In this paper, the classical triple-deck formalism is employed to investigate two instability problems in which an acoustic feedback loop plays an essential role. The first concerns a subsonic boundary layer over a flat plate on which two well-separated roughness elements are present. A spatially amplifying Tollmien–Schlichting (T–S) wave between the roughness elements is scattered by the downstream roughness to emit a sound wave that propagates upstream and impinges on the upstream roughness to regenerate the T–S wave, thereby forming a closed feedback loop in the streamwise direction. Numerical calculations suggest that, at high Reynolds numbers and for moderate roughness heights, the long-range acoustic coupling may lead to absolute instability, which is characterized by self-sustained oscillations at discrete frequencies. The dominant peak frequency may jump from one value to another as the Reynolds number, or the distance between the roughness elements, is varied gradually. The second problem concerns the supersonic ‘twin boundary layers’ that develop along two well-separated parallel flat plates. The two boundary layers are in mutual interaction through the impinging and reflected acoustic waves. It is found that the interaction leads to a new instability that is absent in the unconfined boundary layer.

1. Introduction

Acoustic waves play an important role in boundary-layer transition. On the one hand, they may generate instability waves that eventually lead to transition to turbulence. This so-called receptivity often occurs due to sound waves interacting with the non-parallel flow near the leading edge [1] or with the mean flow induced
by isolated or distributed surface roughness [2–4]. On the other hand, instability waves may radiate sound waves. In the subsonic regime, strong acoustic radiation takes place when instability waves break down into small-scale turbulence [5], or when they are scattered by rapidly varying flows, such as those in the vicinity of a local roughness or trailing edge. The two processes, receptivity and radiation, though occurring at distant streamwise locations, may nevertheless be coupled by acoustic waves. A well-known closed-loop interaction involving acoustic feedback takes place in the boundary layer past an aerofoil, where instability waves are scattered in the vicinity of the trailing edge to radiate dominant sound waves. The latter propagate upstream to regenerate instability modes, which amplify and break down into turbulence (e.g. [6]). As a result of the long-range acoustic coupling, self-sustained oscillations at discrete frequencies are established, and the flow behaves like an absolutely unstable system [7,8].

In a supersonic boundary layer, a perturbation propagating supersonically relative to the free stream radiates a sound (Mach) wave to the far field. If a distant solid surface is present, the Mach wave may be reflected back to impinge on the boundary layer, thereby establishing an acoustic feedback in the transverse direction. Such a form of feedback interaction takes place in super- and hypersonic boundary-layer transition experiments, where the noise emitted by the tunnel wall impinges on and is reflected by the boundary layer on the test model to influence significantly the instability and transition (e.g. [9–11]).

In this paper, I will consider two problems that demonstrate the impact of acoustic feedback on instability characteristics. The first concerns the subsonic boundary layer over a flat plate on which two well-separated compact roughness elements are present. An acoustic feedback arises due to a spatially growing Tollmien–Schlichting (T–S) wave being scattered by the downstream roughness to radiate a sound wave that propagates upstream to interact with the upstream roughness and regenerate the T–S wave. The problem embodies essentially the same mechanisms as in the boundary layer over an aerofoil alluded to above. It was analysed recently by the author [12], and that analysis will be generalized to a more realistic roughness height. The second problem concerns the instability of the supersonic ‘twin boundary layers’ that form along two semi-infinite parallel plates. A transverse acoustic feedback is established as acoustic (Mach) waves are emitted from, and reflected between, the two boundary layers. The problem may be relevant to experiments on super- and hypersonic boundary-layer transition mentioned above.

As with much of my previous research, the present work builds on some seminal papers by Prof. Frank Smith. The mathematical framework to be used is triple-deck theory [13], which is an area where Prof. Smith has made some most remarkable contributions. Owing in great measure to his work, the theory has taken a pivotal position in theoretical fluid mechanics. It is a powerful and elegant mathematical tool that embodies a crucial physical idea, viscous–inviscid interaction, which has proved to be instrumental for understanding a range of subtle and complex phenomena; these include, to name just a few, boundary-layer separation [14–16], impact of sudden changes on boundary layers [17,18], as well as boundary-layer instabilities [19,20] and receptivity [2,3,21]. Recently, this formalism has been employed to develop an asymptotic procedure for calculating sound waves radiated by a certain type of unsteady flows that are governed by a triple-deck structure [22].

### 2. Acoustic feedback in the streamwise direction

I consider the two-dimensional compressible boundary layer over a semi-infinite flat plate. Two roughness elements are present at distances \(l\) and \((l - x_d)\) from the leading edge, respectively, and they are labelled as roughness 1 and roughness 2 in figure 1. The free-stream velocity, density, temperature, viscosity and sound speed are denoted by \(U_\infty, \rho_\infty, T_\infty, \mu_\infty\) and \(a_\infty\), respectively. The Mach number \(M\) and the Reynolds number \(R\) are defined as

\[
M = \frac{U_\infty}{a_\infty} \quad \text{and} \quad R = \frac{\rho_\infty U_\infty l}{\mu_\infty}.
\]
Figure 1. Acoustic feedback between two surface roughness elements.

I will focus on the compressible but subsonic regime with $1 > M = O(1)$, and assume that $R \gg 1$. A small parameter $\epsilon = R^{-1/8}$ is defined for convenience. For simplicity, the viscosity is assumed to obey Chapman’s law, and the Prandtl number is taken to be unity. The flow is described in a Cartesian coordinate system $(x, y)$ with its origin at a distance $l$ from the leading edge, where $x$ and $y$ are along and normal to the surface, and are non-dimensionalized by $l$ and $R^{-1/2}l$, respectively. The time variable $t$ is normalized by $l/U_\infty$. The velocity $\mathbf{u} = (u, v)$, density $\rho$ and temperature $T$ are non-dimensionalized by $U_\infty$, $\rho_\infty$ and $T_\infty$, respectively.

The unperturbed boundary layer has the velocity field $(U(x, y), R^{-1/2}V(x, y))$. Of relevance for the ensuing analysis is the wall behaviour of $U$, 

$$U \to T_w^{-1} \lambda y \quad \text{as} \quad y \to 0,$$  

(2.2)

where $T_w$ is the non-dimensional surface temperature, and $\lambda \approx 0.332$. The dependence of the wall shear on $T_w$ arises from Chapman’s viscosity law.

The roughness shapes are specified as

$$y = \epsilon h F_j(\bar{x}) \quad (j = 1, 2),$$

where $\bar{x} = T_w^{-3/2}x/\epsilon^3$. The local mean-flow distortion and the T–S wave are both described by triple-deck theory [17,20]. The same formalism was also used to analyse the scattering of the T–S wave into a sound wave [22], where the solution for the hydrodynamic field has to be obtained to $O(\epsilon^4)$ accuracy in order to predict the leading-order acoustic far field. Wu [12] later showed that this lengthy calculation can be avoided by constructing a composite theory, in which the unsteady terms in the upper and main decks are retained at leading order in the respective equations. That analysis was based on the assumption $h \ll 1$. In this work, it will be extended to the case $h = O(1)$.

In the main deck, the Blasius boundary layer is perturbed by a small-amplitude perturbation consisting of the mean-flow distortion, the T–S wave and the scattered field. The concern here is the scattered field, and its velocity can be expressed as $\delta(T_w^{3/2}U_s, \epsilon V_s) e^{-i\omega \bar{t}} + \text{c.c.}$, where $\delta \ll 1$ measures the amplitude of the incident T–S wave, $\omega$ is its rescaled frequency, and $\bar{t} = T_w^{-3/2}t/\epsilon^2$.
With the unsteady terms retained, the solution for $U_s$ and $V_s$ is found as [12]

$$U_s = A_s(\tilde{x})U' \quad \text{and} \quad V_s = -\left( \frac{\partial}{\partial \tilde{x}} - i \epsilon \omega \right) A_s(\tilde{x}), \quad (2.3)$$

where $A_s$ is the so-called displacement function, which is to be determined, and the prime denotes the derivative with respect to $y$.

In the upper deck, where $\tilde{y} \equiv \epsilon T^3 y = O(1)$, the pressure of the scattered field is expressed as $\delta \psi_s e^{-i\omega t} + \text{c.c.}$, with $\psi_s(\tilde{x}, \tilde{y})$ being governed by the boundary-value problem

$$M^2 \left(-i \epsilon \omega + \frac{\partial}{\partial \tilde{x}} \right)^2 p_s - \nabla^2 p_s = 0 \quad \text{and} \quad \frac{\partial p_s}{\partial \tilde{y}} \bigg|_{\tilde{y}=0} = \left( \frac{\partial}{\partial \tilde{x}} - i \epsilon \omega \right)^2 A_s, \quad (2.4)$$

where, as indicated above, the higher-order unsteady terms are retained, and the boundary condition follows from matching with the main-deck solution (2.3). In order to simplify the equation, let

$$p_s = e^{-i\phi M^2} \Psi_s, \quad \hat{s} = \frac{\epsilon \omega M}{(1 - M^2)}. \quad (2.5)$$

Then $p_s$ satisfies

$$(1 - M^2) \frac{\partial^2 \Psi_s}{\partial \tilde{x}^2} + \frac{\partial^2 \Psi_s}{\partial \tilde{y}^2} + (1 - M^2) \hat{s}^2 \Psi_s = 0 \quad \text{and} \quad \frac{\partial \Psi_s}{\partial \tilde{y}} \bigg|_{\tilde{y}=0} = e^{i\phi M^2} \left( \frac{\partial}{\partial \tilde{x}} - i \epsilon \omega \right)^2 A_s. \quad (2.6)$$

Taking the Fourier transform with respect to $\tilde{x}$, we can solve the above system to obtain

$$\tilde{p}_s = -\frac{1}{(1 - M^2)^{1/2} \kappa} e^{-(1-M^2)^{1/2} \hat{s}^2 \tilde{k}} e^{[i(\tilde{k} - M\hat{s}) - i \epsilon \omega \tilde{y}]} \tilde{A}_s(k - M\hat{s}), \quad (2.7)$$

where $\tilde{p}_s$ and $\tilde{A}_s$ denote the Fourier transforms of $p_s$ and $A_s$, respectively (with a normalization pre-factor $(2\pi)^{-1/2}$), and

$$\kappa \equiv [k^2 - \hat{s}^2]^{1/2} = \begin{cases} -i[k^2 - \hat{s}^2]^{1/2} & \text{for } -\hat{s} < k < \hat{s}, \\ [k^2 - \hat{s}^2]^{1/2} & \text{for } |k| > \hat{s}. \end{cases} \quad (2.8)$$

Note that the branch is taken to ensure that the wave is outgoing or decaying. For $\tilde{y} = 0$, we may invert (2.7) using the convolution theorem to obtain $\tilde{p}_s$, which is then inserted into (2.5) to give the pressure $\tilde{P}_s$ at $\tilde{y} = 0$:

$$\tilde{P}_s = \frac{1}{\sqrt{1 - M^2}} \int_{-\infty}^{\infty} g(\tilde{x} - \xi; \hat{s}) e^{-iM\tilde{x}(\tilde{x} - \xi)} \left[ \frac{\partial}{\partial \xi} - i \epsilon \omega \right]^2 A_s(\xi) d\xi, \quad (2.9)$$

where

$$g(\xi; \hat{s}) = \frac{1}{\pi} \left\{ -\int_{\hat{s}}^{\infty} \frac{\cos(k\xi)}{\sqrt{k^2 - \hat{s}^2}} dk - i \int_{0}^{\hat{s}} \frac{\cos(k\xi)}{\sqrt{\hat{s}^2 - k^2}} dk \right\}. \quad (2.10)$$

Equation (2.9) is the modified pressure–displacement relation, which accounts for acoustic radiation, a physical aspect that is left out in the usual pressure–displacement relation

$$\tilde{P}_s = \frac{1}{\pi \sqrt{1 - M^2}} \int_{-\infty}^{\infty} \frac{A_s}{\tilde{x} - \xi} d\xi. \quad (2.11)$$

Note that in the limit $\hat{s} \to 0$, $g(\tilde{x} - \xi; \hat{s}) \to (1/\pi) \ln |\hat{s}(\tilde{x} - \xi)|$. Using this in (2.9) with the $O(\epsilon)$ term neglected, and performing integration by parts, one obtains (2.11).
In the lower deck, where \( Y = T_w^{-3/2} y / \epsilon = O(1) \), the velocity and pressure can be written as
\[
\begin{align*}
 u &= T_w^1 \left\{ \epsilon(U_M, \epsilon^2 V_M) + \delta(u_l, \epsilon^2 v_l) e^{i(\alpha \bar{x} - \omega t)} + \delta(\bar{U}_s, \epsilon^2 \bar{V}_s) e^{-i \omega t} + \text{c.c.} + \cdots \right\} \\
p &= \epsilon^2 P_M + \Delta p_{\text{h}} e^{i(\alpha \bar{x} - \omega t)} + \delta \epsilon^2 \bar{P}_s e^{-i \omega t} + \text{c.c.} + \cdots,
\end{align*}
\]
where \( U_M, V_M \) and \( P_M \) denote the distorted mean flow, and \( u_l, v_l \) and \( p_l \) represent the incident T-S wave. The scattered field is governed by the system of equations
\[
\begin{align*}
\frac{\partial \bar{U}_s}{\partial \bar{x}} + \frac{\partial \bar{V}_s}{\partial \bar{y}} &= 0 \\
- i \omega \bar{U}_s + U_M \frac{\partial \bar{U}_s}{\partial \bar{x}} + U_M Y \bar{V}_s + V_M \frac{\partial \bar{U}_s}{\partial \bar{y}} + U_M \bar{\bar{x}} \bar{U}_s &= - \frac{\partial \bar{P}_s}{\partial \bar{x}} + \frac{\partial^2 \bar{U}_s}{\partial \bar{y}^2} + N,
\end{align*}
\]
where
\[
N = -u_l U_M \bar{\bar{x}} - u_l \bar{\bar{x}} U_M - v_l U_M Y - u_l Y V_M
\]
represents the forcing due to the interaction between the incident T-S wave and the roughness-induced mean flow. The no-slip condition on the wall and the matching requirement with the main deck are
\[
\begin{align*}
\bar{U}_s, \bar{V}_s &= -(u_l, v_l) e^{i(\alpha \bar{x} - \omega t)} \quad \text{at } Y = \hbar f_1(\bar{x}) \\
\bar{U}_s &\to \lambda A_s \quad \text{as } Y \to \infty.
\end{align*}
\]
The system (2.13) with (2.9) and (2.14) describes the radiation of sound by a T-S wave interacting with the mean-flow distortion. It should be noted that only the unsteady term has been included, which is the high-order term affecting the leading-order sound, ignoring other high-order terms, which the analysis of Wu & Hogg [22] finds not to affect the dominant acoustic field; those include the high-order inhomogeneous terms arising from the interaction between the mean flow and the T-S wave.

The pressure in the upper deck and acoustic zone, corresponding to \( \bar{x}, \bar{y} = O(1) \) and \( O(\epsilon^{-1}) \), respectively, is given by
\[
p_s = \frac{(1 - M^2)^{-1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1 \times e^{i(k - M \bar{\bar{x}} - (1 - M^2)^{1/2} \bar{\bar{y}})(k - M \bar{\bar{s}}) - \epsilon \omega^2 \bar{\bar{A}}_s(k - M \bar{\bar{s}})} dk.
\]
Use of the stationary phase method shows that, in the far field \( \bar{r} = (\bar{x}^2 + \bar{y}^2)^{1/2} \gg O(\epsilon^{-1}) \),
\[
p_s \sim \frac{2 \bar{\bar{A}}_s(\epsilon k_s)}{(M \epsilon \bar{\bar{r}})^{1/2}} q_s(\theta) \epsilon^2, \quad \epsilon^2 \bar{\bar{A}}_s(\epsilon k_s) \exp \left\{ i \left[ \bar{\bar{s}}(1 - M^2 \sin^2 \theta)^{1/2} \bar{\bar{r}} - \bar{\bar{s}} \bar{\bar{M}} \bar{\bar{x}} - \frac{\pi}{4} \right] \right\},
\]
where \( \theta = \tan^{-1}(\bar{\bar{y}} / \bar{\bar{x}}) \) is the usual observation angle, and
\[
q_s(\theta) = \frac{1}{(1 - M^2 \sin^2 \theta)^{1/4}} \left[ 1 - \frac{M \cos \theta}{(1 - M^2 \sin^2 \theta)^{1/2}} \right]^2,
\]
and
\[
k_s = \frac{M \epsilon}{1 - M^2} \left( -M + \frac{\cos \theta}{\sqrt{1 - M^2 \sin^2 \theta}} \right).
\]
For \( \hbar = O(1) \), the coefficients of (2.13) are dependent on \( \bar{x} \), and thus, if Fourier-transformed, the equation is non-local in the wavenumber space, which signifies energy cascade among different spectral components. A numerical approach is thus required to solve the triple-deck system for the mean-flow distortion [18] as well as the scattering problem (2.13) with (2.9) and (2.14). The numerical algorithm and calculations are being pursued in work in progress.

For the case \( \hbar \ll 1 \), the solution for both the mean-flow distortion and the scattering field can be found analytically by taking the Fourier transform with respect to \( \bar{x} \). Using (2.16), we can evaluate
the pressure in the far field. The radiated sound propagates to reach the site of the upstream roughness at \( x = -x_d \), where the acoustic pressure and streamwise velocity are found as [12]

\[
p_a = T_1 e^{-\nu(\omega/\alpha - 1-M)i \epsilon \bar{x}} p_I \quad \text{and} \quad u_a = -T_1 M e^{-\nu(\omega/\alpha - 1-M)i \epsilon \bar{x}} \tilde{p}_I.
\]

Here \( \bar{x} = T_w^{3/2}(x + x_d)/\epsilon^3 \), \( p_I \) stands for the pressure of the T-S wave approaching the roughness 2, and

\[
T_1 = -\frac{\epsilon^5 T_w^2 M^{1/2} \bar{h}}{(1 - M^2)^{3/2} \epsilon^2 \tilde{p}_0} \exp \left\{ \frac{i R^{1/4} T_w^{-3/2} \omega M x_d}{(1 - M)} \right\}
\]

measures the radiation efficiency, where \( \tilde{p}_0 \) is given by (4.54) of Wu [12] with \( \hat{F} \) replaced by \( \hat{F}_1 \) (the Fourier transform of \( F_1 \))

The acoustic wave drives a viscous one-dimensional motion \( (T_w \hat{u}_s, 0) e^{-\nu(\omega/\alpha - 1-M)i \epsilon \bar{x}} \tilde{T}_1 p_I \), i.e. the Stokes shear wave, in the lower deck, where

\[
\hat{u}_s = -\frac{M}{1 - M} \left[ 1 - \exp \left\{ -\left( -\frac{i \nu}{T_w^{1/2}} \right)^{1/2} \right\} \right].
\]

The Stokes shear wave is scattered by the roughness-induced local mean-flow distortion to generate a T-S wave. The amplitude of the latter at the roughness site is found to be

\[
p_{TS,0} = T_2 \tilde{T}_1 p_I,
\]

where the coupling coefficient [2,3]

\[
T_2 = \frac{\sqrt{2 \pi i (i \nu \lambda)^{2/3} \text{Ai}'(\eta_0)}}{\alpha \Delta(\alpha, \nu) \partial \alpha} \left\{ \hat{u}_{s,Y}(0) \hat{F}_1(1 + \eta_0 R(\eta_0)) + \int_{\eta_0}^{\infty} K(\eta, \eta_0) H_s(Y) \, d\eta \right\}
\]

Here a prime denotes the derivative with respect to \( \eta_0 \), \( \text{Ai} \) is the Airy function and \( \alpha \) is the local wavenumber of the T-S mode, determined by the dispersion relation [20]

\[
\Delta(\alpha; \nu) \equiv \int_{\eta_0}^{\infty} \text{Ai}(\eta) \, d\eta + i \nu (1 - M^2)^{1/2} (i \nu \lambda)^{2/3} \alpha^{-2} \text{Ai}'(\eta_0) = 0
\]

with \( \eta_0 = -i \nu T_w^{-1/2} (i \nu \lambda)^{-2/3} \). The expressions for \( K \) and \( R \) are given by (3.63) and (3.64) of Wu & Hogg [22], respectively, and

\[
H_s = (\nu \lambda^2)^{-1}[i k \lambda \hat{u}_s \hat{U}_m, Y + \hat{u}_{s,Y Y} (\hat{U}_{m,Y Y} - i k \lambda Y \hat{U}_m - i k \hat{P}_m)],
\]

where \( \hat{U}_m \) and \( \hat{P}_m \) denote the Fourier transforms of the streamwise velocity (in the lower deck) and the pressure of the mean-flow distortion caused by roughness 2.

The T-S wave then amplifies exponentially and attains the amplitude

\[
p_{TS}(\omega, x_d) = p_{TS,0} e^N = T(\omega, x_d, R) p_I
\]

at the roughness element downstream. Here

\[
N = i R^{3/8} T_w^{3/2} \int_{-x_d}^{0} \alpha(x) \, dx
\]

is the \( N \)-factor, and

\[
T(\omega, x_d, R) = T_1 T_2 e^N
\]

is the compound transfer function encapsulating the three key elementary physical processes involved in the feedback loop: radiation, receptivity and amplification of the T-S wave. The
regenerated T–S wave would be in phase with the original T–S wave if

\[ \arg(T) = 2n\pi, \]

where \( n \) is an integer. The feedback loop is closed and becomes self-sustained if the onset condition,

\[ T = 1, \]

is satisfied, in which case the condition (2.27) fixes the discrete tones of the oscillation. Once \( |T| \geq 1 \), an accidental perturbation may trigger a T–S wave, which will then be regenerated by its spontaneously emitted sound wave and repeatedly amplified in each feedback cycle. The system thus becomes absolutely unstable via the long-range acoustic feedback even though locally the flow is convectively unstable. Transition may occur even in the absence of any external perturbation (except of course the roughness elements).

Numerical calculations [12] suggest that for \( R = 5 \times 10^6 \), the onset condition (2.28) may be met for a rather small roughness height \( h = O(10^{-1}) \). The discrete tones depend on the parameters \( x_d \), the distance between the roughness elements, and \( R \), the Reynolds number. Figure 2a shows the variation with \( x_d \) of the frequencies of the three strongest tones. The variation with \( R \) is shown in figure 2b, where instead of \( f \) the non-dimensional frequency

\[ \omega_d \equiv \frac{\omega^* l^2}{v} \times 10^{-6} = T_w^{-3/2} \omega R^{15/4} \times 10^{-6} \]

\[ (2.29) \]
is used in order to exhibit the scaling relation with respect to \( U_\infty \). When \( x_d \) or \( R \) is increased, the tonal frequencies decrease (increase) in a piecewise linear manner, but sudden switches occur as \( x_d \) (\( R \)) passes through certain values, at which one of the originally subdominant peaks becomes the most dominant. This leads to a ‘ladder structure’ that is characteristic of an acoustic feedback loop.

3. Acoustic feedback in the transverse direction

In this section, I shall consider the instability of the supersonic ‘twin boundary layers’ that form along two parallel semi-infinite plates. The underlying instability mechanism is closely associated with the acoustic feedback between the two boundary layers. In order to illustrate such a role, I first consider the reflection of a free-stream acoustic wave impinging on the boundary layer.

(a) Reflection of an acoustic wave by a supersonic boundary layer

(i) General formulation

The reflection of acoustic waves by a boundary layer is a problem of interest in its own right, and it can be described by an inviscid theory when their characteristic wavelength is comparable with the local boundary-layer thickness. For our purpose, however, the interest is in the wavelength and frequency that comply with the unsteady triple-deck structure, in which case the reflection is affected by viscous effects. The incident wave may be represented by \( \epsilon^2 \tilde{p}_I(\bar{x}, \bar{y}, \bar{t}) \). A general formulation is presented in this subsection assuming that \( \tilde{p} = O(1) \), for which the response in the lower deck is fully nonlinear. The pressure response \( p \) in the upper deck can be expressed as 

\[
p = \epsilon^2 \tilde{p},
\]

with \( \tilde{p} \) satisfying the equation

\[
(M^2 - 1) \frac{\partial^2 \tilde{p}}{\partial \bar{x}^2} - \frac{\partial^2 \tilde{p}}{\partial \bar{y}^2} = 0 \tag{3.1}
\]

and the far-field condition

\[
\tilde{p} \to \tilde{p}_I(\bar{x}, \bar{y}, \bar{t}) + \tilde{p}_R \quad \text{as} \quad \bar{x}^2 + \bar{y}^2 \to \infty, \tag{3.2}
\]

where \( \tilde{p}_R \) stands for outgoing waves and is to be found as part of the solution. The incident wave satisfies (3.1), and hence may be specified as

\[
\tilde{p}_I = \tilde{p}_I(\bar{x} + \sqrt{M^2 - 1} \bar{y}, \bar{t}). \tag{3.3}
\]

Note that the upper-deck equation is quasi-steady; unlike the subsonic regime, there is no need to include the \( O(\epsilon) \) unsteady terms since the equation is hyperbolic for \( M > 1 \), accommodating the wave behaviour. The general solution for \( \tilde{p} \) is

\[
\tilde{p} = \tilde{p}_I(\bar{x} + \sqrt{M^2 - 1} \bar{y}, \bar{t}) + \tilde{p}_R(\bar{x} - \sqrt{M^2 - 1} \bar{y}, \bar{t}), \tag{3.4}
\]

where \( \tilde{p}_R \) stands for the reflected wave.

The velocity in the main part of the boundary layer takes the form

\[
(u, v) = \epsilon(T_\infty^{3/2} \tilde{A} U', -\epsilon \tilde{A}_z(\bar{x}, \bar{t}) U), \tag{3.5}
\]

where \( \tilde{A} \) is the displacement function to be determined. In the vertical momentum equation, \( \partial p/\partial \bar{y} = -\partial v/\partial \bar{x} \), matching \( \partial p/\partial \bar{y} \) with the main-deck solution for \( v \) gives

\[
\frac{\partial \tilde{p}}{\partial \bar{y}} = \tilde{A}_{\bar{x}} \quad \text{at} \quad \bar{y} = 0. \tag{3.6}
\]

Inserting (3.4) into (3.6) and integrating, we can determine

\[
\tilde{p}_R(\zeta, \bar{t}) = \tilde{p}_I(\zeta, \bar{t}) - (M^2 - 1)^{-1/2} \tilde{A}_z(\zeta, \bar{t}), \tag{3.7}
\]
where $\xi$ is arbitrary. Use of $\tilde{p}_R$ in (3.4) gives the pressure in the upper deck
\[
\tilde{p} = \tilde{p}_I(\tilde{x} + \sqrt{\tilde{M}^2 - 1}\tilde{y}, \tilde{t}) + \tilde{p}_T(\tilde{x} - \sqrt{\tilde{M}^2 - 1}\tilde{y}, \tilde{t}) - (\tilde{M}^2 - 1)^{-1/2}\tilde{A}_x(\tilde{x} - \sqrt{\tilde{M}^2 - 1}\tilde{y}, \tilde{t}).
\] (3.8)

Putting $\tilde{y} = 0$ gives the pressure acting on the main and lower decks
\[
\tilde{P} = 2\tilde{p}_I(\tilde{x}, \tilde{t}) - (\tilde{M}^2 - 1)^{-1/2}\tilde{A}_x(\tilde{x}, \tilde{t}),
\] (3.9)
which is the pressure–displacement relation in the presence of an incident wave.

In the lower deck, the solution for the velocity expands as
\[
(u, v) = \varepsilon T^{1/2}(\bar{U}, e^2\bar{V}) + \cdots,
\] (3.10)
and $(\bar{U}, \bar{V})$ are governed by the fully nonlinear triple-deck equations
\[
\begin{aligned}
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 \\
\frac{\partial \bar{U}}{\partial \tilde{t}} + \bar{U}\frac{\partial \bar{U}}{\partial x} + \bar{V}\frac{\partial \bar{U}}{\partial y} &= -\frac{\partial \tilde{P}}{\partial x} + \frac{\partial^2 \bar{U}}{\partial y^2}.
\end{aligned}
\] (3.11)

The usual no-slip condition on the wall and the matching with the main-deck solution (3.5) are imposed:
\[
\bar{U} = \bar{V} = 0 \quad \text{ at } Y = 0
\] (3.12)
and
\[
\bar{U} \to \lambda(Y + \bar{A}(\tilde{x}, \tilde{t})) \quad \text{as } Y \to \infty.
\]
(3.13)

The triple-deck system consisting of (3.11), (3.9) and (3.12), supplemented by (3.7), describes the reflection of an incident wave by the boundary layer as well as the response induced in the boundary layer.

(ii) Solution of the linearized system

The system (3.11) with (3.9) and (3.12) can be linearized for when $\tilde{p}_I \ll 1$. It suffices to consider each Fourier component in the incident wave, and so the pressure in the upper deck is expressed as
\[
\tilde{p} = [p_I e^{i(M^2 - 1)^{1/2}a\tilde{y}} + p_R e^{-i(M^2 - 1)^{1/2}a\tilde{y}}] e^{i(\alpha\tilde{x} - \omega\tilde{t})} + c.c.,
\] (3.13)
where $p_I$ and $p_R$ represent the amplitudes of the incident and reflected waves, respectively, and $\alpha$ and $\omega$ denote their streamwise wavenumber and frequency.

The lower-deck solution can be written as
\[
(\bar{U}, \bar{V}, \bar{P}, \bar{A}) = (\lambda Y, 0, 0, 0) + (\bar{U}, \bar{V}, \bar{P}, \bar{A}) e^{i(\alpha\tilde{x} - \omega\tilde{t})} + c.c.,
\] (3.14)
where the second term amounts to a small perturbation under the assumption $p_I \ll 1$. Substitution of (3.14) into (3.9), (3.11) and (3.12) leads to a linear triple-deck system. The relation (3.7) reduces to $p_R = p_I - i\alpha(M^2 - 1)^{-1/2}\bar{A}$. A routine calculation shows that the reflection coefficient
\[
R_s \equiv \frac{p_R}{p_I} = \frac{\Delta_+ (\alpha, \omega)}{\Delta_- (\alpha, \omega)},
\] (3.15)
where
\[
\Delta_\pm (\alpha, \omega) = \int_{\eta_0}^{\infty} \text{Ai}(\eta) d\eta \pm \lambda (i\alpha\lambda)^{2/3} \alpha^{-2}(M^2 - 1)^{1/2}\text{Ai}'(\eta_0).
\] (3.16)

(b) Instability of twin supersonic boundary layers

Consider the ‘twin boundary layers’ that develop along two semi-infinite parallel plates, separated by a distance $h$ (figure 3). In order to relate to, and contrast with, the instability of a usual unconfined boundary layer [20], I focus on locations where the scale of the upper deck
Figure 3. Sketch of the twin boundary layers.

is comparable with \( h \), i.e. \( h/l = R^{-3/8}h_s \), where \( l \) is the distance of the location of interest to the leading edge. The lower and upper plates are assumed to be located at \( y = 0 \) and \( y = h \), respectively.

(i) General formulation

Since the time and streamwise scales of the instability modes are of \( O(R^{-1/4}l/U_\infty) \) and \( O(R^{-3/8}l) \), the perturbation is governed by two triple-deck structures, which share the same upper deck. The pressure in the upper deck can be written as \( \tilde{p} \), with \( \tilde{p} \) being given by (3.4). Clearly, \( \tilde{p} \) consists of an incident and a reflected acoustic wave. The solutions in the main layers in the upper and lower boundary layers are given by (3.5) with the displacement function \( \tilde{A} \) being replaced by \( \tilde{A}^\pm \).

Matching the upper-deck pressure gradient \( \partial \tilde{p}/\partial \tilde{y} \) with the main-deck solution \( -v \) in the lower boundary layer gives

\[
\tilde{p}_I(\tilde{x}, \tilde{t}) - \tilde{p}_R(\tilde{x}, \tilde{t}) = (M^2 - 1)^{-1/2} \tilde{A}^- (\tilde{x}, \tilde{t}). \tag{3.17}
\]

The upper wall is located at \( \tilde{y} = h_s \), and a similar consideration of matching with the main-deck solution in the upper boundary layer leads to

\[
- \tilde{p}_I(\tilde{x} + \sqrt{M^2 - 1}h_s) + \tilde{p}_R(\tilde{x} - \sqrt{M^2 - 1}h_s) = (M^2 - 1)^{-1/2} \tilde{A}^+ (\tilde{x}, \tilde{t}). \tag{3.18}
\]

The relations (3.17) and (3.18) are Fourier-transformed to

\[
\hat{p}_I - \hat{p}_R = (M^2 - 1)^{-1/2}i k \hat{A}^- \quad \text{and} \quad - e^{i \sigma k} \hat{p}_I + e^{-i \sigma k} \hat{p}_R = (M^2 - 1)^{-1/2}i k \hat{A}^+,
\]

where \( \sigma = (M^2 - 1)^{1/2}h_s \). From the above equations, we obtain

\[
\hat{p}_I = -ik(M^2 - 1)^{-1/2} e^{-i \sigma k} \hat{A}^- + \hat{A}^+ \quad \text{and} \quad \hat{p}_R = -ik(M^2 - 1)^{-1/2} e^{i \sigma k} \hat{A}^- + \hat{A}^+ \tag{3.19}
\]

The Fourier transforms of the pressures acting on the lower and upper walls are found as

\[
\hat{p}^\pm = -ik(M^2 - 1)^{-1/2}(\cot(\sigma k) \hat{A}^\pm + \csc(\sigma k) \hat{A}^\mp).
\]

These are inverted by using the convolution to obtain the pressure–displacement relations

\[
\hat{p}^\pm = \int_{-\infty}^{\infty} g_1(X - \xi) \hat{A}_\xi^\pm d\xi + \int_{-\infty}^{\infty} g_2(X - \xi) \hat{A}_\xi^\mp d\xi, \tag{3.20}
\]

with \( g_1 \) and \( g_2 \) denoting the Fourier inversions of \( -\cot(\sigma k) \) and \( -\csc(\sigma k) \), respectively.
In the lower decks, the solution for the velocity can be expanded as
\[ (u, v) = \epsilon T_\nu^{1/2} (\tilde{U}^\pm, \epsilon^2 \tilde{V}^\pm) + \cdots, \] (3.21)
and \((\tilde{U}^\pm, \tilde{V}^\pm)\) are governed by the fully nonlinear triple-deck equations
\[ \begin{align*}
\frac{\partial \tilde{U}^\pm}{\partial x} + \frac{\partial \tilde{V}^\pm}{\partial Y} &= 0 \\
\frac{\partial \tilde{U}^\pm}{\partial t} + \tilde{U}^\pm \frac{\partial \tilde{U}^\pm}{\partial x} + \tilde{V}^\pm \frac{\partial \tilde{U}^\pm}{\partial Y} &= -\frac{\partial \tilde{P}^\pm}{\partial x} + \frac{\partial^2 \tilde{U}^\pm}{\partial Y^2},
\end{align*} \] (3.22)
subject to the boundary and matching conditions
\[ \begin{align*}
\tilde{U} = \tilde{V} &= 0 \quad \text{at } Y = 0 \\
\tilde{U}^\pm &\to \lambda (Y + A^\pm(\bar{x}, \bar{t})) \quad \text{as } Y \to \infty.
\end{align*} \] (3.23)
The system (3.22) with (3.20) and (3.23) describes the lower branch nonlinear instability of the twin supersonic boundary layers.

(ii) Acoustic-feedback-induced instability: linear analysis

Before performing a more formal stability analysis using the general formulation presented above, it is illuminating and informative to deduce the dispersion relation from the perspective of reflection of acoustic waves. Note first that, for a small-amplitude perturbation, the pressure of the disturbance in the upper deck can be expressed as
\[ \tilde{p} = [p_I e^{i(M^2 - 1)^{1/2}a^\bar{y}} + p_R e^{-i(M^2 - 1)^{1/2}a^\bar{y}}] e^{i(\alpha \bar{x} - \omega \bar{t})} + \text{c.c.}, \] (3.24)
the same as (3.13). For the lower boundary layer, the first and the second terms represent the incident and reflected waves, respectively. As is shown in §3.1.2, the reflection coefficient is given by (3.15).

Now in the presence of the upper plate at \(\bar{y} = h_s\), I introduce the variable \(\bar{y}^+ = h_s - \bar{y}\) to analyse the ensuing upper boundary layer. In terms of \(\tilde{y}^+\), the disturbance (3.13) may be written as
\[ p = \epsilon^2 [p_I^+ e^{-i(M^2 - 1)^{1/2}a^\bar{y}^+} + p_R^+ e^{i(M^2 - 1)^{1/2}a^\bar{y}^+}], \] (3.25)
with
\[ p_R^+ = e^{i(M^2 - 1)^{1/2}a h_s} p_I \quad \text{and} \quad p_I^+ = e^{-i(M^2 - 1)^{1/2}a h_s} p_R. \] (3.26)
Now the second and the first terms in (3.25) represent, respectively, the incident and reflected waves in the upper boundary layer. The reflection occurs in the same manner as in the lower boundary layer, and hence
\[ \frac{p_R^+}{p_I^+} = \frac{\Delta_+ (\alpha, \omega)}{\Delta_- (\alpha, \omega)}. \] (3.27)
Combining (3.15) with (3.27) and using (3.26), we obtain the required dispersion relation
\[ \Delta (\alpha, \omega) = \Delta_+ (\alpha, \omega) e^{-i(M^2 - 1)^{1/2}a h_s} \pm \Delta_- (\alpha, \omega) = 0, \] (3.28)
from which the instability can be determined. Here the +/- signs correspond to the symmetric/antisymmetric modes, respectively.
The dispersion relation (3.28) can also be derived by a formal analysis, where we decompose the solution as
\[
(\hat{U}^\pm, \hat{V}^\pm, \hat{P}^\pm, \hat{A}^\pm) = (\lambda Y, 0, 0, 0) + (\hat{U}^\pm, \hat{V}^\pm, \hat{P}^\pm, \hat{A}^\pm) e^{i(\alpha x - \omega t)} + \text{c.c.}
\] (3.29)
Substitution of (3.29) into (3.22) leads to linearized triple-deck equations, which can be solved to find
\[
\hat{U}^\pm = C^\pm \int_0^{\eta_0} A_i(\eta) \, d\eta \quad (\eta = (i\alpha \lambda)^{1/3} Y + \eta_0),
\]
where \( C^\pm \) are constants. The boundary and matching conditions (3.23) imply that
\[
C^\pm (i\alpha \lambda)^{2/3} A_i(\eta_0) = i\alpha \hat{P}^\pm \quad \text{and} \quad C^\pm \int_0^\infty A_i(\eta) \, d\eta = \lambda \hat{A}^\pm,
\]
while the pressure–displacement relations (3.20) give
\[
\hat{P}^\pm = -i\alpha (M^2 - 1)^{-1/2} (\cot(\sigma \alpha) \hat{A}^\pm + \csc(\sigma \alpha) \hat{A}^\mp).
\]
Elimination of \( C^\pm \) and \( \hat{P}^\pm \) among the above equations yields the dispersion relation (3.28). It is worth noting that the system admits symmetric (\( \hat{A}^+ = \hat{A}^- \)) and antisymmetric modes (\( \hat{A}^+ = -\hat{A}^- \)), corresponding, respectively, to the plus and minus signs in (3.28).

Comparing the dispersion relation (3.28) with the reflection coefficient (3.15), we note that, in the case of spatial instability,
\[
e^{i(M^2 - 1)^{1/2} h_s \alpha} = \pm R_s \quad \text{and so} \quad e^{-(M^2 - 1)^{1/2} h_s \alpha} = (|R_s|)^n.
\] (3.30)
where \( \alpha = \alpha_r + i\alpha_i \). The above relation indicates that a neutral mode (\( \alpha_i = 0 \)) corresponds to a perfect reflection (\( |R_s| = 1 \)); when over-reflection occurs (\( |R_s| > 1 \)), instability is expected (\( \alpha_i > 0 \)). The instability may therefore be attributed to over-reflection of acoustic waves by the boundary layers. Specifically, neutral modes can be determined from the relations
\[
|R_s(\alpha, \omega)| = 1 \quad \text{and} \quad (M^2 - 1)^{1/2} h_s \alpha = \arg R_s(\alpha, \omega) + n\pi,
\] (3.31)
where \( 0 \leq \arg R_s < 2\pi \), with \( n = 0 \) corresponding to an antisymmetric mode and \( n = 1 \) to a symmetric mode. Higher symmetric (\( n = 3, 5, \ldots \)) and antisymmetric (\( n = 2, 4, \ldots \)) modes also exist, but unstable modes adjacent to them have smaller growth rates.

Computationally, rather than solving for \( \alpha \) and \( \omega \) for a given \( h_s \), it is convenient to find \( \alpha \) for a given \( \omega \) (or vice versa) such that the first relation of (3.31) is satisfied. Then determine from the second relation the value of \( h_s \) for which (\( \alpha, \omega \)) is a neutral mode branching from the perfect reflection. Repeating this for different \( \omega \) leads to a relation or neutral curve, \( h_s = h_s(\omega) \). Figure 4 shows the neutral curves so obtained for \( M = 2 \). For a given distance \( h_s \), there exist symmetric and antisymmetric modes. For \( h_s = 5 \), the modes have (rescaled) frequencies \( \lambda^{-3/2}\omega \approx 3.66, 2.30 \), respectively (with the responding wavenumbers \( \lambda^{-5/4} \approx 2, 0.98 \)). The reflection coefficients are calculated for the above wavenumbers while varying \( \omega \). The result, shown in figure 5a, indicates that over-reflection occurs (\( |R_s| > 1 \)) for frequencies greater than the respective neutral values. From the dispersion relation, the spatial growth rates of the symmetric and antisymmetric modes are computed for \( h_s = 5 \), and they are shown in figure 5b. Clearly, unstable modes emerge in the range of acoustic over-reflection, and symmetric modes have larger growth rates. The instability is found to be of convective nature. Temporal growth would have to be considered had the instability been absolute.

The existence of unstable planar modes is in contrast with the unconfined case, where amplification occurs only for sufficiently oblique modes with spanwise wavenumber \( \beta > (M^2 - 1)^{1/2} \alpha \) [20], for which the eigenfunctions attenuate exponentially in the free stream. The new spectrum of instability for the present twin boundary layers arises as a consequence of acoustic feedback. A somewhat similar mechanism operates in the hypersonic boundary layer over a wedge with a small half-angle [23], where sound waves are reflected between the shock and the boundary layer.
Figure 4. Symmetric and antisymmetric neutral modes branching from perfect reflection for $M = 2$.

Figure 5. (a) Reflection coefficient $R_s$ and (b) the growth rate $(-\alpha_i)$ for $M = 2.0$ and $h_s = 5$. 
4. Concluding remarks

As a typical form of ambient disturbances, external sound waves are known to influence transition via receptivity, an issue that has received much attention. In this paper, I considered two problems and demonstrated that acoustic waves can be an intrinsic part of the instability. In the case of a subsonic boundary layer with two isolated roughness elements on an otherwise flat plate, the sound wave is emitted spontaneously by the T–S wave when the latter is scattered by the mean-flow distortion. The sound wave propagates upstream to interact with the upstream roughness, regenerating a T–S wave. The resulting acoustic feedback in the streamwise direction renders the system absolutely unstable in the sense that it exhibits self-sustained oscillations at discrete frequencies. For the supersonic twin boundary layers formed along two parallel flat plates, the disturbance in the central region corresponds to acoustic waves being reflected between the boundary layers. The transverse acoustic feedback causes a new spectrum of instability in this case.

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References


