_1 = (t_{11}, 0, 0)\) to remove rotations of the lattice and that \(t_2 = (t_{12}, t_{22}, 0)\). The residual three parametric deformations of the lattice can be associated with changing the lengths of the translations and the angle between them. We do not write out the details of the coefficients of \(t'_{11}, t'_{12}\) and \(t'_{22}\) for the infinitesimal deformations of the lattice, but condense the coefficients as \(\mathcal{L}(i, j; m_{ij}, n_{ij})\). We now have a corresponding 2-periodic orbit rigidity matrix \(R_{\langle G_O, m \rangle}(p, t_1, t_2)\) has the schematic form

\[
\begin{pmatrix}
\vdots \\
0 \cdots (*** \cdots) 0 \cdots (*** \cdots) 0 \cdots (*** \cdots) 0 \\
\vdots \\
\{i, j; m_{ij}, n_{ij}\} \\
\vdots
\end{pmatrix}
\]

The 3-space of translations will all be trivial solutions to \(R((G_p, m), p) \times w = 0\). No rotations are trivial solutions because of our assumptions on the placement of the translations. Counting columns, rows and trivial solutions, we have the following necessary condition for forced periodic infinitesimal rigidity: \(c_O \geq 3v_O + 3 - 3v_O\). Changing the representatives of the vertex orbits does not change the rank of this matrix, following the \(T\)-gain procedure of Ross [2]. We see that this count for all periodic motions to be trivial is not sufficient to make all infinitesimal motions of a periodic slab trivial.

(c) Definition of fragments for slabs

A framework \(\mathcal{F} = (F, p)\) is a fragment of a 2-periodic slab \(SI = (\tilde{G}, \Gamma, \tilde{p}; t_1, t_2)\) if there is an injective map \(f\) from the finite graph \(F\) into the vertices and edges of the 2-periodic slab. To make this fragment non-trivial, we require that image of the vertices and edges contains representatives of all the equivalence classes of vertices and edges. Finally, we require that in the embedding there are at least three non-collinear vertices \(p_1, p_2\) and \(p_3\) with \(t_1 + p_1, t_1 + p_2\) and \(t_1 + p_3\) and three non-collinear vertices \(p_4, p_5\) and \(p_6\) with \(t_2 + p_4, t_2 + p_5\) and \(t_2 + p_6\) (perhaps some or all the same as \(p_1, p_2\) and \(p_3\)) also in the embedding. We see below that this overlap is also sufficient for exploiting sequential rigidity.

Given a non-trivial fragment, with the identified independent translations \(t_1\) and \(t_2\) from the slab, the \((t_1, t_2)\)-closure of \(\mathcal{F}\) is the infinite 2-periodic framework generated by repeated application of \(+t_1, -t_1, +t_2\) and \(-t_1\) to the vertices and edges of \(\mathcal{F}\), written as \(\langle \mathcal{F}, t_1, t_2 \rangle\). \(\langle \mathcal{F}, t_1, t_2 \rangle\) is the entire 2-periodic slab \(SI = (\tilde{G}, \Gamma; \tilde{p}; t_1, t_2)\).

Again, giving the independent translations \(t_1, t_2\) and a \((t_1, t_2)\)-compatible labelling of the vertices of a finite framework is sufficient to generate a screw completion and a unique periodic framework. A labelling is \(t_1, t_2\)-compatible when a pair of vertices is labelled as connected by the translations, then geometrically the translation is correct. We also ask that the labelling contains at least three non-collinear vertices \(p_1, p_2\) and \(p_3\) along with \(t_1 + p_1, t_1 + p_2\) and \(t_1 + p_3\), and three non-collinear \(p_4, p_5\) and \(p_6\) with \(t_2 + p_4, t_2 + p_5\) and \(t_2 + p_6\) (perhaps some or all the same as \(p_1, p_2\) and \(p_3\)).

We are also interested in the incidental infinitesimal motions of the same periodic infinite frameworks \((\tilde{G}, S; \tilde{p}; t_1, t_2)\). Here, we no longer require that the velocities preserve the periodic structure. If non-trivial incidental motions exist, some of them will also be found in the fragments.

**Proposition 3.1.** If a 2-periodic slab \((\tilde{G}, \Gamma, \tilde{p}; t_1, t_2)\) has a non-trivial fragment \(\mathcal{F}\) which is infinitesimally rigid (resp. rigid), then the 2-periodic slab is infinitesimally rigid (rigid).

**Proof.** Consider the infinitesimally rigid fragment \(\mathcal{F}\) containing \(p_1, p_2\) and \(p_3\) and \(t_1 + p_1, t_1 + p_2\) and \(t_1 + p_3\) and the \(t_1\) shifted framework \(t_1\mathcal{F}\) which is also infinitesimally rigid. These two frameworks share three non-collinear joints \(t_1 + p_1, t_1 + p_2\) and \(t_1 + p_3\), so the combined framework is also infinitesimally rigid. Similarly, the \(t_2\) shifted framework \(t_2\mathcal{F}\) shares \(t_2 + p_4\),
Figure 9. (a) Sample slabs—the half-octahedral tetrahedral truss and (b) a slab of triangulated cubes. (Online version in colour.)

Figure 10. Periodic versions of the half-octahedral tetrahedral truss: (a) reduced to a minimally periodically rigid 2-periodic slab structure; (b) the full truss generated by two vertices; (c) a subframework which is supercell 2-periodic rigid and (d) a non-periodic infinitesimal motion of the fragment. (Online version in colour.)

$t_2 + p_5$ and $t_2 + p_6$ with the original fragment and is also infinitesimally rigid. When this is repeated with $t_1, t_2$ we have a growing sequence of infinitesimally rigid frameworks which form the infinite slab, in the limit. Any non-trivial infinitesimal motion of the column $(\tilde{G}, \gamma, \tilde{p}, t_1, t_2)$ would have to restrict to a non-trivial motion of some finite framework in this sequence. We conclude that the 2-periodic slab is also infinitesimally rigid.

The identical argument applied with the assumption of rigidity of the fragment demonstrates the rigidity of the 2-periodic slab.

A slab whose rigidity is demonstrated by such a sequence of finite rigid frameworks is *sequentially rigid*. We will see, again, in the examples below that there are rigid (and infinitesimally rigid) 2-periodic slabs which are not sequentially rigid.

(d) Slab examples

A well-studied 2-periodic slab is the *half-octahedral truss* designed by Alexander Graham Bell and independently designed and patented by Buckminster Fuller [22–24] (figures 9a and 10b,c,d). We first consider a set of examples connected to this truss.

Example 3.2 (Minimal forced 2-periodic rigid slab). Consider the truss in figure 10a. It has $v_O = 2$, and $e_O = 6$ and is minimal when extended to a 2-periodic slab and examined for forced periodic rigidity with the orbit matrix. It is clearly not rigid as a fragment, as the vertical and horizontal lines permit rotations.

If we assign alternative directions of rotation around the vertical lines, we create a flex of the supercell periodic framework, with $2t_1$ and $t_2$. Similarly, if we assign alternative directions of rotation to the horizontal lines, we create supercell periodic motions with the periods $t_1$ and $2t_2$. 
Figure 11. (a) A slab of cubes and (b) a slab with with twisted columnar segments. (Online version in colour.)

So, supercell rigidity will require two additional edges in the initial period—the edges in the next example (figure 10b).

**Example 3.3 (Half-octahedral tetrahedral truss).** Consider the full octet truss in figure 10b. As a periodic slab figure 10c, it has \(v_O = 2\) and \(e_O = 8 = 3v_O + 2\). This is 2 overbraced as a 2-periodic slab for periodic motions, but is minimal for supercell infinitesimal rigidity.

As a finite fragment in figure 10d, it has \(e = 30 = 3(12) = 6 = 3v - 6\) and it counts to be rigid. However, figure 10d shows an infinitesimal flex, so this is not independent—it has a self-stress. This truss, both as a fragment and as a periodic slab has a warp to a hyperbolic paraboloid figure 10d [24,25]. This motion is not periodic, on any scale, so it cannot be captured by an addition to the orbit matrix. However, we can add an additional periodic constraint across a square on top or the bottom that blocks this warp [24].

This warp of fragments extends from any fragment to a larger fragment—so the \((t_1, t_2)\)-closure is not infinitesimally rigid. However, the fragments in figure 10d (and larger) are rigid—so the \((t_1, t_2)\)-closure is rigid.

This failure is because of the framework embedding into a complete bipartite graph with vertices on two planes (a quadric) and the added edges being a ruling of this quadric [24]. This is unchanged by any change in the initial pair of vertices or the translation vectors—it is inherent in the 2-periodic graph. If we are going to double the period, even in one direction, and perturb the vertices of one of the new vertex orbits, the fragments will become infinitesimally rigid, as will the \((2t_1, t_2)\) closure.

**Example 3.4 (Cube slab).** Consider the fragment in figure 11a. As each cube in the fragment is a triangulated convex sphere, these are minimally infinitesimally rigid. When combined, the larger fragments of the slab are all rigid and infinitesimally rigid. If we focus on the thick lines and big vertices in figure 11a, and count one edge from each equivalence class, then we have \(v_O = 2, e_O = 8\) with \(e_O = 3v_O + 2\). This provides an example confirming that \(e_O = 3v_O + 2\) can be sufficient for infinitesimal rigidity of the infinite incidentally 2-periodic slab, as well as for supercell infinitesimal rigidity.

**Example 3.5 (Tube-connected slab).** Consider the slab illustrated in figure 11a. This is a crossing version of sets of the quadrilateral tubes in §2d. As a periodic framework, it has \(v_O = 16\) and \(e_O = 54 = 3v_O + 6\)—so it is periodically overbraced by 6. (As an aside, there is a re-count, using the inversive symmetry of the units of the slab, which actually brings this example down to the minimal count for periodic rigidity with inversive symmetry following the approach of Ross et al. [10].)

The behaviour of the small fragment in (b) is also an extension of the behaviour of the fragmentary tubes in example 2.7. The counts of the fragment are \(v = 40, e = 110 = 3v - 10\). The
entire interior core of the fragment is rigid, and the four expected degrees of freedom from this fragment count live only on the four open quadrilateral faces of the fragment. The 2-periodic slab will be sequentially rigid.

From this extreme situation, we can consider a gradual variation of the twist of the two ‘waist’ quadrilaterals directly around the central cube. As in the analysis in example 2.7, continuity guarantees that for small changes in the twists, some motions live across all the vertices of the fragment—but there is an increasing stiffness as we move to the centre of the fragment, with larger motions on the boundary.

We can imagine the same pattern extended to an \( m \times n \) rectangular section of the periodic framework. With the given twist, again all the interior core will be rigid, and the \( 2(m + n) \) open quadrilateral faces will have all the \( 2(m + n) \) motions. A gradual variation of all the twists will keep larger motions on the boundary and an increase in stiffness in the core. These motions appear to extend out to larger fragments, which suggests that the 2-periodic slab will be sequentially flexible. This depends on the geometry of the vertices in the cell. It is possible to perturb the vertices within a period and break this flexibility.

There are many variations of this example, with various combinations of twists, that can be built and should be explored. The interested reader is encouraged to build such a model (say with Polydron framework pieces) and confirm this intriguing behaviour.

**Example 3.6 (Antiparallel beta sheet).** There is one biological structure which also appears as a slab—the antiparallel beta sheet (figure 12a,b). We see that the antiparallel beta sheet can be analysed as a slab fragment, an analysis that captures the translational pattern in a way that is otherwise not accessible to rigidity analysis.

This basic secondary structure of proteins is usually observed with a twist that is (approximately) a parabolic hyperboloid. We have not explored the details of this molecular fragment, which is beyond this paper. Because of its core role in protein rigidity and flexibility, this deserves further analysis and it would be built using molecular (body-bar) counts.

### (e) Summary on incidentally 2-periodic slabs

Consider the summary for 2-periodic slabs in table 2. The table records an essential shift of count from 2-periodic rigidity to supercell 2-periodic rigidity for slabs. We have a smaller collection of interesting examples here, but the half-octahedral-tetrahedral truss of example 3.3 shows that some supercell rigid 2-periodic slabs are not incidentally rigid. This failure is not just because of the geometry of the special realization—but is inherent in the graph and the gains. With different gains, such as \((1, 1)\) in place \((1, 0)\) for the loop on \( v_1 \), but no change in the gain on the loop at \( v_2 \), the fragments would become infinitesimally rigid, and the infinite framework would be sequentially infinitesimally rigid.
Figure 13. (a) A triangulated cubic grid as a fragment and (b) a crystal. (Online version in colour.)

Table 2. Summary of minimal counts for 2-periodic and incidentally 2-periodic slabs.

<table>
<thead>
<tr>
<th>class</th>
<th>forced period</th>
<th>example</th>
<th>supercell</th>
<th>example</th>
<th>incidental period</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>slab</td>
<td>$e_O = 3v_O$</td>
<td>example 3.2</td>
<td>$e_O = 3v_O + 2$</td>
<td>example 3.4</td>
<td>$e_O = 3v_O + 2$</td>
<td>example 3.4</td>
</tr>
</tbody>
</table>

This suggests the general question of which supercell 2-periodic rigid slabs are incidentally rigid. We continue to conjecture that every incidentally rigid 2-periodic slab contains an incidentally rigid 2-periodic slab with $e_O = 3v_O + 2$. Other questions, analogous to those for 1-periodic columns can be posed and we need more examples to clarify these connections.

4. Fragments of crystals

The approach for columns and slabs extends to finite structures with three independent directions of repetition as fragments of the standard periodic frameworks in 3-space with flexible lattices. The goal of such a study would be to better understand the behaviour of crystal structures as we hold them in our hands. The reader can predict the basic set of definitions, and results, by analogy with the previous two sections. The 3-periodic crystals are based on periodic graphs with an automorphism group of rank 3 and three independent translations.

We do not repeat all the details here but mention that a non-trivial fragment should have three vertices and their translates for each of the three translation vectors, in order to guarantee that a rigid fragment will give a sequentially rigid translation closure.

$$\begin{pmatrix}
\vdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & (**) & 0 & 0 & (**) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots 
\end{pmatrix}.$$  

Rigidity requires $\# \text{ rows} \geq \# \text{ columns} - \dim(\text{trivial motions}) e \geq 3v - 6$.

Example 4.1 (Cube crystal). Consider the lattice of triangulated cubes in figure 13a. This can be analysed as a periodic lattice with only one vertex (figure 13b). With $v_O = 1$ and $e_O = 6 = 3v_O + 3$, it is minimal for periodic rigidity. In fragmentary form, it is a rectangular box of cubes—and also clearly rigid.

Example 4.2 (Twisted crystal). We can also extend the models in example 3.5. This will give a crystal fragment with a growing number of motions—but again in this special geometry the non-trivial motions will be restricted to the boundary of the fragment. This in turn confirms the rigidity of the corresponding infinite structure for incidental periodicity and therefore for supercell periodic motions. Note that there are no rigid fragments of this 3-periodic crystal.
Table 3. Summary of minimal counts for 3-periodic and incidentally 3-periodic crystals.

<table>
<thead>
<tr>
<th>class</th>
<th>forced period</th>
<th>example</th>
<th>supercell period</th>
<th>example</th>
<th>incidental period</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>crystal</td>
<td>$e_O = 3v_O + 3$</td>
<td>example 4.1</td>
<td>$e_O = 3v_O + 3$</td>
<td>example 4.1</td>
<td>$e_O = 3v_O + 3$</td>
<td>example 4.1</td>
</tr>
</tbody>
</table>

For 3-periodic crystals, Borcea & Streinu [1] show that a gain graph with $e_O = 3v_O + 3$ and the appropriate inequality on all subgraphs with at least three vertices has some set of gains $M$ which will make it infinitesimally periodically rigid. In fact, this will then hold for ‘almost all’ gains—but this does not provide a combinatorial test for a particular set of gains which would arise for periodic supercell rigidity, or for incidentally periodic rigidity of the infinite framework. This leaves a number of questions to be answered when looking at a repetitive crystal.

Consider the summary in table 3 for 3-periodic crystals. For crystals, the counts are identical for all three categories of infinitesimal rigidity. This does not imply that the same 3-periodic graphs and gain assignments share these properties, specific geometry or for ‘generic periodic’ choices of the placement. Perhaps in the spirit of Borcea & Streinu [1], the same graphs share these properties for almost all choices of gain. Even if this can be verified, this would leave the challenge of which gains work for which category of rigidity.

5. Further work

This paper offers an initial exploration of 1-periodic cylinders and 2-slabs, and of the connections between fragments of periodic structures (tubes, slabs and crystals) and infinite incidentally periodic structures, with the theory of forced periodic structures and supercell rigid periodic structures as underlying pieces. A number of questions remain. Some additional sources of information and possible extensions are briefly presented below.

(a) Expanding examples and inductive explorations

All the examples we presented had small numbers of vertices per period for the periodic frameworks and corresponding fragments. There are a range of inductive techniques which have been used to build larger finite rigid frameworks from smaller ones—and when these are operations local within a period, they extend directly to forced rigidity of periodic frameworks [26,27]. We anticipate many of these same constructions also extend to the rigidity of incidentally rigid periodic frameworks.

One of these inductive techniques is an extended form of vertex splitting which stretches a polygon around a structure out into a cylindrical tube [13]. This type of operation has been used for tubes and can also be used more generally to open up slabs and crystals. In fact, the shift from a cubic slab to the twisted slab can be identified as a periodic sequence of such cycle splits. We anticipate that this type of inductive construction can be modified and adapted and also to preserve periodic rigidity which are generic within these classes. The modifications will have to permit new vertices that are not fully generic, as we want to preserve the repetitive form of the fragments and in the periodic structures.

(b) Applications

We have focused on two new classes of structures: columns and slabs as well as some variants of more standard three-dimensional structures related to crystals. Some prior examples (e.g. the half-octahedral-tetrahedral truss, beta sheets and Guest and Pelligrino’s cylinders) confirm that these fragments appear in practice. There are also some intriguing connections that come up through google searches, for example, the microtubules.

More generally, there are natural and designed materials, often on a nanoscale, which have the associated repetitive structure, and may well have extra symmetries of the type which are now
open to further investigation. It will be interesting to see what current materials are accessible to new insights with this ‘fragment’ lens, as well as what new materials might be designed to have specific desired flexibility/rigidity properties.

(c) Stiffening on the core

We have given special geometric examples of columns, slabs and crystals in which the fragments have all their non-trivial motions restricted to the boundary. By continuity, small variations of these patterns give fragments which have a ‘stiffening’ of the cores as fixed-sized motions on the boundaries produce smaller and smaller velocities as we move to the core of the fragment.

We cannot claim that this stiffening is ‘typical behaviour’ (almost always happens)—and this question is something to be explored theoretically. It would also be interesting to seek experimental situations in which this can be observed (or disconfirmed) beyond the known stiffening observations for microtubules. The study of fragments and slabs offers an additional perspective on new examples and consider the significance of such special behaviours.

(d) Concluding remarks

We hope that the perspective of fragments, combined with the new basic definitions, and theorems for 1-periodic cylinders and 2-periodic slabs offers a set of new questions for further study. The range of examples presented here is but a portion of those that were created, and built, while working on our initial study. In addition, all of the results for 3-space have analogues for the plane, and extensions for higher dimensions, if those are of interest. We hope that the examples and initial results are suggestive of properties that deserve additional investigation.

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