On the flexibility and symmetry of overconstrained mechanisms

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In kinematics, a framework is called overconstrained if its continuous flexibility is caused by particular dimensions; in the generic case, a framework of this type is rigid. Famous examples of overconstrained structures are the Bricard octahedra, the Bennett isogram, the Grünbaum framework, Bottema’s 16-bar mechanism, Chasles’ body–bar framework, Burmester’s focal mechanism or flexible quad meshes.

The aim of this paper is to present some examples in detail and to focus on their symmetry properties. It turns out that only for a few is a global symmetry a necessary condition for flexibility. Sometimes, there is a hidden symmetry, and in some cases, for example, at the flexible type-3 octahedra or at discrete Voss surfaces, there is only a local symmetry. However, there remain overconstrained frameworks where the underlying algebraic conditions for flexibility have no relation to symmetry at all.

1. Introduction

Let \( F = (V, E) \) be a bar-and-joint framework in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) with the vertex set

\[
V = \{ x_1, \ldots, x_v \}, \quad x_i \in \mathbb{R}^d \quad \text{for all } i \in I = \{1, \ldots, v\}
\]

and the edge set

\[
E \subset \{(i, j) \mid i < j, (i, j) \in I\}.
\]

We denote the edge lengths by

\[
l_{ij} := \| x_i - x_j \| \quad \text{for all } (i, j) \in E.
\]

We call each \( v \)-tupel \((x'_1, \ldots, x'_v) \in \mathbb{R}^{vd} \) with the same edge lengths \( l_{ij} \) for all \((i, j) \in E \) a realization of \( F \), and of course we are interested in mutually incongruent realizations (table 1).

\( F \) is called continuously flexible or—in short—flexible, if its spatial form can be changed analytically with respect
This is also valid for singular \( \alpha \) located on the same second conic of the confocal set (note the hyperbolas challenge is to find the geometrical meaning of the algebraic conditions.

2. Flexible bipartite frameworks and Ivory’s theorem

A framework is called bipartite if its edge graph is bipartite, i.e. its vertices (or knots) can be subdivided into two classes \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \) such that the edges (or bars) always connect vertices from different classes. It turns out that the flexibility of bipartite frameworks is always related to Ivory’s theorem [1].

To recall, the two-dimensional version of Ivory’s theorem states that in the orthogonal net of confocal conics each curvilinear quadrangle has diagonals of equal length (figure 1a). In other words, if \( \alpha \) is an affine transformation mapping the conic \( k \) onto a confocal conic \( k' \), whereas the axes of symmetry are kept fixed, then

\[
\|\alpha(x) - y\| = \|x - \alpha(y)\| \quad \text{for all } x, y \in k.
\]

This is also valid for singular \( \alpha \). It turns out that corresponding points \( x \) and \( \alpha(x) \) are always located on the same second conic of the confocal set (note the hyperbolas \( h_1, h_2 \) in figure 1a).

In figure 1b, it is shown how by Ivory’s theorem a planar bipartite framework with \( a_i \in k \) and \( b_j \in k' \) can be transformed into an incongruent realization with \( a_i' \in k' \) and \( b_j' \in k \).

In addition, the converse statement is true—and even in each dimension [2]: for any two incongruent realizations in \( \mathbb{E}^d \), there is a displacement of one of them such that finally the two
realizations are in ‘Ivory position’, i.e. the vertices $a_1, \ldots, a_m, b'_1, \ldots, b'_n$ and $a'_1, \ldots, a'_m, b_1, \ldots, b_n$ are placed on two confocal quadrics of the same type, respectively, and the pairs $a_i \mapsto a'_i$ and $b'_j \mapsto b_j$ are corresponding under an affine transformation. Confocal quadrics in $\mathbb{E}^d, d \geq 3$, are characterized by confocal sections with all hyperplanes of symmetry.

(a) Dixon’s flexible frameworks

Dixon [3] proved in 1899 that there are exactly two types of continuously flexible bipartite frameworks in the Euclidean plane $\mathbb{E}^2$. A new proof based on algebraic methods can be found in [4].

At type I (figure 2a), the two classes of vertices are placed on two orthogonal lines, for example, the two axes of a Cartesian coordinate system. For any given real constant $c$ sufficiently close to 0 the transformation

$$a_i = (x_i, 0) \mapsto a'_i = \left( \sqrt{x_i^2 + c}, 0 \right), \quad b_j = (0, y_j) \mapsto b'_j = \left( 0, \sqrt{y_j^2 - c} \right)$$

preserves all distances $\|a_i - b_j\| = \sqrt{x_i^2 + y_j^2} = \|a'_i - b'_j\|$.

At the second type of flexible bipartite planar framework (figure 2b), the symmetry is essential: the vertices of two rectangles with two common axes of symmetry constitute the classes of vertices. Again, Ivory’s theorem can be used to prove the flexibility (figure 3a).
There is a one-parameter set of conics $k$ passing through $a_1, \ldots, a_4$. They all have common axes of symmetry. For each $k$, there is a confocal conic $k'$ through $b_1, \ldots, b_4$. Hence, by Ivory’s theorem, we can switch to conjugate points thus obtaining a one-parameter set of incongruent realizations of the same framework.

Ivory’s theorem is also true on the sphere $S^2$ (figure 3b). Therefore, the spherical version of the Dixon II framework is again flexible. It is called Bottema’s 16-bar framework [5,6].

(b) Flexible bipartite frameworks in three-space

There is a series of flexible bipartite frameworks in three-space that can be seen as spatial analogues of the planar Dixon frameworks:

— Let $a_1, \ldots, a_{16}$ be corners of two boxes and $b_1, \ldots, b_8$ be corners of a third box such that all three boxes are symmetric with respect to a Cartesian frame [7].
— The vertices $a_1, \ldots, a_m$ are specified on two conics which are located in parallel planes and symmetric with respect to a Cartesian frame, whereas $b_1, \ldots, b_8$ are the corners of a box symmetric with respect to the same frame [7].
— $a_1, \ldots, a_m$ are located on a conic, and $b_1, \ldots, b_n$ are arbitrary points in three-space [8].
— When both classes $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ are coplanar and their carrier planes are orthogonal then the degree of freedom of the bipartite framework is at least $k = 3$ [7].

(c) Henrici’s flexible hyperboloid and Bricard’s octahedra

At the end of this section, we focus on two flexible structures that are related to the bipartite frameworks mentioned before: for any two confocal one-sheet hyperboloids, the affinity according to Ivory’s theorem (analogue to figure 1a) preserves distances along the generators. This is the basis for Henrici’s flexible hyperboloid [9,10].

An arbitrary number of generators of both reguli of a hyperboloid is materialized by rods (figure 4) with a spherical joint at each point of intersection between two rods. In the flat limiting poses, the rods are either tangent to the focal hyperbola or tangent to the focal ellipse, the singular surfaces in the range of confocal hyperboloids. Note that this is no more a pure framework, but there are hidden constraints as the rods remain aligned during the self-motion.

According to Bricard [11], there are exactly three types of flexible octahedra. Octahedra of type 1 have a line symmetry (figure 5) and those of type 2 a planar symmetry with exactly two vertices
Figure 4. Henrici’s flexible hyperboloid (courtesy: G. Glaeser).

Figure 5. Two particular examples of flexible octahedra where two faces are omitted. Both have an axial symmetry ((a) [12, p. 77]; (b) [13]).

located in the plane of symmetry. In both cases, the flexibility can be proved using symmetry arguments, only.

Figure 5 shows two particular examples of flexible octahedra of type 1: in both cases, the faces \( b_1 b_2 a_2 \) and \( b_3 b_4 a_1 \) have been omitted in order to avoid self-intersections. The example in figure 5a is due to Blaschke [12]; the displayed pose is related to a cube. The example in figure 5b is related to a regular octahedron and has been found by Wunderlich [13]. This example admits two flat poses. In figure 6, the unfoldings of these flexible examples are displayed.

The definition of Bricard’s flexible octahedra of type 3, which have no global symmetry, is more complex [14]. One approach is as follows. The octahedron can be seen as a bipartite framework. The apices \( a_1, a_2 \) of the double-pyramid define one class, the vertices \( b_1, \ldots, b_4 \) of the common base quadrangle the other. Therefore, again Ivory’s theorem can be used to prove that Bricard’s examples are the only octahedra which are flexible [15]. Nawratil [16] used a different approach when he recently determined all flexible octahedra with one or more vertices at infinity. One type among them is even free of self-intersections.

Figure 7 shows an arbitrary pose \( a_1, \ldots, b_4 \) with the quadrangular basis on a one-sheet hyperboloid and as second realization a flat pose \( a'_1, \ldots, b'_4 \) where the sides of the quadrangle are tangent to the focal ellipse \( e' \) of the hyperboloid. This flat pose cannot be chosen arbitrarily, but for a given quadrangle \( b'_1, \ldots, b'_4 \), the apices \( a'_1, a'_2 \) must be located on a particular algebraic curve of degree 3, a strophoid [17, theorem 2]. This curve is the locus of focal points of conics tangent to the sides of \( b'_1, \ldots, b'_4 \). The same curve plays a role in Burmester’s focal mechanism (note figure 11).
Figure 6. Nets of the flexible octahedra displayed in figure 5.

Figure 7. Bricard’s flexible octahedron of type 3 and Ivory’s theorem.

Ivory’s theorem is also true in pseudo-Euclidean spaces as well as in spaces of constant curvature [18]. Therefore, there are analogues of the flexible bipartite frameworks in other spaces. For example, the analogues of the three types of flexible octahedra exist also in the hyperbolic three-space [19].

3. Particular overconstrained flexible mechanisms

Here, we present a medley of different flexible structures. Not all are pure frameworks; sometimes, there are hidden constraints such as collinearities or coplanarities of vertices.

(a) Grünbaum’s framework

The initial pose of Grünbaum’s framework is highly symmetrical. It admits the full icosahedral group because it consists of the edges of the 10 regular tetrahedra which can be inscribed into a regular pentagon–dodecahedron. With respect to the dodecahedron, the tetrahedra can be divided into two classes, the left ones $L_1, \ldots, L_5$ and the right ones $R_1, \ldots, R_5$. We choose the indices such that $L_i$ and $R_i$ are complementary tetrahedra inscribed into the same cube.
Each vertex of the dodecahedron is shared by two cubes and two tetrahedra. The left $L_i$ is inscribed into the cube which contains the left diagonals of the adjacent pentagons. The right $R_j$ is inscribed into the other cube. We use the ordered pair of indices $ij$, $i \neq j$, as label of this vertex.

Grünbaum’s framework consists of 20 knots and 60 bars. It is at the same time a body–bar framework, but the included 10 tetrahedra penetrate each other. Though equation (1.1) gives $k = -6$, this framework is flexible. There are two types of one-parameter self-motions which preserve one rotational symmetry.

According to Connelly et al. [20], the tetrahedron $L_1$ can perform a one-parameter motion such that its vertices 12, 13, 14 and 15 remain in the planes $\varepsilon_2, \ldots, \varepsilon_4$ of symmetry through the face axis $f$, respectively (figure 8a). By iterated rotations about $f$ and reflections in planes through $f$, the movement can be continued to all other tetrahedra of the framework. Figure 9a shows the traces of the vertices of $L_1$ under this self-motion which is rational of degree 4 and of type (a) according to the classification given in [21].

Figure 8. (a,b) The two types of one-parameter self-motions of Grünbaum’s framework.

Figure 9. (a) Movement of $L_1$ under a self-motion of the Grünbaum framework preserving the fivefold symmetry about a face axis. (b) Pose of bifurcation into a self-motion with d.f. = 2.
Chasles’ body–bar framework; $a_i \mapsto a'_i$ is a projectivity between the conics $k$ and $k'$.

In a real-world model, the movement stops when different vertices come together. However, one can dissolve some joints, and after reassembling, the structure continues its movement. The 10 tetrahedra fall apart and form a ring. Then, they reach a pose with coinciding pairs of tetrahedra [22]. Here, the five doubled tetrahedra sit on the faces of a flexible five-sided pyramid without basis (figure 9b). This allows a bifurcation from the one-parameter symmetry-preserving self-motion into a two-parameter motion, which is totally unsymmetric. It is surprising that in this framework the degree of freedom increases by waiving the symmetry. However, it makes sense to see pairwise coinciding tetrahedra as particular case of a local symmetry.

In figure 8b, another one-parameter self-motion of the Grünbaum framework is displayed [23]: this time, the threefold symmetry with respect to a vertex axis $v$ is preserved. The movement of $L_1$ is such that $12$ and $13$ remain in the planes $\phi_2$, $\phi_3$ of symmetry, respectively, whereas $14$ and $15$ preserve their distance to the axis $v$. At the same time, the vertices $45$ and $54$ remain on the axis $v$ which therefore is a fixed axis of symmetry for all poses of $L_4, R_4, L_5$ and $R_5$. $120^\circ$ rotations about $v$ and reflections in $\phi_1, \phi_2$ and $\phi_3$ define the positions of all other tetrahedra.

(b) Chasles’ body–bar framework

This framework consists of two planar bodies $\Sigma, \Sigma'$ in $\mathbb{E}^3$ and $n$ connecting bars. The pairwise different anchor points $a_1, \ldots, a_n \in \Sigma$ and $a'_1, \ldots, a'_n \in \Sigma'$ are projectively related and lie on conics $k \subset \Sigma$ and $k' \subset \Sigma'$, respectively (figure 10). When each anchor point represents a spherical joint between bar and body, then a parameter count gives $k = 6 - n$ as degree of freedom.

Chasles [24] recognized with methods of statics that under these conditions for any $n$ there exists a spatial motion of $\Sigma$ relative to $\Sigma'$ which keeps the distances $\|a_i - a'_i\|$ fixed for all $i \in \{1, \ldots, n\}$. Bricard [25, p. 3, footnote 2] emphasized the kinematic meaning of Chasles’ statement—without giving any proof.

We can prove the flexibility by the fact that (in the generic case) the lines connecting corresponding points of two projectively related conics constitute an algebraic ruled surface of degree 4. Such a ruling is always included in a linear line-complex, an argument which is also stressed by Wohlhart [26]. The existence of such a linear line-complex is necessary and sufficient for infinitesimal flexibility. Because each pose of our framework is infinitesimally flexible, we obtain by integration continuous flexibility, provided there is no still stand in the initial pose, e.g. caused by two aligned bars.

For $n = 6$, Chasles’ mechanism is overconstrained. This is also important for robotics, because after replacing the six bars $a_1a'_1$ by telescopic legs, our framework becomes a particular parallel manipulator, a planar Stewart Gough platform (SGP). When at a planar SGP all pairs $a_i \mapsto a'_i$,
Figure 11. Burmester’s focal mechanism; the point triples \((A_0, A', A), (B_0, B', B)\) and \((A, C', B)\) remain collinear during the self-motion.

\[ i = 1, \ldots, 6, \] are corresponding under a projectivity between two conics \(k, k'\), the SGP is singular in each pose, hence architecturally singular. A classification of all architecturally singular SGPs has been given by Karger [27].

The following statement has been presented in [28]: each planar SGP can be extended by additional legs without restricting its mobility. The method applied in the proof of [28] can also be used here to prove the continuous flexibility of Chasles’ framework for any \(n\) directly, i.e. not via infinitesimal flexibility.

(c) Burmester’s focal mechanism

For each four-bar \(A_0B_0BA\) (figure 11), there are points \(F\) such that additional bars connecting \(F\) with appropriate intermediate points on the sides do not restrict the flexibility. This has been found by Burmester [29], and he called it a focal mechanism because point \(F\) must be a focal point of any conic tangent to the sides of the quadrangle \(A_0B_0BA\). The locus of these focal points has already been mentioned before in connection with the flat poses of flexible type-3 octahedra.

Dixon [3] proved that the angle \(\psi = \angle A_0A'F\) is congruent to the angle \(\angle BB'F\). For further relations, see [30, pp. 125–130]. The angles at \(F\) are congruent to the interior angles of the quadrangle, e.g. \(\angle C_0FB' = \angle C_0A_0A\) (figure 11). Hence, the four interior angles in the quadrangle must sum up to 360°, and therefore this mechanism has no spherical analogue! Only the property is preserved on the sphere that under the Dixon condition the composition of the two connected four-bars \(A_0C_0FA'\) and \(C_0B_0BF\) is reducible [31] (compare figure 14b).

(d) The Bennett isogram

According to Bennett [32], there is a flexible kinematic chain consisting of four cyclically ordered systems \(\Sigma_1, \ldots, \Sigma_4\) with revolute joints between any two consecutive bodies. When on each revolute axis two points are fixed, and each body is replaced by a tetrahedron, we obtain a flexible framework with eight vertices and 20 bars. Equation (1.1) gives \(d.f. = 24 - 20 - 6 = -2\).

We give a short proof for its flexibility and start with a skew isogram, i.e. a non-planar quadrangle \(abcd\) where opposite sides are of equal length,

\[ a := \|a - b\| = \|c - d\|, \quad b := \|b - c\| = \|d - a\|. \]

We obtain it from a planar parallelogram by bending about one diagonal through the signed angle \(\gamma\) (figure 12a).

Each skew isogram has a line-symmetry: a rotation through 180° (half-rotation) about the line \(m\) connecting the midpoints of the diagonals exchanges \(a\) with \(c\) as well as \(b\) with \(d\). This can
be concluded from the congruence of the two triangles dab and bcd. It can also be proved by computation as follows: the sum and the difference of the two equations

\[(a - b)^2 - (c - d)^2 = 0 \quad \text{and} \quad (b - c)^2 - (d - a)^2 = 0\]

yields after a factorization like \(x^2 - y^2 = (x + y)(x - y)\)

\[(a - c) \cdot (a - b + c - d) = (b - d) \cdot (a - b + c - d) = 0.\]

This expresses directly that the connection \(m\) of the midpoints of the two diagonals is orthogonal to both diagonals.

The convex hull of this skew isogram is a tetrahedron with the basis dab and the apex c. Let \(\alpha\) and \(\beta\) denote the (signed) dihedral angles along the base-edges ab and ad, respectively (figure 12a). When in the congruent triangles abc or cda, the heights on the sides with lengths \(a\) and \(b\) are denoted by \(h_a\) and \(h_b\), respectively, then the height of the apex c over the base plane can be expressed in two ways as

\[h_a \sin \alpha = h_b \sin \beta, \quad \text{while} \quad ah_a = bh_b.\]

Both sides of the second equation give the doubled area of the triangles mentioned before. After elimination of \(h_a\) and \(h_b\), we get the basic relation

\[a \sin \beta = b \sin \alpha.\]

There is a two-parameter set of mutually incongruent skew isograms sharing the lengths \(a\) and \(b\), because the lengths of the diagonals can be chosen independently—within certain limits. We extract a one-parameter set by keeping the dihedral angle \(\alpha\) fixed. According to the basic relation, \(\beta\) also remains constant. Because \(\alpha\) and \(\beta\) are also the angles between normals of the tetrahedron (figure 12a), our one-parameter set gives flexions of a kinematic revolute chain with four links (figure 12b). Each side of our isogram represents one link \(\Sigma_i\), when at each endpoint the common perpendicular with the neighbouring side serves as the axis of rotation \(I_{ii+1}\) with respect to the neighbour link \(\Sigma_{i+1}\).

According to [33], the Bennett isogram is the only flexible 4R kinematic chain. When \(\Sigma_1\) is kept fixed, then point \(d \in \Sigma_4\) remains on a circle with centre a and axis \(I_{41}\) (figure 12b). On the other hand, according to the rotations about \(I_{21}\) and \(I_{32}\), point \(d \in \Sigma_3\) must be placed on a rotational cyclide [34,35]. Therefore, the mobility of the Bennett isogram is also related to the existence of different families of circles on rotational cyclides.

According to Krames [36], the relative motion between opposite links is line-symmetric; the axes \(m\) of the half-rotations form a regulus of a hyperboloid. The kinematic image of this motion on the study-quadric is a conic [33,37]. Therefore, this motion also serves as a sort of ‘primitive’ in motion design [38].
Figure 13. Kokotsakis mesh for $n = 4$.

(e) 6R kinematic chains

There is a list of about 30 flexible closed 6R chains, including symmetric and non-symmetric ones. They all are overconstrained as equation (1.1) gives $k = 0$.

Recently, Hegedüs et al. [39] could show that the flexibility is related to a factorization of a monic cubic polynomial $P(t)$ over the ring of dual quaternions into linear factors:

$$P(t) = (t - h_1)(t - h_2)(t - h_3),$$

where $h_1, h_2, h_3$ represent rotations. This most interesting result offers a new strategy to find flexible examples, and it reveals clearly that not symmetry but an algebraic property is decisive for continuous flexibility.

4. Flexible Kokotsakis meshes

The following structure is named after Kokotsakis [40,41]: a Kokotsakis mesh is a polyhedral structure consisting of an $n$-sided central polygon $f_0$ surrounded by a belt of polygons (figure 13). Each side $a_i$, $i = 1, \ldots, n$, of $f_0$ is shared by a polygon $f_i$. At each vertex $V_i$ of $f_0$, four faces are meeting.

Each face is a rigid body; only the dihedral angles between adjacent faces can vary. An open problem until recently is: under which conditions is a Kokotsakis mesh with $n \geq 4$ continuously flexible? The answer to this question is of course basic for classifying the flexible compounds of Kokotsakis meshes like quad meshes [42] and for rigid origami, i.e. for exact paper folding [43]. In the case $n = 3$, the posed problem is equivalent to the classification of flexible octahedra (see [16]).

Let us focus on the case $n = 4$. In the kinematic sense, the polygons $f_0, \ldots, f_4$ represent different systems $\Sigma_0, \ldots, \Sigma_4$. We keep $f_0 \subset \Sigma_0$ fixed. The sides $a_i$ of $f_0$ are instantaneous axes $l_i$ of the relative motions $\Sigma_i / \Sigma_0, i = 1, \ldots, 4$. The relative motion $\Sigma_{i+1} / \Sigma_i$ between consecutive systems is a spherical four-bar mechanism.

The lengths of the sides $a_1, \ldots, a_4$ of $f_0$ have no influence on the flexibility of the mesh, only their directions. Hence, we can draw the parallels to all axes through a fixed point in space thus obtaining the spherical image. A Kokotsakis mesh is flexible if and only if its spherical image is flexible.

The faces meeting at $V_i$ correspond in the spherical image to a spherical four-bar. The interior angles $\alpha_i, \beta_i, \gamma_i, \delta_i$ at $V_i$ are equal to the side lengths of the corresponding four-bar (figure 14). A rotation of $f_1$ about $a_1 = l_{10}$ through the angle $\varphi_1$ with respect to $f_0$ corresponds in the spherical
image to a rotation of the bar $I_{10}A_1$ about $I_{10}$ through $\phi_1$. The transmission via $V_1$ to $f_2$ corresponds to the transmission on the sphere via the coupler $A_1B_1$ to the rotation of the bar $I_{20}B_1$ about $I_{20}$ through $\phi_2$.

A Kokotsakis mesh for $n = 4$ is flexible if and only if the transmission from $f_1$ to $f_3$ via $V_1$ and $V_2$ shares a component with the transmission via $V_4$ and $V_3$. In this case, the transmission from the input angle $\varphi_1$ to the output angle $\varphi_3$ of $f_3$ can be decomposed in two ways by two spherical four-bars. Figure 15 shows an example. The composition of the four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ is the same as the product of $I_{10}A_1'B_1''I_{20}$ and $I_{30}A_2'B_2''I_{40}$. As a control, there is a second pose depicted in light blue. By the way, the marked angle $\psi$ is the spherical analogue to the Dixon angle $\psi$ marked in figure 11.

For $n = 4$, there are five types of flexible Kokotsakis meshes known until recently [44]. In the following, we use the term fold for triples of edges where at each vertex $V_i$ opposite edges are combined. In view of figure 13, we can distinguish between two horizontal folds (one includes $V_4$, $V_1$, the other through $V_2$, $V_3$) and two vertical folds (one through $V_4$, $V_3$, the other through $V_1$, $V_2$).

I. Planar-symmetric type: the Kokotsakis mesh has a planar symmetry exchanging $V_1$ with $V_4$ and $V_2$ with $V_3$.

II. Translational type: there is a translation with $V_1 \mapsto V_4$ and $V_2 \mapsto V_3$ mapping $f_2$ to $f_4$.

III. Isogonal type [40]: a Kokotsakis mesh is flexible when at each vertex $V_i$ opposite angles are either equal or complementary, i.e.

\[
\alpha_i = \beta_i, \quad \gamma_i = \delta_i \quad \text{or} \quad \alpha_i = \pi - \beta_i, \quad \gamma_i = \pi - \delta_i.
\]

In this case, the transmission $\varphi_i \mapsto \varphi_{i+1}$ by the spherical four-bars splits into two linear functions, when expressed in terms of the half-angle tangents:

\[
\tan \frac{\varphi_{i+1}}{2} = f_{i+1} \tan \frac{\varphi_i}{2} \quad \text{with} \quad f_{i+1} = \frac{\sin \alpha_i \pm \sin \gamma_i}{\sin(\alpha_i - \gamma_i)}.
\]

Additionally, the product of factors must obey $f_2f_3f_4f_5 = 1$. This condition is fulfilled as soon as there exists a pose which is non-flat at each vertex $V_1, \ldots, V_4$. A discrete Voss surface is a composition of flexible Kokotsakis meshes of this type [42], e.g. Miura-ori. By the way, also at flexible type-3 octahedra (figure 7), the quadruples of faces at each vertex $a_1, \ldots, b_4$ form an isogonal pyramid; hence, there is a local symmetry though these
Figure 15. The transmission from $\varphi_1$ to $\varphi_3$ by the four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ has a common component with that by $I_{10}A_1'B_1'I_{20}$ and $I_{20}A_2'B_2'I_{30}$.

Octahedra are globally unsymmetric. According to Nawratil [45], there is a generalized flexible isogonal type: even if only at two of the four vertices $V_1, \ldots, V_4$ opposite angles are equal or complementary, it can happen that one component of the transmission $\varphi_1 \mapsto \varphi_3$ is decomposable in two ways.

IV. Orthogonal type or T-flat [41]: here the horizontal folds are located in parallel (say horizontal) planes and the vertical folds in vertical planes.

V. Line-symmetric type [44]: a half-rotation maps the pyramid at $V_1$ onto that of $V_4$; another one exchanges the pyramids at $V_2$ and $V_3$. Additionally, $\delta_1 + \delta_2 = \alpha_1 + \beta_2$ must hold together with the ‘Dixon angle condition’ (note angle $\psi$ at $A_1$ and $B_2$ in figure 15)

$$
\sin \alpha_1 \sin \gamma_1 : \sin \beta_1 \sin \delta_1 : (\cos \alpha_1 \cos \gamma_1 - \cos \beta_1 \cos \delta_1) = \pm \sin \beta_2 \sin \gamma_2 : \sin \alpha_2 \sin \delta_2 : (\cos \alpha_2 \cos \delta_2 - \cos \beta_2 \cos \gamma_2).
$$

According to Kokotsakis [40,42], the tessellation of the plane by congruent convex quadrangles, generated by iterated point reflections, yields a flexible quad mesh [46]. Here, each included Kokotsakis mesh is of type V.

5. Conclusion

We presented a list of overconstrained frameworks and related flexible structures and we explained why they are flexible. Of course, we did not claim completeness. Personal interest and some common properties were the only motivation for selecting only these examples and not others.

Sometimes, the flexibility is a consequence of a symmetry, and the self-motion preserves this symmetry, e.g. Dixon I, Grünbaum’s framework, Bricard’s flexible octahedra of types 1 and 2, and the Bennett isogram. However, in the majority of cases, particular algebraic properties are responsible for the fact that a structure with particular dimensions is continuously flexible though in the generic case the structure is rigid.

Some of the presented examples date back to the nineteenth century; others have been detected recently. It turned out that in many cases the flexibility can be concluded from Ivory’s theorem.
As this holds for each dimension and in different metrics, the related mechanisms often have analogues in other spaces. But there are also examples without any counterparts in other spaces, e.g. Burmester’s focal mechanism.

Generally speaking, there is no ‘kings-road’ for proving the flexibility of overconstrained structures. Different types require different methods. But this is always a challenge for kinematicians.

Funding statement. This research is supported by grant no. I 408-N13 of the Austrian Science Fund FWF within the project ‘Flexible polyhedra and frameworks in different spaces’, an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

References