Thin aerofoils are prone to localized flow separation at their leading edge if subjected to moderate angles of attack $\alpha$. Although ‘laminar separation bubbles’ at first do not significantly alter the aerofoil performance, they tend to ‘burst’ if $\alpha$ is increased further or if perturbations acting upon the flow reach a certain intensity. This then either leads to global flow separation (stall) or triggers the laminar–turbulent transition process within the boundary layer flow. This paper addresses the asymptotic analysis of the early stages of the latter phenomenon in the limit as the characteristic Reynolds number $\text{Re} \to \infty$, commonly referred to as marginal separation theory. A new approach based on the adjoint operator method is presented that enables the fundamental similarity laws of marginal separation theory to be derived and the analysis to be extended to higher order. Special emphasis is placed on the breakdown of the flow description, i.e. the formation of finite-time singularities (a manifestation of the bursting process), and on its resolution being based on asymptotic arguments. The passage to the subsequent triple-deck stage is described in detail, which is a prerequisite for carrying out a future numerical treatment of this stage in a proper way. Moreover, a composite asymptotic model is developed in order for the inherent ill-posedness of the Cauchy problems associated with the current flow description to be resolved.

1. Introduction

The investigation of laminar–turbulent boundary layer transition is of fundamental importance in respect of the understanding of the complicated structure of turbulence and also to develop appropriate engineering models for the prediction of flow characteristics. Of crucial
theoretical as well as practical interest is the accurate calculation of lift and drag forces acting on aerodynamic bodies, which requires comprehensive knowledge on whether the flow is laminar or turbulent, attached or separated. The theory of boundary layer flows and, in particular, the examination of its most important issues, namely separation and transition when the Reynolds number is asymptotically large, have been a field of active research since Prandtl [1] presented his theory for laminar steady two-dimensional flows in 1904. Cornerstones of that development are, among others, the discovery of the singular behaviour and breakdown of the classical boundary layer equations near a point of vanishing skin friction (separation point in the case of steady flows) by Landau & Lifshitz [2] and Goldstein [3], and that of viscous–inviscid interaction independently made by Stewartson [4], Messiter [5] and Neiland [6] in the late 1960s, which has generally become known as the triple-deck theory. In conventional triple-deck problems, abrupt changes of boundary conditions or singular behaviour of the imposed pressure gradient initiate the interaction mechanism.

On the contrary, in cases of the so-called marginal separation, a (moderate) increase in the smooth imposed adverse pressure gradient controlled by a characteristic parameter leads to the onset of the interaction process and in further consequence to localized separation. In the early 1980s, Ruban [7,8] and Stewartson, Smith and Kaups [9] formulated a rational description of the local interaction mechanism now commonly referred to as the theory of marginal separation. It serves as the foundation of this work, which deals with the investigation of the early stages of the transition process triggered by the presence of laminar separation bubbles. As is well known, the theory of marginal separation predicts an upper bound of the control parameter for the existence of strictly steady, i.e. unperturbed, flows. The incorporation of unsteady effects led to the conclusion that the onset of the bursting process is associated either with exceeding the critical value of the control parameter or with the presence of a sufficient perturbation level in the case of below-critical conditions. Within the framework of the existing theory, vortex shedding from the rear of the separation bubble manifests itself in the occurrence of a finite-time singularity. Surprisingly, in that case recent findings strongly suggest the development of a unique blow-up pattern in leading order, entirely independent of the previous history of the flow [10]. The associated breakdown of the flow description implies the emergence of shorter scales, and the subsequent evolution of the flow then is described by a fully nonlinear triple-deck interaction, which seems to suffer finite-time breakdown as well [11,12]. The tracking of this ‘breakdown cascade’ is of particular interest and a main focus of the present investigation since it reflects the successive genesis of shorter spatio-temporal scales, which is a distinctive feature of the vortex generation process in transitional flows.

To highlight the main issues, we restrict ourselves to the most simple case of planar incompressible flows. Furthermore, it is assumed that the reader is familiar with the basic concept of marginal separation theory, which is well established. A very detailed description can be found in for example [13]. Our main focus is placed on the extension of the existing theory to higher orders (§2) for the purpose of formulating proper initial conditions for the triple-deck stage (§3) and addressing the observed ill-posedness of initial value problems associated with the current asymptotic flow description (§4).

2. Adjoint operator method

In the following study, we reinvestigate the fundamental lower-deck (LD) problem of marginal separation theory. Here, however, we use an alternative method to that employed in the original studies [7–9,14] to derive the well-known similarity laws and their higher-order corrections, which govern the flow in the sublayer region LD (surrounded by a solid line in figure 1). To this end, non-dimensional quantities are introduced in the form

\[
(\hat{x}, \hat{y}) = \left( \frac{\tilde{x}}{\tilde{L}}, \frac{\tilde{y}}{\tilde{L}} \right), \quad \hat{t} = \frac{\tilde{t}}{\tilde{L}}, \quad p = \frac{\tilde{p} - \tilde{p}_\infty}{\tilde{\rho}_\infty \tilde{u}_\infty^2} \quad \text{and} \quad \psi = \frac{\tilde{\psi}}{\tilde{u}_\infty \tilde{L}},
\]  

(2.1)
where \( \hat{x} \) and \( \hat{y} \) denote the coordinates in the streamwise direction and normal to the wall, \( t \) the time, \( p \) the pressure and \( \psi \) the stream function. The specific flow problem under consideration is characterized by the dimensional reference quantities: length \( \tilde{L} \), velocity \( \tilde{u}_\infty \), pressure \( \tilde{p}_\infty \), density \( \tilde{\rho}_\infty \) and kinematic viscosity \( \tilde{\nu}_\infty \), respectively (i.e. the unperturbed free stream values).

Particularly, we are interested in the behaviour of the boundary layer characteristics, i.e. displacement thickness \( \delta^* (\hat{x}) \) and wall shear stress \( \tau_w(\hat{x}) \),

\[
\delta^* = \int_0^\infty \left( 1 - \frac{U}{u_w} \right) dy_m \quad \text{and} \quad \tau_w = \frac{\partial U}{\partial y_m} \bigg|_{y_m=0} = Re^{-1/2} \frac{\partial^2 \psi}{\partial \hat{y}^2} \bigg|_{\hat{y}=0},
\]

in the vicinity of a laminar separation bubble in the limit as the characteristic Reynolds number \( Re \) of the flow problem tends to infinity,

\[
Re = \frac{\tilde{u}_\infty \tilde{L}}{\tilde{v}_\infty} \rightarrow \infty.
\]

Here, \( u_w(\hat{x}) \) is the velocity of the outer inviscid flow at the solid wall, \( y_m \) the boundary layer (wall normal) coordinate and \( U \) the velocity distribution in the viscous boundary layer in the \( \hat{x} \)-direction, which obeys the matching condition \( U(\hat{x}, y_m \to \infty) \to u_w(\hat{x}) \). Furthermore, the usual scalings

\[
y_m = Re^{1/2} \frac{\hat{y}}{\tilde{L}}, \quad (U, u_w) = \left( \frac{\tilde{U}}{\tilde{u}_\infty}, \frac{\tilde{u}_w}{\tilde{u}_\infty} \right), \quad \delta^* = Re^{1/2} \frac{\tilde{\delta}^*}{\tilde{L}} \quad \text{and} \quad \tau_w = Re^{1/2} \frac{\tilde{\tau}_w}{\tilde{\rho}_\infty \tilde{u}_\infty^2}
\]

have been used.

The expansions of the stream function and the pressure gradient in terms of the perturbation parameter

\[
\epsilon := Re^{-1/20} \to 0
\]

according to the original papers [7–9] for planar flow in the LD region in essence read

\[
\psi = \epsilon^{13} p_{00} \frac{\Psi^3}{6} + \epsilon^{16} \psi_1 + \epsilon^{19} \psi_2 + \epsilon^{22} \psi_3 + \cdots
\]

and

\[
\frac{\partial p}{\partial \hat{x}} = p_{00} + \epsilon^4 p_{01} \hat{x} + \epsilon^6 \frac{\partial p_1}{\partial \hat{x}} + \epsilon^9 \frac{\partial p_2}{\partial \hat{x}} + \cdots
\]
Here, the suitably scaled independent variables are denoted by
\[ x = e^{-4(\bar{x} - \bar{x}_0)}, \quad y = e^{-11\bar{y}} \text{ and } t = \bar{t}, \]
and the origin \( \bar{x}_0 \) of the streamwise coordinate is chosen such that it coincides with the point where, according to classical boundary layer theory, the wall shear stress vanishes (and immediately recovers downstream) as the parameter \( \alpha \) controlling separation attains its critical value \( \alpha_c \). Furthermore, \( p_{00}y^3/6 \) represents the separation profile, where \( p_{00} > 0 \) is the imposed leading-order adverse pressure gradient; \( \psi_n(x,y,t) \) and \( \nu_n(x,t) \) characterize the perturbation stream functions and the induced pressures to be determined at the levels \( n=1,2,\ldots \) of the approximation.

Substitution of (2.6) into the Navier–Stokes equations yields to leading order
\[ \mathcal{L}\psi_1 := \frac{\partial^3 \psi_1}{\partial y^3} - p_{00} \frac{y^2}{2} \frac{\partial^2 \psi_1}{\partial x \partial y} + p_{00} y \frac{\partial \psi_1}{\partial x} = 0, \]
(2.8)
supplemented with the no-slip condition \( \psi_1 = \partial \psi_1 / \partial y = 0 \) at \( y = 0 \). In order to close the boundary value problem for (2.8), it is sufficient to require that \( \psi_1 \) does not show exponential growth as \( y \to \infty \). Then \( \psi_1 = A_1 y^2/2 + \cdots \) as \( y \to \infty \) follows from (2.8), where the displacement function \( A_1(x,t) \) remains arbitrary at this stage. As a consequence, the homogeneous solution (eigenfunction) of (2.8) is \( \psi_1 = A_1 y^2/2 \). The function \( A_1 \) is related to the displacement thickness and the wall shear stress (2.2) via the expansions
\[ \delta^s \sim \delta^s(x_0) + Re^{-1/5} [q_1 A_1(x,t) + q_2 x] + \cdots \quad \text{and} \quad \tau_w \sim Re^{-1/5} q_3 A_1(x,t) + \cdots \]
in the limit as \( \bar{x} \to \bar{x}_0 \) and \( \alpha \to \alpha_c \). Here, \( \delta^s(x_0) \) represents the value of the displacement thickness according to classical boundary layer theory in the limit \( \alpha = \alpha_c \) at \( \bar{x} = \bar{x}_0 \), and the values of the constants \( q_1 < 0, q_2, q_3 > 0 \) depend on the specific flow problem under consideration.

According to Fredholm’s alternative, solutions of the inhomogeneous higher-order problems
\[ \mathcal{L}\psi_n = b_n, \quad n = 2, 3, \ldots \]
(2.10)
exist if and only if the right-hand sides \( b_n \) are orthogonal to the eigenfunction \( \ell \) of the adjoint to \( \mathcal{L} \) for the eigenvalue 0. This solvability condition (for \( n = 2 \)) determines \( A_1 \) uniquely. To be specific, we multiply (2.8) with the yet unknown function \( \ell(x,y) \) from the left and perform integration over the whole flow domain. Furthermore, we make use of the Fourier transform \( \mathcal{F} \) and Parseval’s theorem for square-integrable functions \( f \) and \( g \) (a bar denotes the complex conjugate)
\[ \mathcal{F}(f(x)) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \hat{f}(x)g(x) \, dx = \int_{-\infty}^{\infty} \hat{\psi}(k) \overline{\tilde{g}(k)} \, dk \].
(2.11)

Then multiple application of integration by parts yields
\[ \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \mathcal{L}\psi_1 \, dy = \int_{-\infty}^{\infty} dk \int_{0}^{\infty} \bar{\ell}\mathcal{L}\psi_1 \, dy = \int_{-\infty}^{\infty} dk \left[ \int_{0}^{\infty} \psi_1 \bar{\ell}^t \tilde{\ell} \, dy + \text{b.t.} (\bar{\ell}, \psi_1) \right] = 0, \]
(2.12)
thus leading to the adjoint operator \( \mathcal{L}^t \tilde{\ell} \) defined by
\[ \mathcal{L}^t \tilde{\ell} := \frac{\partial^3 \tilde{\ell}}{\partial y^3} - ikp_{00} \frac{y^2}{2} \frac{\partial^2 \tilde{\ell}}{\partial y} - 2ikp_{00} y \tilde{\ell} = 0, \]
(2.13)
and the boundary terms
\[ \text{b.t.} (\bar{\ell}, \psi_n) := \left[ \bar{\ell} \left( ikp_{00} \frac{y^2}{2} \psi_n - \frac{\partial^2 \psi_n}{\partial y^2} \right) + \frac{\partial \bar{\ell}}{\partial y} \frac{\partial \psi_n}{\partial y} - \frac{\partial^2 \bar{\ell}}{\partial y^2} \psi_n \right]_{y=0}^{\infty}, \quad n = 1, 2, \ldots \]
(2.14)
The general solution of (2.13) may be written as
\[ \tilde{\ell}(k,y) = c_1(k) F_2(1;1/2,3/4;\Omega y^4) + \frac{y^3}{2} \left[ c_2(k)I_{-1/4}(2\sqrt{\Omega} y^2) + c_3(k)I_{1/4}(2\sqrt{\Omega} y^2) \right], \]
(2.15)
where \( pFq(a_1, \ldots, a_p; b_1 \ldots b_q; z) \) denotes the generalized hypergeometric function, \( I_v(z) \) the modified Bessel function of the first kind and

\[
\Omega := \frac{ikp_{00}}{32}.
\]

From the requirement of vanishing boundary terms b.t.\((\tilde{\ell}, \tilde{\psi}_1) = 0\) in (2.12), we deduce the homogeneous boundary condition \( \tilde{\ell}(k, 0) = 0 \) from (2.14), which leads to \( c_1(k) = 0 \). Furthermore, the suppression of exponential growth of \( \tilde{\ell} \) as \( y \to \infty \) is ensured if \( c_3(k) = -c_2(k) \), which actually results in strong decay. Using the relation \( K_v(z) = \pi [I_{-v}(z) - I_v(z)]/[2\sin(v\pi)] \) for the modified Bessel function of the second kind, one may write

\[
\tilde{\ell} = c(k)\tilde{h}(k, y), \quad \tilde{h} := (ik)^{1/8}y^{3/2}K_{1/4}(2\sqrt{\Omega}y^2).
\]

Application of the procedure underlying (2.12) to higher-order problems (2.10) results in

\[
\int_{-\infty}^{\infty} dk \int_{0}^{\infty} \tilde{\ell} \tilde{\psi}_n dy = \int_{-\infty}^{\infty} c(k)b.t. (\tilde{h}, \tilde{\psi}_n) dk = \int_{-\infty}^{\infty} c(k) \left[ \int_{0}^{\infty} \tilde{h} b_n dy \right] dk.
\]

Since \( c(k) \) remains undetermined, we infer the solvability condition in Fourier space to be

\[
\left[ \frac{\partial^2 \tilde{h}}{\partial y^2} \psi_n - \frac{a_k \partial \tilde{\psi}_n}{\partial y} \frac{\partial \tilde{h}}{\partial y} \right]_{y=0} - \int_{0}^{\infty} \tilde{h} b_n dy = 0
\]

by using the explicit expression for the boundary term (2.14). Alternatively, inverse Fourier transform and application of the convolution theorem give the solvability condition in physical space as

\[
2^{-3/8} \Gamma\left(\frac{1}{4}\right) p_{00}^{-1/8} \left. \frac{\partial \psi_n}{\partial y} \right|_{y=0} - 2^{11/8} p_{00}^{1/8} \left. \frac{1}{(x-\xi)^{1/4}} \frac{\partial \psi_n}{\partial \xi} \right|_{y=0} \, d\xi
- \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} h(x - \xi, y) b_n(\xi, y, t) \, d\xi = 0,
\]

where

\[
h(x, y) = \frac{p_{00}^{1/8} \sqrt{\pi}}{2^{9/8} y^{8/4}} \exp \left( -\frac{p_{00}y^4}{32x} \right) H(x),
\]

with \( H(x) \) denoting Heaviside’s step function (figure 2).

The applicability of the procedure described above, in particular in respect of the use of the Fourier transform, requires sufficiently strong decay of \( \psi_n \) as \( x \to \pm \infty \). To ensure this prerequisite, a shift of the form

\[
\psi_2 = \psi^*_2 + A_2(x, t) \frac{y^2}{2} + \frac{a_0^2 y^5}{5!} + \frac{p_{00} a_0^2 y^9}{8!} + \frac{(A_1^2 - a_0^2 x^2 - 2a_0 a_1 k_1)y}{2p_{00}}
\]

is performed, where \( a_0, a_1 \) are flow-problem-specific constants and \( k_1 = e^{-8(\alpha - \alpha_c)} \sim O(1) \) is the scaled control parameter; see [7,8] for details. From the matching to the up- and downstream boundary layer region, it is known that \( A_1 \sim a_0 |x| + a_1 k_1/|x| + \cdots \) as \( x \to \pm \infty \); this far-field behaviour can be deduced from a local analysis of the classical (non-interactive) boundary layer equations near a point of vanishing skin friction [7]. The corresponding problem now reads

\[
\mathcal{L} \psi^*_2 = b^*_2, \quad b^*_2 = \frac{\partial \psi^*_2}{\partial x} + y \frac{\partial A_1}{\partial t},
\]

with the modified no-slip condition \( \psi^*_2 |_{y=0} = 0 \), \( \partial \psi^*_2/\partial y = -(A_1^2 - a_0^2 x^2 - 2a_0 a_1 k_1)/(2p_{00}) \) at \( y = 0 \).

From (2.20) with the use of (2.21) and

\[
\int_{0}^{\infty} y^{2+m} \exp \left( -\frac{p_{00}y^4}{32x} \right) \, dy = 2^{(7+5m)/4} \Gamma\left(\frac{3+m}{4}\right) p_{00}^{-(3+m)/4} x^{(3+m)/4}, \quad m > -3,
\]

(2.24)
one immediately obtains—without the necessity to know the solution \( \psi^* \)—the well-known fundamental equation for \( A_1 \),

\[
A_1^2 - a_0^2 r^2 - 2a_0 a_1 k_1 = - \sqrt{p_{00} \lambda} \int_{-\infty}^x \frac{\partial p_1 / \partial \xi}{\sqrt{x - \xi}} \, d\xi - \frac{p_{00}^{1/4}}{\nu} \int_{-\infty}^x \frac{\partial A_1 / \partial t}{(x - \xi)^{1/4}} \, d\xi,
\]

with the abbreviations

\[
\lambda = \frac{\Gamma(3/4)}{\sqrt{2} \Gamma(5/4)} \quad \text{and} \quad \gamma = \frac{2^{3/4}}{\Gamma(5/4)}.
\]

See [8,9] for the steady and [11,15] for the unsteady flow case. For the incompressible flow case considered here, Eq. (2.25) has to be supplemented with the upper-deck (UD) solution, i.e. the interaction law

\[
\frac{\partial p_1}{\partial x} = \frac{U_{00}^2}{\rho_0 \pi} \int_{-\infty}^\infty \frac{\partial^2 A_1 / \partial \xi^2}{x - \xi} \, d\xi,
\]

where \( U_{00} \) is the velocity at the outer edge of the boundary layer evaluated at the separation point \( x = 0 \).

Self-evidently, Eq. (2.19) can also be evaluated. Taking into account Eq. (2.17) and

\[
\int_0^\infty y^{3/2 + m} K_{1/4}(2 \sqrt{\Omega} y^2) \, dy = 2(1+10m)^8 (ikp_{00})^{-(5+2m)/8} \Gamma \left( \frac{2 + m}{4} \right) \Gamma \left( \frac{3 + m}{4} \right), \quad m > -2,
\]

one then obtains

\[
\mathcal{F}(A_1^2 - a_0^2 r^2 - 2a_0 a_1 k_1) = - \sqrt{p_{00} \Gamma(1/2) \lambda} \mathcal{F} \left( \frac{\partial p_1}{\partial x} \right) - \frac{p_{00}^{1/4} \Gamma(3/4) \gamma}{(ik)^{3/4}} \mathcal{F} \left( \frac{\partial A_1}{\partial t} \right).
\]

The Fourier transform of the pressure gradient (2.30) may be written as

\[
\mathcal{F} \left( \frac{\partial p_1}{\partial x} \right) = \frac{U_{00}^2}{\rho_0} \frac{i k |k| \hat{A}_1}{\lambda}.
\]
By applying the Fourier transforms of the Weyl fractional integrals

\[ \mathcal{F}\left( \frac{1}{f(x)} \int_{-\infty}^{x} (x - \xi)^{\nu-1} f(\xi) \, d\xi \right) = (ik)^{-\nu} F(k) \]

and

\[ \mathcal{F}\left( \frac{1}{f(x)} \int_{x}^{\infty} (\xi - x)^{\nu-1} f(\xi) \, d\xi \right) = (-ik)^{-\nu} F(k) \] (2.31)

with the restriction \( 0 < \nu < 1 \), one immediately recovers (2.25). Commonly, affine transformations are introduced to eliminate the problem-specific constants \( a_0, a_1, p_{00}, U_{00} \) and to underline the similarity law character of the fundamental equation (2.25) in combination with (2.27). With respect to the passage of the ‘marginal separation stage’ into the triple-deck stage (via finite-time blow-up), we prefer to keep these constants in the corresponding equations.

Moreover, it should be emphasized that the effects of flow control elements (‘smart structures’), such as surface-mounted obstacles and/or suction/blowing devices, can easily be incorporated into the analysis, for the latter, e.g. the second term in (2.20), does not vanish. The additional terms thus resulting may also be used for the formulation of appropriate initial value problems, as studied in [10] or §4.

We are now in the position to extend the theory of marginal separation to second order. To this end, the explicit solution \( \psi_2^* \) of (2.23) is required, which is known in closed form in Fourier space only and may be found by means of a power series ansatz [13]:

\[
\tilde{\psi}_2^*(k, y, t) = \mathcal{F}\left( \frac{A_1^2 - a_0^2 y^2 - 2a_0 a_1 k_1}{2p_{00}} \right) \Gamma\left( \frac{5}{4} \right) \sum_{n=0}^{\infty} \frac{\Omega^n y^{4n+1}}{\Gamma(n + 5/4)(4n - 1)n!} \\
+ \mathcal{F}\left( \frac{\partial p_1}{\partial x} \right) \frac{\sqrt{\pi}}{16} \Gamma\left( \frac{3}{4} \right) \sum_{n=0}^{\infty} \frac{\Omega^n y^{4n+3}}{\Gamma(n + 3/2)\Gamma(n + 7/4)(4n + 1)} \\
+ \mathcal{F}\left( \frac{\partial A_1}{\partial t} \right) \frac{\Gamma(3/4)}{32} \sum_{n=0}^{\infty} \frac{\Omega^n y^{4n+4}}{\Gamma(n + 2)\Gamma(n + 7/4)(2n + 1)} + b \frac{y^2}{2}.
\] (2.32)

The arbitrary function \( b(k, t) \) is chosen such that \( \tilde{\psi}_2^* \) does not grow algebraically \( \sim O(y^2) \) as \( y \to \infty \), and, in combination with (2.29), one can alternatively rewrite (2.32) in terms of generalized hypergeometric functions:

\[
\tilde{\psi}_2^*(k, y, t) = \mathcal{F}\left( \frac{\partial A_1}{\partial t} \right) \left[ -\frac{\pi^{3/2} y^2}{\sqrt{ikp_{00}}\Gamma(1/4)^2} + \frac{1 - 1F_2(-1/2; 1/2, 3/4; \Omega y^4)}{ikp_{00}} \right] \\
+ \frac{2^{5/4} \Gamma(3/4)^2 1F_2(-1/4; 3/4, 5/4; \Omega y^4)y}{\pi (ikp_{00})^{3/4}} \mathcal{F}\left( \frac{\partial p_1}{\partial x} \right) \left[ -\frac{\pi^{3/2} y^2}{21/4 (ikp_{00})^{1/4} \Gamma(1/4)^2} \right] \\
+ \frac{\sqrt{2} \pi \Gamma(3/4) 1F_2(-1/4; 3/4, 5/4; \Omega y^4)y}{\sqrt{ikp_{00}} \Gamma(1/4)} + 2F_3\left( 1/4, 1; 5/4, 3/2, 7/4; \Omega y^4 \right) y^3/6.
\] (2.33)

The far-field behaviour of \( \tilde{\psi}_2^* \) and its contribution to the wall shear stress are consequently given by

\[
\tilde{\psi}_2^* \bigg|_{y \to \infty} \sim \frac{1}{ikp_{00}} \mathcal{F}\left( \frac{\partial A_1}{\partial t} \right) + \frac{2}{3ikp_{00}} \mathcal{F}\left( \frac{\partial p_1}{\partial x} \right) \frac{1}{y} + O(y^{-5})
\]

and

\[
\frac{\partial^2 \tilde{\psi}_2^*}{\partial y^2} \bigg|_{y=0} = \tilde{b} = -\frac{2\pi^{3/2}}{\sqrt{ikp_{00}} \Gamma(1/4)^2} \mathcal{F}\left( \frac{\partial A_1}{\partial t} \right) - \frac{2^{3/4} \pi^2}{(ikp_{00})^{1/4} \Gamma(1/4)^2} \mathcal{F}\left( \frac{\partial p_1}{\partial x} \right).
\] (2.34)
In order to determine the second-order correction displacement function $A_2$, we follow the procedure described above and introduce, similar to (2.22), the shift

$$
\psi_3 = \psi^*_3 + A_3(x, t) \frac{y^2}{2} + \frac{A_1 A_2 y}{p_{00}} + \frac{A_1}{p_{00}} \frac{\partial \psi^*_2}{\partial y} + \frac{a_0^2 y^4}{24 p_{00}} \left( A_1 + a_0 x + \frac{a_1 k_1}{x} \right) + \frac{a_0^2 y^6 A_1}{480} + \frac{1}{p_{00}^2} \left[ - \frac{A_1^3}{3} - \frac{a_0^2 x^3}{3} - a_0^2 a_1 k_1 x + \frac{A_1}{2} (A_1^2 - a_0^2 x^2 - 2 a_0 a_1 k_1) \right] + \frac{a_0^2 y^8}{4032} + \frac{a_0^2 p_{00} y^{12}}{1774080},
$$

(2.35)

which leads to

$$
\mathcal{L} \psi_3^* = b_3^*, \quad b_3^* = \frac{\partial p_2}{\partial x} + y \frac{\partial A_2}{\partial t} + \frac{\partial^2 \psi^*_2}{\partial y^2},
$$

(2.36)

and the boundary conditions

$$
y = 0: \quad \psi_3^* = \frac{A_3^3 + a_0^3 x^3 + 3 a_0^2 a_1 k_1 x}{3 p_{00}^2}, \quad \frac{\partial \psi_3^*}{\partial y} = - \frac{A_1}{p_{00}} \left( A_2 + \frac{\partial^2 \psi^*_2}{\partial y^2} \right)_{y=0}.
$$

(2.37)

Substitution of the Fourier-transformed versions of (2.36) and (2.37) into the solvability condition (2.19), evaluation by means of (2.32), (2.34) and (2.28), and application of the inverse Fourier transform yields the forced linear, fundamental equation for $A_2$:

$$
2 A_1 A_2 + \sqrt{p_{00} \lambda} \int_{-\infty}^{x} \frac{\partial p_2}{\partial \xi} d \xi + \frac{1}{\sqrt{x - \xi}} \int_{-\infty}^{x} \frac{\partial A_2}{\partial t} (x - \xi)^{1/4} d \xi = A_1 \left[ p_{00}^{-1/4} c_1 \int_{-\infty}^{x} \frac{\partial p_1}{\partial \xi} (x - \xi)^{1/4} d \xi + p_{00}^{-1/2} \lambda \int_{-\infty}^{x} \frac{\partial A_1}{\partial t} (x - \xi)^{1/4} d \xi \right] + p_{00}^{-3/4} \gamma \int_{-\infty}^{x} A_2^3 \frac{\partial A_1}{\partial \xi} + a_0^2 x^2 + a_0^2 a_1 k_1 (x - \xi)^{1/4} d \xi - p_{00}^{-1/4} c_2 \int_{-\infty}^{x} (x - \xi)^{1/4} \frac{\partial^2 A_1}{\partial t^2} d \xi - c_3 \frac{\partial p_1}{\partial t}.
$$

(2.38)

Here

$$
c_1 = \frac{2^{1/4} \Gamma(3/4)^3}{\pi}, \quad c_2 = \frac{8 \times 2^{3/4} \left[ \pi^{3/2} \Gamma(-1/4) + \pi^2 \Gamma(1/4) \right]}{\Gamma(1/4)^4},
$$

and

$$
c_3 = \frac{\sqrt{2}}{\Gamma(1/4)^3} \left[ 2 \pi^2 \Gamma \left( -\frac{1}{4} \right) \right] + \pi^{3/2} \Gamma \left( \frac{1}{4} \right) \sum_{m=0}^{\infty} \left( \frac{\Gamma(1/4 + m) \Gamma(1 + m)}{\Gamma(3/4 + m) \Gamma(3/2 + m)} - \frac{\Gamma(m - 1/4) \Gamma(1/2 + m)}{\Gamma(1/4 + m) \Gamma(1 + m)} \right)
$$

(2.39)

are positive constants ($c_2 \approx 0.661009$, $c_3 \approx 0.108380$). For the closure of (2.38), in addition to (2.27) a relationship between $p_2$ and $A_2$ is required. Investigation of the UD region (figure 1) leads to

$$
\frac{\partial p_2}{\partial x} = \frac{U_{00}^2}{p_{00} \pi} \int_{-\infty}^{x} \frac{\partial^2 A_2}{\partial \xi^2} (x - \xi) d \xi,
$$

(2.40)

see (4.15). An investigation similar to that performed in the appendix of [10] yields the far-field behaviour $A_2 \sim O(|x|^{-3/4})$ as $x \to -\infty$ and $A_2 \sim O(x^{7/4})$ as $x \to \infty$.

### 3. Finite-time blow-up and the subsequent triple-deck stage

As is known from [11], solutions of the fundamental equation for $A_1$ may blow up at a finite time $t_s$, at a single point $x_s$ under certain conditions (e.g. sufficiently strong forcing for below-critical control parameter conditions or above-critical conditions even without any forcing, e.g. [16]). The
specific behaviour is given by

$$A_1 \sim \tau^{-2/3} \hat{A}_1(\hat{x}) + \cdots, \quad p_1 \sim \tau^{-10/9} \hat{p}_1(\hat{x}) + \cdots \quad \text{and} \quad x - x_s = \tau^{4/9}\hat{x}$$

(3.1)

as $\tau := t_s - t \to 0^+$. Recent investigations show that the blow-up profile $\hat{A}_1$ or, equivalently, $\hat{p}_1$ is unique [10] (figure 3), and consequently the following question arises. How do the initial conditions, effects of flow control devices, etc. (the ‘history’ of the flow) enter the matching condition to the subsequent fully nonlinear triple-deck stage first studied in [11,12] and sketched in figure 1? With the appropriate rescalings in the LD region

$$\hat{x} - \hat{x}_0 - \varepsilon^4 x_s = \sigma^2 X, \quad \hat{y} = \sigma^4 Y, \quad \hat{t} - \varepsilon^{-1} t_s = \sigma T$$

(3.2)

and

$$\psi \sim \sigma^5 \Psi(X, Y, T) + \cdots, \quad \frac{\partial p}{\partial x} \sim p_{00} + \frac{\partial P}{\partial X} + \cdots$$

(3.3)

as $\sigma := Re^{-1/7} \to 0$, one obtains, after substitution into the Navier–Stokes equations, the fundamental LD problem

$$\frac{\partial^2 \Psi}{\partial Y^2} + \frac{\partial \Psi}{\partial Y} \frac{\partial^2 \Psi}{\partial Y \partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial^2 \Psi}{\partial Y^2} = -\left(p_{00} + \frac{\partial P}{\partial X}\right) + \frac{\partial^3 \Psi}{\partial Y^3},$$

(3.4)

subject to the no-slip boundary conditions $\Psi = \partial \Psi / \partial Y = 0$ at $Y = 0$ and the far-field behaviour

$$\Psi \sim \frac{p_{00}}{6} (Y + A)^3 + \int_{-\infty}^X \frac{\partial A}{\partial T} \, d\xi + \frac{2}{3p_{00}} \frac{P}{Y} + O(Y^{-2})$$

(3.5)

as $Y \to \infty$ and $A, P \to 0$ as $X \to -\infty$. In addition, the interaction law connecting the displacement function $A(X, T)$ and the induced pressure $P(X, T)$ in the UD for incompressible flows

$$P = \frac{U_{00}^2}{p_{00} \pi} \int_{-\infty}^{\infty} \frac{\partial A}{\partial \xi} \frac{d\xi}{X - \xi}$$

(3.6)
is recovered. As expected, the local displacement effect and the action of the wall shear stress (2.2) become intensified

$$\delta^* = \delta^*_0(\hat{x}_0) + Re^{-1/4} q_4 A(X, T) + \cdots \quad \text{and} \quad \tau_w \sim Re^{-1/4} q_5 \frac{\partial^2 \psi}{\partial Y^2} \bigg|_{Y=0} + \cdots, \quad (3.7)$$

cf. (2.9). Here $q_4 < 0$, $q_5 > 0$ again denote problem-specific constants.

Combining expansions (2.6) and the (blow-up) scalings (3.1)–(3.3) results in the initial/matching condition as $T \to -\infty$, $\tau \to 0^+$:

$$\Psi \sim |T|^{1/3} \left[ p_{00} \hat{y}^3 \frac{2}{6} + |T|^{-7/9} \hat{y}^2 \hat{A}_1(\hat{x}) + |T|^{-11/9} \hat{y}^2 \hat{\psi}_1(\hat{x}) \right. \right.$$

$$\left. + |T|^{-14/9} \hat{\psi}_2(\hat{x}, \hat{y}) + |T|^{-16/9} \hat{y}^2 \hat{c}_2(\hat{x}) + |T|^{-18/9} \hat{\psi}_3(\hat{x}, \hat{y}) \right]$$

$$\left. + |T|^{-21/9} \hat{\psi}_4(\hat{x}, \hat{y}) + |T|^{-23/9} \hat{\psi}_2(\hat{x}, \hat{y}) + \cdots \right], \quad (3.8)$$

and

$$\mathcal{P} \sim |T|^{-10/9} \hat{p}_1(\hat{x}) + |T|^{-14/9} \hat{p}_{01}(\hat{x}) + |T|^{-17/9} \hat{p}_2(\hat{x}) + |T|^{-19/9} \hat{p}_{12}(\hat{x}) + \cdots \right].$$

Here, $X = |T|^{4/9} \hat{x}$, $Y = |T|^{1/9} \hat{y}$, and $\hat{c}_1$, $\hat{c}_2$ denote eigenfunctions resulting from the ansatz $A_2 \sim \tau^{-2} \hat{c}_\mu(\hat{x})$ for the homogeneous part of (2.38) with eigenvalues $\mu = (10/9, 15/9)$, respectively. Furthermore, $\hat{\psi}_2$ represents the limiting solution $\psi_2$ as $\tau \to 0$ and, similar to (2.22), we introduce

$$\hat{\psi}_2 = \hat{\psi}_2^* + \hat{A}_2(\hat{x}) \hat{y}^2 \frac{2}{3}, \quad (3.9)$$

with a yet unknown correction displacement function $\hat{A}_2$, which enters the asymptotic representation of the triple-deck displacement function

$$\mathcal{A} \sim p_{00}^{-1} |T|^{-6/9} \hat{A}_1 + |T|^{-10/9} \hat{c}_1 + |T|^{-13/9} \hat{A}_2 + |T|^{-15/9} \hat{c}_2 + \cdots \right]$$

as $T \to -\infty$. An asymptotic expansion of (3.4) based on (3.8) immediately leads to

$$\mathcal{E} \hat{\psi}_2^* = \hat{p}_1' + \frac{2\hat{y}}{3} \left( \hat{A}_1 + \frac{2}{3} \hat{\psi}_1(\hat{x}) \right) + \frac{\hat{y}^2}{4} (\hat{A}_1')'$$

with $\hat{\psi}_2^* = \partial \hat{\psi}_2^*/\partial \hat{y} = 0$ at $\hat{y} = 0$. The solvability condition (2.20) then yields the equation for the blow-up profile

$$\hat{A}_1^2 = -\frac{p_{00} A}{\sqrt{\lambda}} \int_{-\infty}^{\hat{x}} \frac{\hat{p}_1'}{\sqrt{\hat{x} - \xi}} \left[ \hat{A}_1 + \frac{2}{3} \hat{\psi}_1(\hat{x}) \right] \frac{d\xi}{(\hat{x} - \xi)^{1/4}}, \quad (3.12)$$

where $\hat{p}_1$ and $\hat{A}_1$ are related via (3.6) (figure 3). Similarly, we obtain

$$2 \hat{A}_1' \hat{\psi}_1 + \frac{p_{00} A}{\sqrt{\lambda}} \int_{-\infty}^{\hat{x}} \frac{\hat{p}_1'}{\sqrt{\hat{x} - \xi}} \left[ \hat{A}_1 + \frac{2}{3} \hat{\psi}_1(\hat{x}) \right] \frac{d\xi}{(\hat{x} - \xi)^{1/4}} = \frac{\mu \hat{\psi}_1 + \frac{4}{9} \xi \hat{\xi}}{(\hat{x} - \xi)^{1/4}}, \quad \mu = \left( \frac{10}{9}, \frac{15}{9} \right). \quad (3.13)$$

Here, the indeterminate amplitudes of $\hat{c}_1$ and $\hat{c}_2$ carry the ‘history’ of the flow and may be converted into a shift of the blow-up point $x_s \to x_s + \Delta x_s$ and $t_s \to t_s + \Delta t_s$ with $\Delta x_s, \Delta t_s \ll 1$, respectively (formulation (3.4) is invariant with respect to a shift in $X$ and $T$). Furthermore, the shapes of $\hat{c}_1$ and $\hat{c}_2$ correspond to $x$ and $t$ derivatives of $A_1$ in the limit as $\tau \to 0$ (figure 3)

$$\hat{c}_1 \propto \hat{A}_1' \quad \text{and} \quad \hat{c}_2 \propto \left( \hat{A}_1 + \frac{2}{3} \hat{\psi}_1(\hat{x}) \right). \quad (3.14)$$

To determine the blow-up profile $\hat{A}_2$, analogously to (2.35), $\hat{\psi}_3$ is written as

$$\hat{\psi}_3 = \hat{\psi}_3^* + \hat{A}_3(\hat{x}) \hat{y}^2 \frac{2}{3} + \frac{\hat{A}_1 \hat{A}_2 \hat{y}}{p_{00}} + \frac{\hat{A}_1}{p_{00}} \frac{\partial \hat{\psi}_2^*}{\partial \hat{y}} + \frac{\hat{A}_1^3}{6p_{00}^2}, \quad (3.15)$$
which, on further expansion of (3.4), leads to

\[ \mathcal{L} \dot{\psi}^*_3 = \dot{\rho}^*_2 - \frac{2\hat{A}_1}{3p_{00}} \left( \hat{A}_1 + \frac{2}{3} \hat{A}_1^* \right) + \int^\infty_\lambda \dot{\varphi}^2 + \frac{4}{9} \frac{\partial \dot{\psi}^*_2}{\partial \varphi} + \frac{\hat{y}}{9} \frac{\partial^2 \psi^*_2}{\partial \varphi^2}, \]

supplemented with the modified no-slip boundary conditions

\[ y = 0: \dot{\psi}^*_3 = -\frac{\hat{A}_1}{6p_{00}} \frac{\partial \dot{\psi}^*_3}{\partial \varphi} = -\frac{\hat{A}_1}{p_{00}} \frac{\partial^2 \psi^*_3}{\partial \varphi^2}, \]

Application of the procedure that led to (2.38) here then yields

\[ 2\hat{A}_1 \hat{A}_2 + \sqrt{p_{00}} \int^\infty_{-\infty} \dot{\rho}^*_2 \frac{1}{\sqrt{\varphi}} \varphi \, d\varphi = \frac{13}{9} \hat{A}_1 + \frac{2\hat{A}_1^*}{(\varphi - \xi)^{1/4}} \int^\infty_{-\infty} \frac{3\hat{A}_1 + 2\hat{A}_1^*}{\sqrt{\varphi - \xi}} \, d\varphi, \]

Alternatively, one can derive (3.18) from (2.38) by taking the limit \( t \to t \) and using the blow-up scalings (3.1), (3.8) and (3.10).

With the blow-up profiles \( \hat{A}_1, \hat{A}_2 \), the eigenfunctions \( \hat{e}_1, \hat{e}_2 \) (with arbitrary amplitudes) and the stream function \( \psi^*_2 \) determined numerically, we are able to properly start the triple-deck computations. This involved task is currently under investigation.

In general, the Cauchy problems associated with the fundamental problem (2.25) and (2.27) and the triple-deck stage (3.4)–(3.8) are known to be ill-posed [12,17], but can be regularized if the streamline curvature is taken into account; see [18–20] and the following section for a detailed analysis.

### 4. Regularization terms

In a recent study by the present authors [10], initial value problems based on (2.25) together with (2.27), i.e.

\[ A_1^2 - a_0^2 x^2 - 2a_0 a_1 k_1 = \frac{U_{00}^2}{\sqrt{p_{00}}} \int^\infty_{-\infty} \frac{\partial^2 A_1 / \partial \xi^2}{\sqrt{\xi - \hat{x}}} \, d\xi - p_{00} \frac{1}{4} \int^\infty_{-\infty} \frac{\partial A_1 / \partial t}{(\xi - \hat{x})^{1/4}} \, d\xi + f(x,t), \]

were addressed numerically. Here, the additional term \( f(x,t) \) accounts for the forcing due to control devices such as a surface-mounted obstacle or a suction slot, respectively, and \( A_{10}(x) \) is the solution to the steady and unforced version of (4.1). Numerical solutions to the steady problem for different values of the parameters \( k_1, a_0, a_1, p_{00} \) and \( U_{00} \), which by application of an affine transformation to (4.1) can be combined into a single control parameter, are presented, for example, in the original works [8,9]. In the analysis [10], special emphasis was placed on solutions that terminate in the form of finite-time singularities, and comprehensive numerical computations convincingly demonstrated the development of a unique blow-up structure, entirely independent of the particular choice of initial data. As already outlined before, further support for this important result was provided by an asymptotic analysis of (4.1) near the blow-up point, where
the resulting equation admits a unique solution that is in perfect agreement with the numerical findings.

On the other hand, some inconsistencies associated with the proposed Cauchy problem (4.1) were discovered in [15], and their repercussions on the solvability in general are discussed in detail in [17]. Both works mention the occurrence of instabilities in the high-wavenumber regime, with the latter, however, putting this in the context of an ill-posed Cauchy problem. More precisely, it was shown that the absolute instability of the velocity field against short-scale disturbances and, entailed by that, the incorrectness of the problem essentially are a direct consequence of the abnormal dispersion relation governing the linearized problem in the limit of very high wavenumbers. Using a Fourier approach in the sense of a global stability analysis such that

$$A_1(x, t) = A_{10}(x) + \delta e^{i\omega t} \mathcal{F}^{-1}(\tilde{A}_{11}(k)) + \cdots, \quad \delta \ll 1,$$

where \(k\) is real, one can derive from (2.29) and (2.26) the asymptotic result

$$k \to \pm \infty: \quad \omega(k) = -\frac{23/4 \pi \frac{1}{2} \lambda_{00}^2}{4 \pi^{3/4} \frac{3}{0} k |k|^{1/4} + \cdots}.$$  (4.3)

This yields that, to leading order in the high-wavenumber limit, the left-hand side of (2.29) does not affect the dispersion relation and, above all, that the growth rates for short-scale instabilities

$$k \to \pm \infty: \quad -\Im(\omega) = D_1 |k|^{5/4} + \cdots, \quad D_1 > 0,$$  (4.4)

are not bounded from above. Therefore, the Cauchy problem as given in (4.1) turns out to be ill-posed and has to be regularized.

The essence of the above discussion is that higher-order terms in \(\varepsilon\) not included so far must play a distinctive role in the evolution of the unsteady marginal separation process, given the bounded spectrum that the unsteady Navier–Stokes equations are assumed to have, but which the leading-order problem fails to deliver. In the light of this, the absence of short-scale instabilities in the numerical solutions presented in [10] has to be attributed to the applied (semi-)implicit scheme. Obviously, the small but, nonetheless, always present terms that result from the discretization and that are of the order of the truncation error proved very effective in regularizing the problem. For the same reason, the steady solutions serving as initial conditions had to be destabilized by sufficiently strong forcing in order for the blow-up to be triggered. On physical grounds, however, the unrealistic growth of self-excited waves should be avoided by analysing the Navier–Stokes equations up to higher orders rather than implementing a numerical scheme designed for producing the desired results. In anticipation of what follows, the solutions presented in [10] will nevertheless turn out to be consistent with the then well-posed problem.

To the best of the authors’ knowledge, there does not exist a rigorous asymptotic theory for the regularization of otherwise ill-posed Cauchy problems. The error introduced by omitting higher-order terms of the original Navier–Stokes equations falls beyond the scope of an asymptotic leading-order approach. In other words, the filtering process entailed by the temporal and spatial scales on which the asymptotic analysis is based may lead to the undesired result that the terms needed for regularizing the Cauchy problem at a certain level of approximation cannot be incorporated into it. They will thus enter the equations of higher order and, by that, form the higher-order forcing terms. Thus, the only way to regularize the original problem is to set up a composite asymptotic model, where terms of higher order are taken as part of the leading-order problem such that they can effectively contribute to stabilizing its homogeneous solution against short-scale disturbances.

This method has proved to be very successful in the past (e.g. [19,21,22] and references therein) and will also be applied here. As revealed in these studies, for higher-order asymptotic terms to play the role of regularization terms, they must at least generate derivatives with respect to \(t\) and \(x\) of orders that are higher than those already present in the system. Inspection of the inhomogeneous terms in equation (2.38) for \(A_2\) when they are considered as part of the leading-order problem shows that, although they then do represent derivatives of higher order, essentially
the same high-wavenumber limit (4.3), merely multiplied with a positive factor, is obtained. The expression for the growth rates (4.4) thus remains unchanged. However, the pressure induced by the second-order displacement and, to even higher order in \( \varepsilon \), by the LD stream function \( \psi_2 \) has not been examined until now, and from the arguments put forward in [21,22] it can be deduced that precisely the higher-order terms associated with \( \psi_2 \) as \( y \to \infty \) will give rise to physical phenomena that have a beneficial effect on the regularization of the leading-order Cauchy problem.

In the main-deck (MD) region of the marginal separation stage, where \( y_m = \varepsilon^{-10} \tilde{y} \), the expansion of the stream function resulting from the LD solutions \( \psi_1 \) and \( \psi_2 \) as \( y \to \infty \), see (2.34), thus assumes the form

\[
\psi_m(x, y_m) = \varepsilon^{10} \psi_{00}(y_m) + \varepsilon^{14} \psi_{01} + \varepsilon^{17} \psi_{02} + \varepsilon^{18} \psi_{03} + \varepsilon^{19} \psi_{04} + \varepsilon^{20} \psi_{05} + \cdots. \tag{4.5}
\]

Substitution of (4.5) into the horizontal momentum equation then yields to leading order

\[
\mathcal{L}_2 \psi_{m1} := \psi_{00} \frac{\partial^2 \psi_{m1}}{\partial x \partial y_m} - \psi_{00} \frac{\partial \psi_{m1}}{\partial x} = -p_{00} + \psi_{00}^\prime, \tag{4.6}
\]

and additionally,

\[
\begin{align*}
\mathcal{L}_2 \psi_{m2} &= 0, \\
\mathcal{L}_2 \psi_{m3} &= -xp_{01} + \frac{\partial^3 \psi_{m1}}{\partial y_m^3} - \frac{\partial \psi_{m1}}{\partial y_m} \frac{\partial^2 \psi_{m1}}{\partial x \partial y_m} + \frac{\partial^2 \psi_{m1}}{\partial y_m^2} \frac{\partial \psi_{m1}}{\partial x}, \\
\mathcal{L}_2 \psi_{m4} &= -\frac{\partial^2 \psi_{m1}}{\partial y_m \partial t} \quad \text{and} \quad \mathcal{L}_2 \psi_{m5} = -\frac{\partial p_1}{\partial x}.
\end{align*} \tag{4.7}
\]

The last equation in (4.7) is interesting insofar as that here the induced pressure component of leading order \( O(\varepsilon^{10}) \) enters. Furthermore, in the main layer, \( \tilde{y}/\tilde{x} = \varepsilon^6 y_m/x \) and the pressure induced by \( \psi_{m5} \) in the UD and acting in the LD forms the component of \( O(\varepsilon^{16}) \) in the pressure expansion. These facts suggest a pressure variation in the vertical direction for precisely this term of \( O(\varepsilon^{16}) \) to come into play, since its normal gradient will be comparable in size with the leading-order convective term in the vertical momentum equation. Consequently, the extended version of the horizontal pressure gradient in the MD is given by

\[
\frac{\partial p_m}{\partial x} = p_{00} + \varepsilon^4 p_{01} x + \varepsilon^6 \frac{\partial p_1}{\partial x} + \varepsilon^9 \frac{\partial p_2}{\partial x} + \varepsilon^{10} \frac{\partial p_3}{\partial x} + \varepsilon^{11} \frac{\partial p_4}{\partial x} + \varepsilon^{12} \frac{\partial p_{m5}}{\partial x}(x, y_m, t) + \cdots, \tag{4.8}
\]

where, as in (2.6), all terms representing Taylor expansions of orders higher than \( O(\varepsilon^4) \) have been omitted since they will not contribute to the results presented in the following. Most important, all terms in this expansion, except for \( p_{m5}(x, y_m, t) \), are identical to the corresponding components in the LD. As already indicated by the above order-of-magnitude argument, the asymptotic expansion of the vertical momentum equation then leads to the relationship

\[
\psi_{00} \frac{\partial^2 \psi_{m1}}{\partial x^2} = \frac{\partial p_{m5}}{\partial y_m}, \tag{4.9}
\]

revealing the intrusion of a normal pressure gradient. As will be shown in the following, the last equation in (4.7) together with (4.9) will induce a pressure response in the LD that is proportional to the curvature of the streamlines, i.e. the second derivative of the displacement function \( A_1 \) with respect to the streamwise coordinate.
After applying the rules for matching with the LD and extracting the singular parts of the resulting integrals, one can write the solutions to (4.6) and (4.7) as

\[
\begin{align*}
\frac{\partial \psi_m}{\partial x} &= \psi''_0 \left[ 1 \frac{\partial A_1}{\partial x} + \gamma_m \left( \frac{\psi''_m - p_{00}}{(\psi'_0)^2} + \frac{p_{00}}{U^2_{00}} \right) \right], \\
\frac{\partial \psi_m}{\partial x} &= \psi'_0 \frac{\partial A_2}{\partial x'}, \\
\frac{\partial \psi_m}{\partial x} &= \psi'_0 \left[ \gamma_m \left( \frac{1}{(\psi'_0)^2} \left( \frac{\partial^3 \psi_m}{\partial \eta^2} - x \eta \frac{\partial^2 \psi_m}{\partial x \partial \eta} + \frac{\partial^2 \psi_m}{\partial \eta^2} \frac{\partial \psi_m}{\partial x} \right) \right) \\
&+ \frac{2}{p_{00}^2} \left( A_1 \frac{\partial A_1}{\partial x} - a^2 \right) + \frac{x}{U^2_{00}} \left( \eta^2 + \frac{p^2_{00} + p_{01} U^2_{00}}{U^2_{00}} \right) \right] d\eta \\
&+ \psi'_0 \left[ \frac{2}{p^2_{00} \gamma_m} \left( A_1 \frac{\partial A_1}{\partial x} - a^2 \right) - \frac{x}{U^2_{00}} \left( \eta^2 + \frac{p^2_{00} + p_{01} U^2_{00}}{U^2_{00}} \right) \right], \\
\frac{\partial \psi_m}{\partial t} &= \frac{1}{U^2_{00}} \frac{\partial A_1}{\partial t}, \\
\frac{\partial \psi_m}{\partial x} &= -\psi'_0 \frac{\partial p_1}{\partial x} \left[ \gamma_m \left( \frac{1}{(\psi'_0)^2} \left( \frac{\partial^3 \psi_m}{\partial \eta^2} - \frac{4}{p_{00}^2} \frac{1}{U^2_{00}} \right) \right) d\eta \right] - \frac{4}{\gamma_m^2} \frac{\partial^3 \psi_m}{\partial x^3} + \gamma_m U^2_{00} \right]. 
\end{align*}
\]

(4.10)

Note that for the derivation of these expressions, the expansions \( y_m \to 0: \psi_{00} = p_{00} y^2_{m}/6 + (2p_{00} p_{01}/7) y^2_{m} + \ldots \) and \( y_m \to \infty: \psi_{00} = U_{00} y_m + \ldots \) have been used (e.g., [9]). In addition, the equation for the normal pressure gradient (4.9) results in the expression

\[
\frac{\partial p_{m5}}{\partial x} = \frac{\partial p_5(x, t)}{\partial x} + \frac{1}{p_{00}} \frac{\partial^3 A_1}{\partial x^3} \left[ \gamma_m \left( \frac{4}{(\psi'_0)^2} - \frac{1}{U^2_{00}} \right) d\eta + U^2_{00} y_m \right]. 
\]

(4.11)

Here, \( \partial p_5/\partial x \) denotes the induced pressure gradient of order \( O(\varepsilon^{12}) \) acting in the LD, see (2.6), which remains to be determined later.

As mentioned above, the purpose of the analysis presented here is the identification of those pressure terms induced by the leading-order displacement function \( A_1 \) that are promising candidates for regularizing the Cauchy problem stated in (4.1). Therefore, and for simplicity, LD displacement effects of orders higher in \( \varepsilon \) than that represented by \( A_2 \) have been omitted in the MD solutions. For the same reason, furthermore, the pressure generated by the component \( \psi_{m3} \) will not be considered in the following, since neither the linear nor the quadratic nonlinear terms contained in \( \psi_{m3} \) will lead to derivatives of \( A_1 \) with respect to \( x \) or \( t \) that are of higher order than that already present in the leading-order term \( \psi_{m1} \).

The thus simplified UD expansions, with the appropriate coordinate normal to the wall now being \( y_u = -\varepsilon^{-4} \tilde{y} \), are

\[
\psi_u(x, y_u) = \varepsilon^4 U_{00} y_u + \ldots + \varepsilon^{14} \psi_{u1} + \varepsilon^{17} \psi_{u2} + \ldots + \varepsilon^{19} \psi_{u4} + \varepsilon^{20} \psi_{u5} + \ldots
\]

(4.12)

and

\[
p_u(x, y_u) = \frac{1 - U^2_{00}}{2} + \ldots + \varepsilon^{10} p_{u1} + \varepsilon^{13} p_{u2} + \ldots + \varepsilon^{15} p_{u4} + \varepsilon^{16} p_{u5} + \ldots.
\]

(4.13)

Through substitution into the momentum equations, it is easy to verify that the components of the stream function and the pressure satisfy

\[
\Delta p_{u1} = 0, \quad \frac{\partial p_{u1}}{\partial y_u} = U_{00} \frac{\partial^2 \psi_{u1}}{\partial x^2}, \\
\Delta p_{u2} = 0, \quad \frac{\partial p_{u2}}{\partial y_u} = U_{00} \frac{\partial^2 \psi_{u2}}{\partial x^2},
\]

and

\[
\Delta p_{u4} = 0, \quad \frac{\partial p_{u4}}{\partial y_u} = \frac{\partial^2 \psi_{u4}}{\partial x \partial t} + U_{00} \frac{\partial^2 \psi_{u4}}{\partial x^2},
\]

\[
\Delta p_{u5} = 0, \quad \frac{\partial p_{u5}}{\partial y_u} = U_{00} \frac{\partial^2 \psi_{u5}}{\partial x^2}.
\]

(4.14)
where $\Delta$ is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$. Relying on the matching principles and taking into account that apart from $p_{m5}$ the MD pressure terms do not depend on $y_m$, one then obtains from (4.14) and (4.10)

\[
\begin{align*}
\frac{\partial p_1}{\partial x} &= \frac{U_{00}^2}{p_{00}} \mathcal{H} \left( \frac{\partial^2 A_1}{\partial x^2} \right), \\
\frac{\partial p_2}{\partial x} &= \frac{U_{00}^2}{p_{00}} \mathcal{H} \left( \frac{\partial^2 A_2}{\partial x^2} \right) \quad \text{and} \\
\frac{\partial p_4}{\partial x} &= 2 \frac{U_{00}^2}{p_{00}} \mathcal{H} \left( \frac{\partial^2 A_1}{\partial x \partial t} \right).
\end{align*}
\] (4.15)

Here, $\mathcal{H}(\cdot)$ denotes the Hilbert transform, i.e.

\[
\mathcal{H}(f(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi} \, d\xi, \quad \mathcal{H}^2(f(x)) = -f(x).
\] (4.16)

Consequently, the well-known leading-order result (2.27) for $p_1$ is recovered and also the problem for $A_2$ as given in (2.38) can now be stated in closed form. Furthermore, it should be noted here that, as a matter of course, the singular parts of the solutions (4.10) as $y_m \to \infty$ match seamlessly with the corresponding counterparts in the UD. Moreover, the last equations in (4.14) and (4.10), respectively, together with (4.11) lead to

\[
\begin{align*}
\frac{\partial p_{m5}}{\partial x}(x, t, y_m = 0) &= -\mathcal{H} \left( \frac{\partial^2 p_1}{\partial x^2} \right) I_1 = \frac{\partial p_5(x, t)}{\partial x} - \frac{U_{00}^2}{p_{00}} \partial^3 A_1 \partial x^3 I_2, \\
I_1 &= \int_0^\infty \left( \frac{U_{00}^2}{(\psi_{00}')^2} - \frac{4U_{00}^2}{p_{00}^4} - 1 \right) \, d\eta, \quad I_2 = \int_0^\infty \left( 1 - \frac{(\psi_{00}')^2}{U_{00}^2} \right) \, d\eta,
\end{align*}
\] (4.17)

and from (4.15) and (4.16), the LD pressure gradient $\partial p_5/\partial x$ is inferred to be given by

\[
\frac{\partial p_5}{\partial x} = \frac{U_{00}^2}{p_{00}} \partial^3 A_1 \partial x^3 (l_1 + l_2). 
\] (4.18)

Interestingly, results very similar to this and that for $\partial p_4/\partial x$ were found in the triple-deck analyses of the Blasius boundary layer with respect to the so-called lower branch stability [23] and, respectively, the associated acoustic radiation [24].

In the following, it will be assumed that $\psi_{00}$ is smaller than $U_{00}$ throughout the MD such that both $l_1$ and $l_2$ are positive quantities. As a consequence, the induced pressure gradient in the LD replacing $\partial p_1/\partial x$ from (2.27) in order to form a composite model equation when introduced into (2.25) can be written as

\[
\frac{\partial p_c}{\partial x} = \frac{U_{00}^2}{p_{00}} \mathcal{H} \left( \frac{\partial^2 A_1}{\partial x^2} \right) + \varepsilon^2 \frac{U_{00}^2}{p_{00}} \mathcal{H} \left( \frac{\partial^2 A_1}{\partial x \partial t} \right) + \varepsilon^6 \frac{U_{00}^2}{p_{00}} \partial^3 A_1 \partial x^3 C, \quad C = (l_1 + l_2) > 0. 
\] (4.19)

Accordingly, the modified dispersion relation derived from (2.29) for the compound model reads

\[
k \to \pm \infty: \quad \omega(k) = \frac{\varepsilon U_{00} C}{2} |k| - \frac{U_{00}}{2k^3} k + \frac{2^{1/4} p_{00}^{3/4} C}{2k^3 \pi^{1/2}} i(k)^{3/4} + O(|k|^{-1/4}),
\] (4.20)

leading to

\[
k \to \pm \infty: \quad \Im(\omega) = D_2 |k|^{3/4} + O(|k|^{-1/4}), \quad D_2 > 0. 
\] (4.21)

From this, it can be concluded that the growth rates $-\Im(\omega)$ are bounded from above and tend to $-\infty$ in the limit of very high wavenumbers, which, in turn, gives good reason to assume that the Cauchy problems based on this corrected model are well-posed.

In order to test this assumption, we re-examined the numerical solutions presented in [10] for a different initial value problem by using the extended model for the pressure gradient (4.19). In all cases, the temporal evolution of the leading-order displacement function $A_1$ as well as the development of the blow-up structure could be recalculated without difficulties. The results
perfectly agree with those already given in that paper, even when other numerical schemes were applied that definitely would have failed due to the lack of inherent numerical regularization terms if the uncorrected model had been used.

5. Conclusion

This paper addresses an adjoint operator approach to the calculation of higher-order displacement effects in marginal separation theory. In further consequence, this method is used to determine the initial conditions necessary for a proper formulation of the triple-deck stage, which is initiated due to a finite-time blow-up event in the marginal separation stage. Moreover, it is shown that the application of a composite asymptotic model which accounts for higher-order effects, such as streamline curvature, successfully leads to a regularization of the ill-posedness associated with initial value problems in marginal separation theory.

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References


